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Analytical and Solutions of Fourth Order Difference Equations

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Abstract
In this article, we presented the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(\pm 1 \pm x_{n-2}x_{n-3})},$$

where the initial conditions x_{-3} , x_{-2} , x_{-1} and x_0 are arbitrary real numbers. Also, we studied some dynamic behavior of these equations.

Keywords: Difference equation, Stability, Global attractor, Linearized stability, Periodicity
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1. Introduction

Recently, there has been an increasing interest in the study of global behavior of rational difference equations. The reason behind that is because difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, etc. See [1]. Rational difference equations is an important class of difference equations where they have many applications in real life, for example, the difference equation $x_{n+1} = \frac{a+bx_n}{c+x_n}$, which is known by Riccati Difference Equation has an application in optics and mathematical biology. For more results of the investigation of the rational difference equation see ([2]-[36]) and the references therein.

Karatas [37] examined the global behavior of higher order difference equation

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{b + cx_{n-2k}x_{n-(2k+1)}}.$$

In [38] Gumus et al. studied behavior of a third order difference equation

$$x_{n+1} = \frac{\alpha x_n}{\beta + \gamma x_{n-1}^p x_{n-2}^q}.$$

Elsayed [39] investigated the global of a higher order rational difference equation

$$x_{n+1}=a+\frac{bx_{n-l}+cx_{n-k}}{dx_{n-l}+ex_{n-k}}.$$

In [40], Kulenovic has got the global stability, periodic nature and gave the solution of non-linear difference equation

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}}{A+Bx_{n-1}}.$$

Elsayed [41] obtained periodic solution of period two and three of the difference equation

$$x_{n+1}=\alpha+\frac{\beta x_n}{x_{n-1}}+\frac{\gamma x_{n-1}}{x_n}.$$

Al-Shabi and Abo-Zeid [42] studied the global stability, periodic and boundedness of the positive solutions of the difference equation

$$x_{n+1}=\frac{Ax_{n-2r-1}}{B+Cx_{n-2l}x_{n-2k}}.$$

Amleh and Drymonis [43] investigated the global character of solution of a certain rational difference equation

$$x_{n+1}=\frac{(\alpha x_n+\beta x_n x_{n-1}+\gamma x_{n-1})x_n}{Ax_n+Bx_n x_{n-1}+Cx_{n-1}}.$$

Nirmaladevi and Karthikeyan [44] studied periodicity solution and the global stability of nonlinear difference equation

$$y_{n+1}=Py_n+Qy_{n-k}+Ry_{n-l}+\frac{by_{n-k}}{dy_{n-k}-ey_{n-l}}.$$

Elsayed and El-Dessoky [45] investigated behavior of the rational difference equation of the fourth order

$$x_{n+1}=ax_n+\frac{bx_n x_{n-2}}{cx_{n-2}+dx_{n-3}}.$$

In this paper we investigate the global asymptotic behavior and the form of the solutions of the solutions of the following recursive sequences

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(\pm 1\pm x_{n-2}x_{n-3})},$$

where the initial conditions $x_{-3},\ x_{-2}\ ,x_{-1}$ and x_0 are arbitrary real numbers.

Here, we will review some of the definitions and theorems used in solving special cases of difference equations:

Definition 1.1. *Let I be some interval of real numbers and let*

$$F: I^{k+1}\to I,$$

be a continuously differentiable function. Then for every set of initial condition $x_{-k},\ x_{-k+1},\ \ldots,\ x_0\in I$, *the difference equation*

$$x_{n+1}=F(x_n,x_{n-1},x_{n-2},\ldots,x_{n-k}),\quad n=0,1,\ldots, \tag{1.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.2. *A point $x^* \in I$ is called an equilibrium point of (1.1) if*

$$x^* = F(x^*),$$

that is,

$$x_n = x^* \ for\ all \quad n \geq -k.$$

*is a solution of (1.1), or equivalently, x^* is **a fixed point** of F.*

Definition 1.3. *(Stability)*

Let x^* be an equilibrium point of (1.1).

*(i) The equilibrium point x^* of (1.1) is called **locally stable** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\{x_n\}_{n=-k}^{\infty}$ is a solution of (1.1) and*

$$|x_{-k} - x^*| + |x_{1-k} - x^*| + ... + |x_0 - x^*| < \delta,$$

then

$$|x_n - x^*| < \varepsilon \quad for\ all \quad n \geq 0.$$

*(ii) The equilibrium point x^* of (1.1) is called **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of (1.1) and*

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = x^*.$$

*(iii) The equilibrium point x^* of (1.1) is called **a global attractor** if for every solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) we have*

$$\lim_{n\to\infty} x_n = x^*$$

*(iv) The equilibrium point x^* of (1.1) is called **globally asymptotically stable** if it is locally stable and global attractor of (1.1).*

*(v) The equilibrium point x^* of (1.1) is called **unstable** if x^* is not **locally stable**.*

2. Linearized stability analysis

Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point x^*. Let

$$p_i = \frac{\partial F}{\partial u_i}(x^*, x^*, ..., x^*) \quad for \quad i = 0, 1, ..., k,$$

denote the partial derivatives of $F(u_0, u_1, ...u_k)$ evaluated at the equilibrium x^* of (1.1).

Then the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + ... + p_k y_{n-k}\ , \quad n = 0, 1, ..., \tag{2.1}$$

is called **the linearized equation associated** of (1.1) about the equilibrium point x^* and the equation

$$\lambda^{k+1} - p_0 \lambda^k - ... - p_{k-1}\lambda - p_k = 0, \tag{2.2}$$

is called the characteristic equation of (2.1) about x^*.

The following result known as the Linear Stability Theorem is very useful in determining the local stability character of the equilibrium point x^* of (1.1).

Theorem 2.1. *[46] Assume that $p_0, p_2, ..., p_k$ are real numbers such that*

$$|p_0| + |p_1| + ... + |p_k| < 1,$$

or

$$\sum_{i=1}^{k} |p_i| < 1.$$

Then all roots of (2.2) lie inside the unit disk.

3. Qualitative behavior of solutions of $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(1+x_{n-2}x_{n-3})}$

In this part, we study the some qualitative properties for the recursive equation in the form:

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(1+x_{n-2}x_{n-3})}, \tag{3.1}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary positive real numbers.

Theorem 3.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of difference equation (3.1). Then for $n=0,1,...$*

$$\begin{aligned}
x_{6n-3}&=d\prod_{i=0}^{n-1}\frac{(1+2icd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+2ibc)},\\
x_{6n-2}&=c\prod_{i=0}^{n-1}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+2iab)}\frac{(1+2ibc)}{(1+(2i+1)bc)},\\
x_{6n-1}&=b\prod_{i=0}^{n-1}\frac{(1+(2i+2)cd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n}&=a\prod_{i=0}^{n-1}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+1)bc)},\\
x_{6n+1}&=\frac{cd}{a(1+cd)}\prod_{i=0}^{n-1}\frac{(1+(2i+2)cd)}{(1+(2i+3)cd)}\frac{(1+(2i+2)ab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n+2}&=\frac{ab(1+cd)}{d(1+bc)}\prod_{i=0}^{n-1}\frac{(1+(2i+3)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+3)bc)},
\end{aligned}$$

where $x_{-3}=d, x_{-2}=c, x_{-1}=b, x_0=a$.

Proof. For $n=0$, the result holds. Now, assume that $n>0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-9}&=d\prod_{i=0}^{n-2}\frac{(1+2icd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+2ibc)},\\
x_{6n-8}&=c\prod_{i=0}^{n-2}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+2iab)}\frac{(1+2ibc)}{(1+(2i+1)bc)},\\
x_{6n-7}&=b\prod_{i=0}^{n-2}\frac{(1+(2i+2)cd)}{(1+(2i+1)cd)}\frac{(1+2iab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n-6}&=a\prod_{i=0}^{n-2}\frac{(1+(2i+1)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+1)bc)},\\
x_{6n-5}&=\frac{cd}{a(1+cd)}\prod_{i=0}^{n-2}\frac{(1+(2i+2)cd)}{(1+(2i+3)cd)}\frac{(1+(2i+2)ab)}{(1+(2i+1)ab)}\frac{(1+(2i+1)bc)}{(1+(2i+2)bc)},\\
x_{6n-4}&=\frac{ab(1+cd)}{d(1+bc)}\prod_{i=0}^{n-2}\frac{(1+(2i+3)cd)}{(1+(2i+2)cd)}\frac{(1+(2i+1)ab)}{(1+(2i+2)ab)}\frac{(1+(2i+2)bc)}{(1+(2i+3)bc)}.
\end{aligned}$$

From (3.1) that

$$
\begin{aligned}
x_{6n-3} &= \frac{x_{6n-6}x_{6n-7}}{x_{6n-4}\left(1+x_{6n-6}x_{6n-7}\right)} \\
&= \frac{ab\prod\limits_{i=0}^{n-2}\left(\frac{1+2iab}{1+(2i+2)ab}\right)}{\frac{ab(1+cd)}{d(1+bc)}\prod\limits_{i=0}^{n-2}\left(\frac{1+(2i+3)cd}{1+(2i+2)cd}\right)\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)\left(\frac{1+(2i+2)bc}{1+(2i+3)bc}\right)\left(1+ab\prod\limits_{i=0}^{n-2}\left(\frac{1+2iab}{1+(2i+2)ab}\right)\right)} \\
&= \frac{d(1+bc)}{(1+cd)}\frac{\left[\frac{(1+2ab)(1+4ab)...(1+(2n-6)ab)(1+2(n-2)ab)}{(1+2ab)(1+4ab)...(1+(2n-4)ab(1+(2n-2)ab)}\right]}{\prod\limits_{i=0}^{n-2}\left(\frac{1+(2i+3)cd}{1+(2i+2)cd}\right)\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)\left(\frac{1+(2i+2)bc}{1+(2i+3)bc}\right)\left(1+\frac{ab}{1+(2n-2)ab}\right)} \\
&= d\frac{\left(\frac{1}{1+(2n-2)ab}\right)}{\prod\limits_{i=0}^{n-1}\left(\frac{1+(2i+1)cd}{1+2icd}\right)\left(\frac{1+2ibc}{1+(2i+1)bc}\right)\prod\limits_{i=0}^{n-2}\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)\left(\frac{1+(2n-1)ab}{1+(2n-2)ab}\right)} \\
&= \frac{d}{(1+(2n-1)ab)\prod\limits_{i=0}^{n-1}\left(\frac{1+(2i+1)cd}{1+2icd}\right)\left(\frac{1+2ibc}{1+(2i+1)bc}\right)\prod\limits_{i=0}^{n-2}\left(\frac{1+(2i+1)ab}{1+(2i+2)ab}\right)} \\
&= \frac{d}{\prod\limits_{i=0}^{n-1}\left(\frac{1+(2i+1)cd}{1+2icd}\right)\left(\frac{1+2ibc}{1+(2i+1)bc}\right)\left(\frac{1+(2i+1)ab}{1+2iab}\right)}.
\end{aligned}
$$

Consequently, we have

$$
x_{6n-3} = d\prod_{i=0}^{n-1}\left(\frac{1+2icd}{1+(2i+1)cd}\right)\left(\frac{1+2iab}{1+(2i+1)ab}\right)\left(\frac{1+(2i+1)bc}{1+2ibc}\right).
$$

Similarly, other relations can be obtained and thus, the proof has been proved. □

Theorem 3.2. *The difference equation* (3.1) *has a unique equilibrium point which is* 0 *and it is not locally asymptotically stable.*

Proof. For the equilibrium points of (3.1), is given by

$$
\begin{aligned}
x^* &= \frac{x^{*2}}{x^*\left(1+x^{*2}\right)}, \\
1+x^{*2} &= 1,
\end{aligned}
$$

or

$$
x^{*2} = 0.
$$

Then the unique equilibrium point is $x^* = 0$.
Let F be function defined by

$$
F(u,v,w) = \frac{vw}{u(1+vw)}.
$$

Therefore it follows that

$$
F_u(u,v,w) = \frac{-vw}{u^2(1+vw)},\ F_v(u,v,w) = \frac{w}{u(1+vw)^2},\ F_w(u,v,w) = \frac{v}{u(1+vw)^2},
$$

we see that

$$
F_u(x^*,x^*,x^*) = -1,\ F_v(x^*,x^*,x^*) = 1,\ F_w(x^*,x^*,x^*) = 1.
$$

This completes the proof by using Theorem 2.1. □

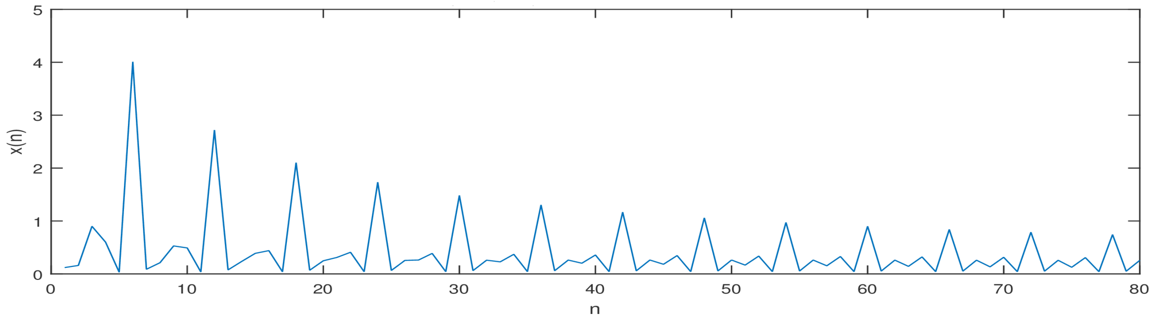

Figure 3.1. Plot of $x(n+1)=\frac{x(n-2)x(n-3)}{x(n)(1+x(n-2)x(n-3))}$

Numerical Examples

For statement the results of this part, we take into account numerical examples which illustrate different types of solutions to (3.1).

Example 3.3. *See Figure 3.1, since* $x_{-3}=0.12,\ x_{-2}=0.16,\ x_{-1}=0.9$ *and* $x_0=0.6$.

4. Qualitative behavior of solutions of $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1+x_{n-2}x_{n-3})}$

Here, we obtain the solution of the following difference equation

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n\left(-1+x_{n-2}x_{n-3}\right)},\quad n=0,1,..., \tag{4.1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0.are arbitrary real numbers with $x_{-2}x_{-3}\neq 1,\ x_{-1}x_{-2}\neq 1$ and $x_0x_{-1}\neq 1$.

Theorem 4.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of difference equation of* (4.1)*. Then the equation* (4.1) *has unboundedness solutions and for* $n=0,1,...$

$$\begin{aligned}
x_{6n-3}&=\frac{d\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n},\quad x_{6n-2}=\frac{c\left(-1+cd\right)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n},\\
x_{6n-1}&=\frac{b\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n},\quad x_{6n}=\frac{a\left(-1+cd\right)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n},\\
x_{6n+1}&=\frac{cd\left(-1+bc\right)^n}{a\left(-1+ab\right)^n\left(-1+cd\right)^{n+1}},\quad x_{6n+2}=\frac{ab\left(-1+cd\right)^{n+1}\left(-1+ab\right)^n}{d\left(-1+bc\right)^{n+1}},
\end{aligned} \tag{4.2}$$

where $x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b$ *and* $x_0=a$.

Proof. For $n=0$ the conclusion holds. Now, assume that $n>0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-9}&=\frac{d\left(-1+bc\right)^{n-1}}{\left(-1+cd\right)^{n-1}\left(-1+ab\right)^{n-1}},\quad x_{6n-8}=\frac{c\left(-1+cd\right)^{n-1}\left(-1+ab\right)^{n-1}}{\left(-1+bc\right)^{n-1}},\\
x_{6n-7}&=\frac{b\left(-1+bc\right)^{n-1}}{\left(-1+cd\right)^{n-1}\left(-1+ab\right)^{n-1}},\quad x_{6n-6}=\frac{a\left(-1+cd\right)^{n-1}\left(-1+ab\right)^{n-1}}{\left(-1+bc\right)^{n-1}},\\
x_{6n-5}&=\frac{cd\left(-1+bc\right)^{n-1}}{a\left(-1+ab\right)^{n-1}\left(-1+cd\right)^{n}},\quad x_{6n-4}=\frac{ab\left(-1+cd\right)^{n}\left(-1+ab\right)^{n-1}}{d\left(-1+bc\right)^{n}}.
\end{aligned}$$

Now we proof some of relations of (4.2).

$$
\begin{aligned}
x_{6n-2} &= \frac{x_{6n-5}x_{6n-6}}{x_{6n-3}\left(-1+x_{6n-5}x_{6n-6}\right)} = \frac{(cd)\left(-1+cd\right)^{-1}}{\frac{d(-1+bc)^n}{(-1+cd)^n(-1+ab)^n}\left(-1+cd\left(-1+cd\right)^{-1}\right)} \\
&= \frac{c\left(-1+cd\right)^{n-1}\left(-1+ab\right)^n}{\left(-1+bc\right)^n\left(-1+\frac{cd}{(-1+cd)}\right)} = \frac{c(-1+cd)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
x_{6n-1} &= \frac{x_{6n-4}x_{6n-5}}{x_{6n-2}\left(-1+x_{6n-4}x_{6n-5}\right)} \\
&= \frac{\left(bc\left(-1+bc\right)^{n-1}\right)l\left(-1+bc\right)^n}{(c(-1+cd)^n(-1+ab)^n/(-1+bc)^n)\left(-1+\frac{bc}{-1+bc}\right)} \\
&= \frac{b\left(-1+bc\right)^{n-1}}{\left(-1+cd\right)^n\left(-1+ab\right)^n\left(\frac{1}{-1+bc}\right)} = \frac{b\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n}.
\end{aligned}
$$

Similarly, other relations can be obtained and thus, the proof has been proved. □

Theorem 4.2. *The difference equation (4.1) has a periodic solution of periodic six iff $ab = 2$ and $b = d$ and we will take the form:*

$$
\left\{d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},\ldots\right\}.
$$

Proof. Assume that there exists a prime period six solution of (4.1):

$$
d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},d,c,b,a,\frac{cd}{a\left(-1+cd\right)},\frac{ab}{d},\ldots .
$$

From (4.2), we get

$$
\begin{aligned}
x_{6n-3} &= d = \frac{d\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n}, \quad x_{6n-2} = c = \frac{c\left(-1+cd\right)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n}, \\
x_{6n-1} &= b = \frac{b\left(-1+bc\right)^n}{\left(-1+cd\right)^n\left(-1+ab\right)^n}, \quad x_{6n} = a = \frac{a\left(-1+cd\right)^n\left(-1+ab\right)^n}{\left(-1+bc\right)^n}, \\
x_{6n+1} &= \frac{cd}{a\left(-1+cd\right)} = \frac{cd\left(-1+bc\right)^n}{a\left(-1+ab\right)^n\left(-1+cd\right)^{n+1}}, \\
x_{6n+2} &= \frac{ab}{d} = \frac{ab\left(-1+cd\right)^{n+1}\left(-1+ab\right)^n}{d\left(-1+bc\right)^{n+1}}.
\end{aligned}
$$

Then we can see that

$$
ab = 2 \text{ and } b = d.
$$

Conversely, suppose that $ab=2$ *and* $b=d$. Then we see that

$$\begin{aligned}
x_{6n-3} &= \frac{d\left(-1+bc\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}} = \frac{d\left(-1+cd\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+2\right)^{n}} = d,\\
x_{6n-2} &= \frac{c\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}{\left(-1+bc\right)^{n}} = \frac{c\left(-1+cd\right)^{n}\left(-1+2\right)^{n}}{\left(-1+cd\right)^{n}} = c,\\
x_{6n-1} &= \frac{b\left(-1+bc\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}} = \frac{b\left(-1+cd\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+2\right)^{n}} = b,\\
x_{6n} &= \frac{a\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}{\left(-1+bc\right)^{n}} = a,\\
x_{6n+1} &= \frac{cd\left(-1+bc\right)^{n}}{a\left(-1+ab\right)^{n}\left(-1+cd\right)^{n+1}} = \frac{cd}{a\left(-1+cd\right)},\\
x_{6n+2} &= \frac{ab\left(-1+cd\right)^{n+1}\left(-1+ab\right)^{n}}{d\left(-1+bc\right)^{n+1}} = \frac{ab}{d}.
\end{aligned}$$

Thus we obtained a periodic solution of period six. □

Theorem 4.3. *Equation* (4.1) *has a periodic solution of period two iff* $ab=bc=cd=2$ *(It also means* $a=c,\ b=d$*) and we will take the form:*

$$\{d,c,d,c,\ldots\}.$$

Proof. First assume that there exists a prime period two solution of (4.1):

$$d,c,d,c,\ldots.$$

We see from the form of the solutions of (4.1) that

$$\begin{aligned}
x_{6n-3} &= d = \frac{d\left(-1+bc\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}, \quad x_{6n-2} = c = \frac{c\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}{\left(-1+bc\right)^{n}},\\
x_{6n-1} &= d = \frac{b\left(-1+bc\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}, \quad x_{6n} = c = \frac{a\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}{\left(-1+bc\right)^{n}},\\
x_{6n+1} &= d = \frac{cd\left(-1+bc\right)^{n}}{a\left(-1+ab\right)^{n}\left(-1+cd\right)^{n+1}}, \quad x_{6n+2} = c = \frac{ab\left(-1+cd\right)^{n+1}\left(-1+ab\right)^{n}}{d\left(-1+bc\right)^{n+1}}.
\end{aligned}$$

Thus we see that $ab=bc=cd=2$.
Second suppose that $ab=bc=cd=2$. Then we obtain

$$\begin{aligned}
x_{6n-3} &= \frac{d\left(-1+bc\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}} = \frac{d\left(-1+2\right)^{n}}{\left(-1+2\right)^{n}\left(-1+2\right)^{n}} = d,\\
x_{6n-2} &= \frac{c\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}{\left(-1+bc\right)^{n}} = \frac{c\left(-1+2\right)^{n}\left(-1+2\right)^{n}}{\left(-1+2\right)^{n}} = c,\\
x_{6n-1} &= \frac{b\left(-1+bc\right)^{n}}{\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}} = \frac{b\left(-1+2\right)^{n}}{\left(-1+2\right)^{n}\left(-1+2\right)^{n}} = d,\\
x_{6n} &= \frac{a\left(-1+cd\right)^{n}\left(-1+ab\right)^{n}}{\left(-1+bc\right)^{n}} = c,\\
x_{6n+1} &= \frac{cd\left(-1+bc\right)^{n}}{a\left(-1+ab\right)^{n}\left(-1+cd\right)^{n+1}} = d,\\
x_{6n+2} &= \frac{ab\left(-1+cd\right)^{n+1}\left(-1+ab\right)^{n}}{d\left(-1+bc\right)^{n+1}} = c.
\end{aligned}$$

Thus we obtained a periodic solution of period two. □

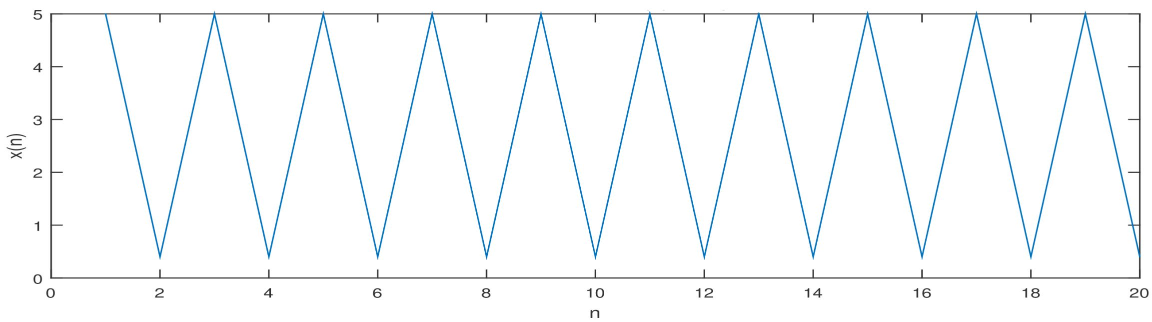

Figure 4.1. Plot of $x(n+1) = \frac{x(n-2)x(n-3)}{x(n)(-1+x(n-2)x(n-3))}$

Theorem 4.4. *Difference equation (4.1) has equilibrium points which are $0, \pm\sqrt{2}$ such that they are not locally asymptotically stable.*

Proof. We can write

$$x^* = \frac{x^{*^2}}{x^*\left(-1+x^{*^2}\right)}, \quad or \quad x^{*2}\left(x^{*2}-2\right) = 0,$$

consequently $0, \pm\sqrt{2}$ are the equilibrium points.

Suppose that $F : (0,\infty)^3 \to (0,\infty)$ be function defined by

$$F(u,v,w) = \frac{vw}{u(-1+vw)},$$

then

$$F_u(u,v,w) = \frac{-vw}{u^2(-1+vw)}, \; F_v(u,v,w) = \frac{-w}{u(-1+vw)^2}, \; F_w(u,v,w) = \frac{-v}{u(-1+vw)^2},$$

we see that,

$$F_u(x^*,x^*,x^*) = -1, \; F_v(x^*,x^*,x^*) = -1, \; F_w(x^*,x^*,x^*) = -1.$$

This completes the proof by using Theorem 2.1. □

Numerical Examples

We put some numerical examples which illustrate different types of solutions of (4.1).

Example 4.5. *When we put $x_{-3} = 5, \; x_{-2} = 2/5, \; x_{-1} = 5$ and $x_0 = 2/5$. See Figure 4.1.*

5. Qualitative behavior of solutions of $x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(1-x_{n-2}x_{n-3})}$

In this section, we get the expressions of the solution of the difference equation in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n(1-x_{n-2}x_{n-3})}, \quad n = 0,1,..., \tag{5.1}$$

where the initial conditions $x_{-3}, \; x_{-2}, \; x_{-1}$ *and* x_0 are arbitrary real numbers.

Theorem 5.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of the difference equation of (5.1). Then for $n = 0,1,...$*

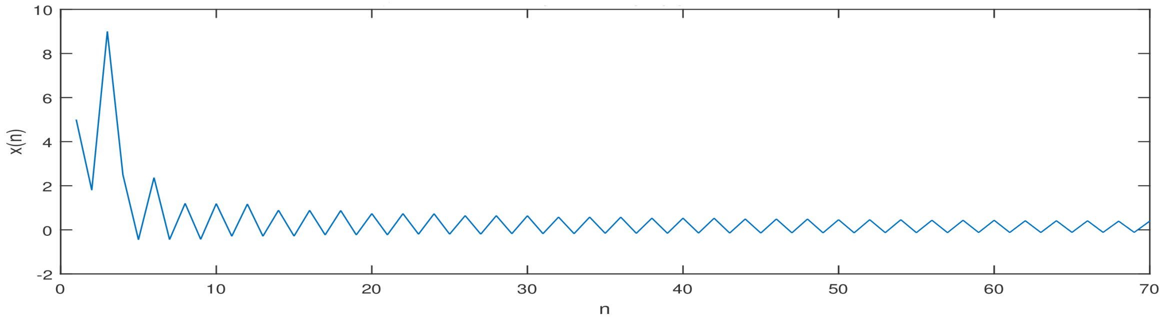

Figure 5.1. Plot of $x(n+1)=\frac{x(n-2)x(n-3)}{x(n)(1-x(n-2)x(n-3))}$

$$x_{6n-3}=d\prod_{i=0}^{n-1}\frac{(1-2icd)}{(1-(2i+1)cd)}\frac{(1-2iab)}{(1-(2i+1)ab)}\frac{(1-(2i+1)bc)}{(1-2ibc)},$$
$$x_{6n-2}=c\prod_{i=0}^{n-1}\frac{(1-(2i+1)cd)}{(1-(2i+2)cd)}\frac{(1-(2i+1)ab)}{(1-2iab)}\frac{(1-2ibc)}{(1-(2i+1)bc)},$$
$$x_{6n-1}=b\prod_{i=0}^{n-1}\frac{(1-(2i+2)cd)}{(1-(2i+1)cd)}\frac{(1-2iab)}{(1-(2i+1)ab)}\frac{(1-(2i+1)bc)}{(1-(2i+2)bc)},$$
$$x_{6n}=a\prod_{i=0}^{n-1}\frac{(1-(2i+1)cd)}{(1-(2i+2)cd)}\frac{(1-(2i+1)ab)}{(1-(2i+2)ab)}\frac{(1-(2i+2)bc)}{(1-(2i+1)bc)},$$
$$x_{6n+1}=\frac{cd}{a(1-cd)}\prod_{i=0}^{n-1}\frac{(1-(2i+2)cd)}{(1+(2i+3)cd)}\frac{(1-(2i+2)ab)}{(1+(2i+1)ab)}\frac{(1-(2i+1)bc)}{(1+(2i+2)bc)},$$
$$x_{6n+2}=\frac{ab(1-cd)}{d(1-bc)}\prod_{i=0}^{n-1}\frac{(1-(2i+3)cd)}{(1-(2i+2)cd)}\frac{(1-(2i+1)ab)}{(1-(2i+2)ab)}\frac{(1-(2i+2)bc)}{(1-(2i+3)bc)}.$$

Theorem 5.2. *Equation (5.1) has a unique equilibrium point which is* 0 *and it is not locally asymptotically stable.*

Example 5.3. *See Figure 5.1, we let* $x_{-3}=5,\ x_{-2}=1.8,\ x_{-1}=9$ *and* $x_{-0}=2.5$.

6. Qualitative behavior of solutions of $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1-x_{n-2}x_{n-3})}$

In this part, we obtain the form of solution of the following difference equation

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1-x_{n-2}x_{n-3})},\quad n=0,1,..., \tag{6.1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary real numbers with $x_{-2}x_{-3}\neq-1,\ x_{-1}x_{-2}\neq-1$ and $x_0x_{-1}\neq-1$.

Theorem 6.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (6.1)*. Then* (6.1) *has the following solution for* $n=0,1,...$

$$x_{6n-3}=\frac{d(-1-bc)^n}{(-1-cd)^n(-1-ab)^n},\quad x_{6n-2}=\frac{c(-1-cd)^n(-1-ab)^n}{(-1-bc)^n},$$
$$x_{6n-1}=\frac{b(-1-bc)^n}{(-1-cd)^n(-1-ab)^n},\quad x_{6n}=\frac{a(-1-cd)^n(-1-ab)^n}{(-1-bc)^n},$$
$$x_{6n+1}=\frac{cd(-1-bc)^n}{a(-1-ab)^n(-1-cd)^{n+1}},\quad x_{6n+2}=\frac{ab(-1-cd)^{n+1}(-1-ab)^n}{d(-1-bc)^{n+1}}.$$

Theorem 6.2. *Difference equation* (6.1) *has a periodic solution of period six iff* $ab=-2$ *and* $b=d$ *and we will take the form:*

$$\left\{d,c,b,a,\frac{cd}{a(-1-cd)},\frac{ab}{d},d,c,b,a,\frac{cd}{a(-1-cd)},\frac{ab}{d},...\right\}.$$

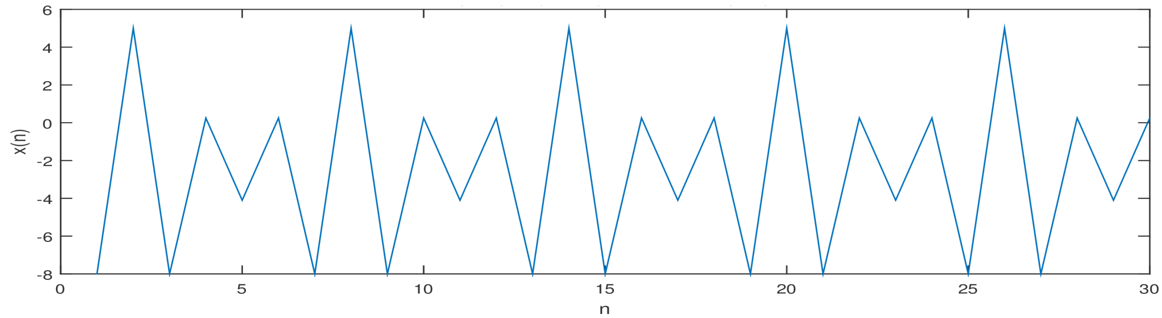

Figure 6.1. Plot of $x(n+1) = \frac{x(n-2)x(n-3)}{x(n)(-1-x(n-2)x(n-3))}$

Theorem 6.3. *Equation* (6.1) *has a periodic solution of period two iff* $ab = bc = cd = -2$ *and takes the form:* $\{d, c, d, c, ...\}$.

Theorem 6.4. *Difference equation* (6.1) *has equilibrium point which is* 0 *and it is not locally asymptotically stable.*

Example 6.5. *Figure 6.1 shows the period six solutions of* (6.1) *since* $x_{-3} = -8$, $x_{-2} = 5$, $x_{-1} = -8$ *and* $x_0 = 1/4$.

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