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α-Supraposinormality of Operators in Dense Norm-Attainable Classes

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Abstract

The notion of supraposinormality was introduced by Rhaly in a superclass of posinormal operators. In this paper, we give an extension of this notion of supraposinormality to α-supraposinormality of operators in the dense norm-attainable class.

1. Introduction

Characterization of normality has been done in different aspects by many mathematicians. In [1, 2, 3] and the references therein, they showed characterizations of posinormality and gave some spectral properties of posinormal operators. The relationship between a hyponormal operator and a posinormal operator has also been considered [1]. The author in [2] further introduced a superclass of the posinormal operators and determined sufficient conditions for this superclass to be posinormal and hyponormal. The idea of norm-attainabilty has also been considered by quite a number of authors, for instance, [4, 5] considered conditions for norm-attainability for elementary operators. In this paper, we are interested in characterizing $\alpha-$supraposinormal operators in dense norm-attainable classes. At this point, we give some useful notations. From [1] it is known that an operator A on a Hilbert space H is posinormal if and only if $\gamma^2 A^*A \geq AA^*$ for some $\gamma \geq 0$. A is hyponormal when $\gamma = 1$. The operator A is dominant if $Ran(A-\lambda) \subset Ran(A-\lambda)^*$ for all λ in the spectrum of A; A is dominant if and only if $A-\lambda$ is posinormal for all complex numbers λ. Hyponormal operators are necessarily dominant. If A is posinormal, then $KerA \subset KerA^*$. Moreover, A is norm-attainable if there exists a unit vector $x \in H$ such that $\|Ax\| = \|A\|$, where $\|.\|$ is the usual operator norm [5]. The class of all norm-attainable operators is denoted by $NA(H)$. In this work, without loss of generality, $NA(H)$ is taken to be norm dense and separable unless otherwise stated and $NA(H) \subseteq B(H)$.

2. Preliminaries

In this section, we give some definitions and auxiliary results which are useful in the sequel.

Definition 2.1. *Let $A \in NA(H)$, we say that A is supraposinormal if there exist positive operators S and T on H such that $ASA^* = A^*TA$, where at least one of S, T has dense range. The ordered pair (T,S) is called an interrupter pair associated with A.*

Definition 2.2. *Let $A \in NA(H)$, then for some positive integer α we say that A is α-supraposinormal if there exist positive invertible operators S and T on H such that $A^{\alpha}SA^* = A^{\alpha *}TA$, where at least one of S, T has a separable range and A is self-adjoint. For simplicity we denote an α-supraposinormal operator by A^{α}.*

Definition 2.3. *Let $A \in NA(H)$, we say that A is totally supraposinormal if A-λ is supraposinormal for all complex numbers λ.*

We know that the superclass of operators contains all operators which are posinormal, hyponormal, invertible, positive, coposinormal and norm-attainable [3]. If A is posinormal, then $AA^* = A^*PA$ for some positive operator P, so A is supraposinormal with interrupter pair (I,P). If A is coposinormal, then $A^*A = AQA^*$ for some positive operator Q, so A is supraposinormal with interrupter pair (Q,I).

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Remark 2.4. *Analogously from [3], the collection $\mathscr{S}$ of all supraposinormal operators on H forms a cone in $NA(H)$, and $\mathscr{S}$ is involutive. Indeed, it is easy to see that $\mathscr{S}$ is closed under scalar multiplication, so $\mathscr{S}$ contains all αA for $A \in \mathscr{S}$ and $\alpha \geq 0$, and therefore $\mathscr{S}$ is a cone. Moreover, it is equally easy to see that A is supraposinormal if and only if A^* is supraposinormal, so $\mathscr{S}$ is closed under involution since $NA(H)$ is a C^*-algebra.*

3. Main Results

In this section, we give the main results in this paper. We begin with the following proposition.

Lemma 3.1. *Let $A \in NA(H)$ satisfy $A^{\alpha}QA^* = A^{\alpha *}PA$ for positive invertible operators $P, Q \in NA(H)$ and a positive integer α. The following conditions hold:*

(i). If Q has separable and norm dense range, then A is supraposinormal and $KerA^{\alpha} \subset KerA^{\alpha *}$.
(ii). If P has separable dense range, then A is supraposinormal and dominant. Moreover, $KerA^{\alpha} \subset KerA^{\alpha *}$.
(iii). If Q is positive invertible and norm-attainable, then the α-supraposinormal operator A is α-posinormal and hence α-hyponormal.
(iv). If P is positive invertible and norm-attainable, then the α-supraposinormal operator A is α-coposinormal.
(v). If P and Q are both positive invertible and norm-attainable, then A is both posinormal and coposinormal with $KerA^{\alpha} = KerA^{\alpha *}$ and $RanA^{\alpha} = RanA^{\alpha *}$.
(vi). If P and Q are both positive invertible, norm-attainable and either is dominant, then A is both α-coposinormal and norm-attainable with $KerA^{\alpha} \cap KerA^{\alpha *} = RanA^{\alpha} \cap RanA^{\alpha *}$.

Proof. Proofs of $(i)-(v)$ follow analogously from [3]. For the proof of (vi), We consider the orthogonal complements of $KerA^{\alpha} \cap KerA^{\alpha *}$ and $RanA^{\alpha} \cap RanA^{\alpha *}$. Since $NA(H)$ is a C^*-algebra, normality and norm-attainabilty of P and Q are necessary. Hence, Fugledge-Putman theorem for posinormal and norm attainable class suffices. This completes the proof. □

Theorem 3.2. *Let $A^{\alpha} - \lambda$ be supraposinormal for distinct real values $\lambda = r_1, r_2, ..., r_k$, and assume that the same interrupter pair (Q, P) serves $A^{\alpha} - \lambda$ in each value of the sequence. Then $Q = P$ and $Ker(A^{\alpha} - \lambda) = Ker(A^{\alpha} - \lambda)^*$ when $\lambda = r_1, r_2, ..., r_k$*

Proof. We first consider three cases when $\lambda = 0, r_1$, and r_2, as in [3]. For any positive integer α, $(A^{\alpha} - \lambda)Q(A^{\alpha} - \lambda)^* = (A^{\alpha} - \lambda)^* P(A^{\alpha} - \lambda)$ for we find that for $k = 1$ and 2, $(A - r_k)Q(A - r_k)^* = (A^{\alpha} - r_k)^* P(A^{\alpha} - r_k)$ reduces to $PA^{\alpha} + A^{\alpha} * P + r_k Q = QA^{\alpha} * + A^{\alpha} Q + r_k P$. Therefore, $(r_1 - r_2)Q = (r_1 - r_2)P$, so $Q = P$. The fact that $Ker(A^{\alpha} - \lambda) = Ker(A^{\alpha} - \lambda)^*$ for $\lambda = 0, r_1$, and r_2 follows from [2], Corollary 3.2. For the complete sequence upto r_k, we consider Caratheodory's extension theorem and by Proposition (**??**), the proof is complete. □

For a generalization consider the following corollary.

Corollary 3.3. *If $A^{\alpha} \in B(H)$ is totally supraposinormal and the same two positive operators $Q, P \in B(H)$ form an interrupter pair (Q, P) for $A^{\alpha} - \lambda$ for all complex numbers λ, then $Q = P$; it also follows that $Ker(A^{\alpha} - \lambda) = Ker(A^{\alpha} - \lambda)^*$ for all λ if and only if $A^{\alpha} = Ker(A^{\alpha *}$.*

Proof. The proof is analogous to that of [3], Corollary 3. □

4. Conclusion

We conclude with the following open question: Does there exist an operator A^{α} that is totally α-supraposinormal but neither norm-attainable nor dominant/codominant in a non-separable space?

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