MORE ON TRANSLATIONALLY SLOWLY VARYING SEQUENCES

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Abstract. We define and study an equivalence relation in the class \( \text{Tr}(SV_s) \) of translationally slowly varying positive real sequences and its relations with selection principles and game theory. We also prove a game-theoretic result for translationally rapidly varying sequences.

1. Introduction

Throughout the paper \( \mathbb{N} \) will denote the set of natural numbers, \( \mathbb{R} \) the set of real numbers, \( \mathcal{S} \) the set of sequences of positive real numbers.

The theory of regular variation, including in particular slow variation, was initiated in 1930 by J. Karamata [8]. Nowadays this branch of asymptotic analysis of divergent processes is known as Karamata’s theory of regular variation. Another kind of variation, called rapid variation, was introduced and first studied in 1970 by de Haan [7]. These two theories are developed for functions and sequences and have various applications in several mathematical disciplines: number theory, differential and difference equations, probability theory, \( q \)-calculus, and so on. For more information about the theory of regular variation and the theory of rapid variation we refer the reader to the book [1]. In this article we are interested in two classes of sequences related to slow and rapid variations.

We recall first the definitions of slowly and rapidly varying sequences.

**Definition 1.1.** ([1] [2] [12]) A sequence \( c = (c_n)_{n \in \mathbb{N}} \in \mathcal{S} \) is slowly varying (respectively, rapidly varying) if for each \( \lambda > 0 \) (respectively, \( \lambda > 1 \)) the following is satisfied:

\[
\lim_{n \to \infty} \frac{c \lfloor \lambda n \rfloor}{c_n} = 1,
\]

(1.1) (respectively,

\[
\lim_{n \to \infty} \frac{c \lfloor \lambda n \rfloor}{c_n} = \infty,
\]

(1.2)

where for \( x \in \mathbb{R}, \lfloor x \rfloor \) denotes the greatest integer part of \( x \).

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The classes of slowly varying and rapidly varying sequences are denoted by $SV_s$ and $R_{s,\infty}$, respectively.

In what follows we work with the following two classes of sequences.

**Definition 1.2.** ([3][11]) A sequence $c = (c_n)_{n \in \mathbb{N}} \in S$ is *translationally slowly varying* (respectively, *translationally rapidly varying*) if for each $\lambda \geq 1$ the following asymptotic condition is satisfied:

\[
\lim_{n \to \infty} \frac{c_{n+\lambda}}{c_n} = 1 \quad \text{(1.3)}
\]

(respectively,

\[
\lim_{n \to \infty} \frac{c_{n+\lambda}}{c_n} = \infty \quad \text{(1.4)}
\]

$Tr(SV_s)$ denotes the class of translationally slowly varying sequences, and $Tr(R_{s,\infty})$ denotes the class of translationally rapidly varying sequences (see [2, 3, 4, 5]).

Observe that $R_{s,\infty} \cap Tr(SV_s) \neq \emptyset$, $R_{s,\infty} \setminus Tr(SV_s) \neq \emptyset$, $Tr(SV_s) \setminus R_{s,\infty} \neq \emptyset$, and $Tr(R_{s,\infty}) \subset R_{s,\infty}$.

In this paper we define and study a new equivalence relation in the class $Tr(SV_s)$, in particular its relations with selection principles and game theory. We also provide a game-theoretic result concerning the class $Tr(R_{s,\infty})$.

### 2. Results

We begin this section with definitions of concepts we use in this article.

**Definition 2.1.** Sequences $c = (c_n)_{n \in \mathbb{N}}$ and $d = (d_n)_{n \in \mathbb{N}}$ from $S$ are mutually *translationally slowly equivalent*, denoted by

\[
c_n \sim_{ts} d_n, \text{ as } n \to \infty,
\]

if

\[
\lim_{n \to \infty} \frac{c_{n+\lambda}}{d_n} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{d_{n+\lambda}}{c_n} = 1 \quad \text{(2.1)}
\]

hold for each $\lambda \geq 1$.

**Definition 2.2.** Sequences $c = (c_n)_{n \in \mathbb{N}}$ and $d = (d_n)_{n \in \mathbb{N}}$ from $S$ are mutually *translationally rapidly equivalent*, denoted by

\[
c_n \sim_{tr} d_n, \text{ as } n \to \infty,
\]

if

\[
\lim_{n \to \infty} \frac{c_{n+\lambda}}{d_n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{d_{n+\lambda}}{c_n} = \infty \quad \text{(2.2)}
\]

hold for each $\lambda \geq 1$.

**Theorem 2.1.** Let sequences $c = (c_n)_{n \in \mathbb{N}}$ and $d = (d_n)_{n \in \mathbb{N}}$ be elements from $S$. If $c_n \sim_{ts} d_n$, as $n \to \infty$, then $c \in Tr(SV_s)$ and $d \in Tr(SV_s)$.

Proof. For $\lambda \geq 1$ we have

\[
\lim_{n \to \infty} \frac{c_{n+\lambda}}{c_n} = \lim_{n \to \infty} \left( \frac{c_{n+1}}{c_n} \right)^{[\lambda]}
\]

if the limit on the right side exists. Further, since $c_n \sim_{ts} d_n$, we have

\[
\lim_{n \to \infty} \frac{c_{n+2}}{c_n} = \lim_{n \to \infty} \left( \frac{c_{n+2}}{d_{n+1}} \cdot \frac{d_{n+1}}{c_n} \right) = \lim_{n \to \infty} \frac{c_{n+2}}{d_{n+1}} \cdot \lim_{n \to \infty} \frac{d_{n+1}}{c_n} = 1.
\]
Therefore
\[ 1 = \lim_{n \to \infty} \left( \frac{c_{n+2}}{c_{n+1}} \cdot \frac{c_{n+1}}{c_n} \right) = \lim_{k \to \infty} \left( \frac{c_{k+1}}{c_k} \right)^2, \]

hence
\[ \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 1. \]

This means that
\[ \lim_{n \to \infty} \frac{c_{[n+\lambda]}}{c_n} = 1 \text{ for each } \lambda \geq 1, \]
i.e. \( c \in \text{Tr}(SV_s) \).

Similarly we prove \( d \in \text{Tr}(SV_s) \).

In a similar way, by suitable modifications in the proof, we prove the following result.

**Theorem 2.2.** Let \( c = (c_n)_{n \in \mathbb{N}} \) and \( d = (d_n)_{n \in \mathbb{N}} \) be sequences in \( S \). If \( c_n \overset{ts}{\sim} d_n \), as \( n \to \infty \), then \( c \in \text{Tr}(R_{S,\infty}) \) and \( d \in \text{Tr}(R_{S,\infty}) \).

**Theorem 2.3.** Relation \( \overset{ts}{\sim} \) is an equivalence relation on \( \text{Tr}(SV_s) \).

**Proof.**
1. (Reflexivity) Let \( c \in \text{Tr}(SV_s) \). Then \( \lim_{n \to \infty} \frac{c_{[n+\lambda]}}{c_n} = 1 \) for each \( \lambda \geq 1 \), that is, \( c_n \overset{ts}{\sim} c_n \) as \( n \to \infty \), and so reflexivity holds.

2. (Symmetry) It follows from the definition of relation \( \overset{ts}{\sim} \).

3. (Transitivity) Let \( c = (c_n)_{n \in \mathbb{N}}, d = (d_n)_{n \in \mathbb{N}} \) and \( e = (e_n)_{n \in \mathbb{N}} \) be elements from \( \text{Tr}(SV_s) \) such that \( c_n \overset{ts}{\sim} d_n, n \to \infty \), and \( d_n \overset{ts}{\sim} e_n, n \to \infty \). Then we have
\[ \lim_{n \to \infty} \frac{c_{n+2}}{e_n} = \lim_{n \to \infty} \frac{c_{n+2}}{d_{n+1}} \cdot \lim_{n \to \infty} \frac{d_{n+1}}{e_n} = 1. \]

We conclude
\[ 1 = \lim_{n \to \infty} \left( \frac{c_{n+2}}{e_{n+1}} \cdot \frac{e_{n+1}}{e_n} \right). \]

Because of \( e \in \text{Tr}(SV_s) \), we obtain
\[ \lim_{n \to \infty} \frac{c_{n+1}}{e_n} = 1. \]

It follows from here that for each \( \lambda \geq 1 \) it holds
\[ \lim_{n \to \infty} \frac{c_{[n+\lambda]}}{e_n} = 1. \]

In a similar way one proves
\[ \lim_{n \to \infty} \frac{e_{[n+\lambda]}}{c_n} = 1, \lambda \geq 1, \]
which means \( c_n \overset{ts}{\sim} e_n \). \( \square \)

**Remark.** Let a sequence \( c = (c_n)_{n \in \mathbb{N}} \) belong to the class \( \text{Tr}(SV_s) \) and let \( d = (d_n)_{n \in \mathbb{N}} \in \mathcal{S} \) be such that \( c_n \overset{ts}{\sim} d_n \). Then
\[ \lim_{n \to \infty} \frac{c_n}{d_n} = \lim_{n \to \infty} \left( \frac{c_n}{c_{n+1}} \cdot \frac{c_{n+1}}{d_n} \right) = 1 \]
and we conclude that sequences \( c \) and \( d \) are strongly asymptotically equivalent (see, for instance. [1, 6]), i.e. \( \lim_{n \to \infty} \frac{c_n}{d_n} = 1 \).
Recall the definition of selection principles, which we need in what follows (see [9][10]).

**Definition 2.3.** Let \( A \) and \( B \) be subfamilies of the set \( S \). The symbol \( \alpha_i(A, B) \), \( i \in \{2, 3, 4\} \), denotes the following selection hypotheses: for each sequence \((A_n)_{n \in \mathbb{N}}\) of elements from \( A \) there is an element \( B \in B \) such that:

1. \( \alpha_2(A, B) \): the set \( \text{Im}(A_n) \cap \text{Im}(B) \) is infinite for each \( n \in \mathbb{N} \);
2. \( \alpha_3(A, B) \): the set \( \text{Im}(A_n) \cap \text{Im}(B) \) is infinite for infinitely many \( n \in \mathbb{N} \);
3. \( \alpha_4(A, B) \): the set \( \text{Im}(A_n) \cap \text{Im}(B) \) is nonempty for infinitely many \( n \in \mathbb{N} \), where \( \text{Im} \) denotes the image of the corresponding sequence.

The following infinitely long game is related to \( \alpha_2 \) (see [9][10]).

**Definition 2.4.** Let \( A \) and \( B \) be nonempty subfamilies of \( S \). The symbol \( G_{\alpha_2}(A, B) \) denotes the following infinitely long game for two players, I and II, who play a round for each natural number \( n \). In the first round I chooses an arbitrary element \( A_1 = (A_1, j)_{j \in \mathbb{N}} \) from \( A \), and II chooses a subsequence \( y_{r_1} = (A_1, r_1(j))_{j \in \mathbb{N}} \) of the sequence \( A_1 \). At the \( k^{th} \) round, \( k \geq 2 \), I chooses an arbitrary element \( A_k = (A_k, j)_{j \in \mathbb{N}} \) from \( A \) and II chooses a subsequence \( y_{r_k} = (A_k, r_k(j))_{j \in \mathbb{N}} \) of the sequence \( A_k \), such that \( \text{Im}(r_k(j)) \cap \text{Im}(r_{p(j)}) = \emptyset \) is satisfied, for each \( p \leq k - 1 \). II wins a play

\[
A_1, y_{r_1}, \ldots; A_k, y_{r_k}, \ldots
\]

if and only if all elements from \( Y = \bigcup_{k \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} A_k, r_k(j) \), with respect to second index, form a subsequence \( y = (y_m)_{m \in \mathbb{N}} \in B \).

A strategy \( \sigma \) for the player II is a *coding strategy* if II remembers only the most recent move by I and by II before deciding how to play the next move.

Observe, that if II has a winning strategy in the game \( G_{\alpha_2}(A, B) \), then the selection principle \( \alpha_2(A, B) \) is true. Also, \( \alpha_2(A, B) \Rightarrow \alpha_3(A, B) \Rightarrow \alpha_4(A, B) \).

Let \( c = (c_n)_{n \in \mathbb{N}} \in S \). Then we define

\[
[c]_{ts} = \{d = (d_n)_{n \in \mathbb{N}} \in S : c_n \xrightarrow{ts} d_n, n \to \infty\}
\]

(2.3) as the equivalence class of \( c \) in \( \text{Tr}(SV_s) \).

**Theorem 2.4.** For a fixed element \( c \in \text{Tr}(SV_s) \), the player II has a winning coding strategy in the game \( G_{\alpha_2}([c]_{ts}, [c]_{ts}) \).

**Proof.** (1\textsuperscript{st} round): Let \( \sigma \) be the strategy of the player II. The player I chooses a sequence \( x_1 = (x_{1,n})_{n \in \mathbb{N}} \in [c]_{ts} \) arbitrary. Then the player II chooses the subsequence \( \sigma(x_1) = (x_{1,k_1(n)})_{n \in \mathbb{N}} \) of the sequence \( x_1 \), where \( \text{Im}(k_1) \) is the set of natural numbers greater of or equal to \( n_1 \in \mathbb{N} \) which are divisible by 2 and not divisible by \( 2^2 \), and \( 1 - \frac{1}{2} \leq \frac{c_n}{x_{m,n}} \leq 1 + \frac{1}{2} \) holds for each \( n \geq n_1 \).

\((m\textsuperscript{th} \text{ round, } m \geq 2)\): The player I chooses a sequence \( x_m = (x_{m,n})_{n \in \mathbb{N}} \in [c]_{ts} \). Then the player II chooses the subsequence

\[
\sigma(x_m, (x_{m-1,k_{m-1}(n)})_{n \in \mathbb{N}}) = (x_{m,k_m(n)})_{n \in \mathbb{N}}
\]
of the sequence \( x_m \), so that \( \text{Im}(k_m) \) is the set of natural numbers greater of or equal to \( n_m \in \mathbb{N} \), which are divisible by \( 2^m \), and not divisible by \( 2^{m+1} \), and \( 1 - \frac{1}{2^m} \leq \frac{c_n}{x_{m,n}} \leq 1 + \frac{1}{2^m} \) holds for each \( n \geq n_m \).
Consider now the set \( Y = \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} x_{m,k_m(n)} \in \mathcal{S} \) indexed by the second index \( k_m(n) \). This set we can consider as the subsequence of the sequence \( y = (y_i)_{i \in \mathbb{N}} \) given by:

\[
y_i = \begin{cases} 
  x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}; \\
  c_i, & \text{otherwise}.
\end{cases}
\]

By the construction \( y \in \mathcal{S} \). Also, the intersection of \( y \) and \( x_m, m \in \mathbb{N} \), is an infinite set.

Let us prove that \( y_m \xrightarrow{\varepsilon} c_m, as \, m \to \infty \). Let \( m \) be the smallest natural number such that \( \frac{1}{2m} \leq \varepsilon \). For each \( k \in \{1, 2, \ldots, m - 1\} \) there is \( n_k^* \in \mathbb{N} \), so that \( 1 - \varepsilon \leq \frac{c_k}{x_{k,n_k^*}} \leq 1 + \varepsilon \) for each \( n \geq n_k^* \). Set \( n^* = \max\{n_1^*, n_2^*, \ldots, n_m^* - 1\} \). For each \( i \geq n^* \) we have \( 1 - \varepsilon \leq \frac{c_i}{y_i} \leq 1 + \varepsilon \). Therefore, \( \lim_{n \to \infty} \frac{c_i}{y_i} = 1 \). It follows

\[
\lim_{i \to \infty} \frac{c_{i+1}}{y_i} = \lim_{i \to \infty} \left( \frac{c_{i+1}}{c_i} \cdot \frac{c_i}{y_i} \right) = 1
\]

because \( c \in \text{Tr} (\mathcal{S} \mathcal{V}_s) \). In a similar way we prove

\[
\lim_{i \to \infty} \frac{y_{i+1}}{c_i} = 1.
\]

One concludes that for each \( \lambda \geq 1 \)

\[
\lim_{i \to \infty} \frac{y_{i+\lambda}}{c_i} = \lim_{i \to \infty} \frac{c_{i+\lambda}}{y_i} = 1
\]

i.e. \( y = (y_i)_{i \in \mathbb{N}} \in \{c\}_{ts} \). The theorem is proved.

**Corollary 2.5.** The selection principle \( \alpha_2 ([c]_{ts}, [c]_{ts}) \) holds for each fixed element \( c \in \text{Tr} (\mathcal{S} \mathcal{V}_s) \). Consequently, \( \alpha_3 ([c]_{ts}, [c]_{ts}) \) and \( \alpha_4 ([c]_{ts}, [c]_{ts}) \) also hold.

We end the paper by proving a result about mutually translationally rapidly equivalent sequences.

Let \( c = (c_n)_{n \in \mathbb{N}} \in \mathcal{S} \). Then we define

\[
[c]_{tr} = \{d = (d_n)_{n \in \mathbb{N}} \in \mathcal{S} : c_n \overset{\varepsilon}{\sim} d_n, n \to \infty\}. \tag{2.4}
\]

**Theorem 2.6.** The player II has a winning coding strategy in the game \( \mathcal{G}_{\alpha_2} ([c]_{tr}, [c]_{tr}) \), for any fixed element \( c \in \text{Tr} (\mathcal{R}_s \mathcal{V}_s) \).

**Proof.** Let \( \sigma \) be the strategy of II.

\((m^{th} \text{ round}, m \geq 1)\): The player I chooses a sequence \( x_m = (x_m(n))_{n \in \mathbb{N}} \in [c]_{tr} \).

Then the player II chooses the subsequence

\[
\sigma(x_m, (x_{m-1,k_{m-1}(n)})_{n \in \mathbb{N}}) = (x_{m,k_m(n)})_{n \in \mathbb{N}}
\]

of the sequence \( x_m \), so that \( \text{Im}(k_m) \) is the set of natural numbers greater or equal to \( n_m \), which are divisible with \( 2^m \), and not divisible with \( 2^{m+1} \), \( n_m \in \mathbb{N} \), and \( \frac{c_{n+1}}{x_{m,n+1}} \geq 2^m \) and \( \frac{x_{m,n+1}}{c_{n}} \geq 2^m \) for each \( n \geq n_m \). Let \( \lambda \geq 1 \). Since \( c \in \text{Tr} (\mathcal{R}_s \mathcal{V}_s) \), we have \( \frac{c_{n+1}}{c_n} \geq 1 \) for sufficiently large \( n \). Then

\[
\frac{c_{n+\lambda}}{x_{m,n}} = \frac{c_{n+\lambda}}{c_{n+\lambda}-1} \cdot \frac{c_{n+\lambda}-1}{c_{n+\lambda}-2} \cdots \frac{c_{n+1}}{c_n} \geq 2^m
\]

for each \( n \geq n_m \). Since \( x_{m,n} \overset{\varepsilon}{\sim} c_n \), as \( n \to \infty \), we have \( x_m \in \text{Tr} (\mathcal{R}_s \mathcal{V}_s) \) (Theorem 2.2). In a similar way we prove \( \frac{x_{m,n+\lambda}}{c_n} \geq 2^m \) for all \( n \geq n_m \).
Form the set $Y = \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} x_{m,k_m(n)}$ of positive real numbers indexed by the second index. This set is a subsequence of the sequence $y = (y_i)_{i \in \mathbb{N}}$ defined by:

$$y_i = \begin{cases} x_{m,k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}; \\ c_i, & \text{otherwise.} \end{cases}$$

Evidently, $y \in S$ and the intersection of the set $y$ and $x_m$, $m \in \mathbb{N}$, is an infinite set.

We prove $y_m \sim c_m$, as $m \to \infty$. Let $M > 0$. Choose the smallest $m \in \mathbb{N}$ such that $2^m > M$. For each $k \in \{1, 2, ..., m-1\}$ there is $n_k^* \in \mathbb{N}$, so that $\frac{c(n+\lambda - k)}{c_{n_k}^*} \geq M$ and $\frac{\lambda + (n+\lambda)}{c_{n_k}^*} \geq M$ for each $\lambda \geq 1$ and each $n \geq n_k^*$. Let $n^* = \max\{n_1^*, ..., n_{m-1}^*\}$. Therefore, the inequalities $\frac{c(n+\lambda - k)}{y_i} \geq M$ and $\frac{\lambda + (n+\lambda)}{c_i} \geq M$ hold for each $\lambda \geq 1$ and each $i \geq n^*$. As $M$ was arbitrary, one concludes $y_i \sim c_i$, as $i \to \infty$. In other words, $y \in [c]_{tr}$.

**Corollary 2.7.** The selection principle $\alpha_2([c]_{tr}, [c]_{tr})$ holds for each fixed element $c \in \text{Tr}(\mathbb{R}, \infty)$, and thus $\alpha_3([c]_{tr}, [c]_{tr})$ and $\alpha_4([c]_{tr}, [c]_{tr})$ hold.

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