A New Theorem on The Existence of Positive Solutions of Singular Initial-Value Problem for Second Order Differential Equations

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Abstract
We proved a new theorem on the existence of positive solutions to initial-value problems for second-order nonlinear singular differential equations. The existence of solutions is proven under considerably weaker than previously known conditions.

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1. Introduction
We consider the problem

\[(py')' + pqg(y) = 0, \quad t \in [0, T] \tag{1.1}\]
\[y(0) = a > 0,\]
\[\lim_{t \to 0^+} p(t)y'(t) = 0\]

and

\[(py')' + pqg(y) = 0, \quad t \in [0, T] \tag{1.2}\]
\[y(0) = a > 0,\]
\[y'(0) = 0,\]

where \(0 < T < \infty, p \geq 0, q \geq 0\) and \(g : [0, \infty) \to [0, \infty)\).

Agarwal and O’Regan [1] established the existence theorems for the positive solution of the problem (1.1) and (1.2):

Theorem 1.1. [1] Suppose the following conditions are satisfied

\[p \in C[0, T] \cap C^1(0, T) \quad \text{with} \quad p > 0 \text{ on } (0, T) \tag{1.3}\]
\( q \in L^1_p[0,t^*] \) for any \( t^* \in (0,T) \) with \( q > 0 \) on \( (0,T) \), \hspace{1cm} (1.4)

where \( L^1_p[0,a] \) is the space of functions \( u(t) \) with \( \int_0^a |u(t)|^r dt < \infty \),

\[
\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)dxds < \infty \text{ for any } t^* \in (0,T)
\]

and

\[
g : [0,\infty) \to [0,\infty) \text{ is continuous, nondecreasing on } [0,\infty) \text{ and } g(u) > 0 \text{ for } u > 0.
\]

(1.6)

Let

\[
H(z) = \int_z^a \frac{dx}{g(x)} \text{ for } 0 < z \leq a
\]

and assume

\[
\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)dxds < a \text{ for any } t^* \in (0,T),
\]

(1.7)

here

\[
\tau(x) = g \left( H^{-1} \left( \int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)dzdw \right) \right).
\]

Then (1.1) has a solution \( y \in C(0,T) \) with \( py' \in C(0,T) \), \( (py')' \in L^1_{pq}(0,T) \) and \( 0 < y(t) \leq a \) for \( t \in [0,T] \). In addition if either

\[
p(0) \neq 0
\]

or

\[
p(0) = 0 \text{ and } \lim_{t \to 0^+} \frac{p(t)q(t)}{p'(t)} = 0
\]

holds, then \( y \) is a solution of (1.2).

The condition (1.7) in connection with the definition of the function \( \tau(x) \), makes this theorem difficult for an application. In [2] we proved more easy and applicable theorem:

**Theorem 1.2.** Suppose (1.3)-(1.5) hold. In addition, we assume

\[
\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(x)dxds < a
\]

for any \( t^* \in (0,T_0) \). Then

a) (1.1) has a solution \( y \in C(0,T_0) \) with \( py' \in C(0,T_0) \), \( (py')' \in L^1_{pq}(0,T_0) \) and \( 0 < y(t) \leq a \) for \( t \in [0,T_0] \).

b) If \( \int_0^{T_1} \frac{1}{p(s)} \int_0^s p(x)q(x)g(T_0)dxds < y(T_0) \), and conditions (1.3)-(1.6) satisfied then solution can be extended into the interval \( [0,T_1] \).

In this paper we generalized the Theorem 1.2.
2. Main result

**Theorem 2.1.** Suppose the following conditions are satisfied

\[ p \in C[0, T) \cap C^1(0, T) \text{ with } p > 0 \text{ on } (0, T], \]
\[ q \geq 0, \]
\[ \int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)dxds < \infty \text{ for any } t^* \in (0, T], \]
\[ g : [0, \infty) \to [0, \infty) \text{ is continuous, nondecreasing on } [0, \infty). \]

and assume

\[ \int_0^t \frac{1}{p(s)} \int_0^s p(x)g(a - \varphi(x))dxds \leq a - \varphi(t), \]
\[ \int_0^t \frac{1}{p(s)} \int_0^s p(x)g(\varphi(x))dxds \geq \varphi(t) \]

for some \( \varphi(t) \in C[0, T], \) with \( 0 \leq \varphi(t) \leq a. \) Then (1.1) has a solution \( y \in C[0, T] \) with \( py' \in C[0, T], \) \( (py')' \in L_{pq}^1(0, T) \) and \( 0 < y(t) \leq a \text{ for } t \in [0, T]. \) In addition if either

\[ p(0) \neq 0 \]

or

\[ p(0) = 0 \text{ and } \lim_{t \to 0^+} \frac{p(t)q(t)}{p'(t)} = 0 \]

holds, then \( y \) is a solution of (1.2).

**Remark 2.2.** The case \( \varphi(t) \equiv 0, \) corresponds to the case of inequality

\[ \int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)g(a)dxds < a \text{ for any } t^* \in (0, T]. \]

**Proof of Theorem 2.1.** Consider the sequence \( \{y_n(t)\}, n = 0, 1, 2, ... \) with \( y_0(t) \equiv a - \varphi(t), \)

\[ y_n(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{n-1}(x))dxds, \quad n = 1, 2, ..., t \leq T. \]

We have

\[ y_0(t) = a - \varphi(t), \]
\[ y_1(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(a - \varphi(x))dxds \geq \varphi(t), \]
\[ y_2(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_1(x))dxds \]
\[ \leq a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(\varphi(x))dxds \]
\[ \leq a - \varphi(t), \]
\begin{align*}
y_3(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_2(x))dxds \geq \varphi(t) \\
y_4(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_3(x))dxds \leq a - \varphi(t),...
\end{align*}

The sequences \(\{y_{2n}(t)\}\) and \(\{y_{2n+1}(t)\}\) are equicontinuous. Indeed we have

\[|y_n(t) - y_n(r)| = \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{n-1}(x))dxds \leq M \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds,\]  

(2.1)

where

\[M = \max\{g(u) : 0 \leq u \leq a\}\]

and the right hand side of (2.1) can be taken \(\epsilon < |t - r| < \delta\), regardless of the choice of \(t\) and \(r\): the function \(\int_0^\delta \frac{1}{p(s)} \int_0^s p(x)q(x)dxds\) is (uniformly) continuous on \([0,T]\). It follows from Ascoli Arzela Theorem that the sequence \(\{y_{2n}(t)\}\) has the (uniformly) convergent subsequence, \(y_{2n_k}(t) \to u(t)\). The Lebesgue dominated theorem guarantees that

\begin{align*}
y_{2n+1}(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{2n_k}(x))dxds \to v(t), \\
v(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds \\
and \quad u(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(v(x))dxds.
\end{align*}

If \(u(t) = v(t)\) we have that the function \(u(t)\) is the solution of the problem (1.1), indeed it follows from

\[u(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds\]

that

\begin{align*}
u'(t) &= -\frac{1}{p(t)} \int_0^t p(x)q(x)g(u(x))dx, \\
p\nu' &= -\int_0^t p(x)q(x)g(u(x))dx, \\
(pu')' &= -pqg(u).
\end{align*}

So we suppose \(u(t) \neq v(t)\). We have \(u(0) = v(0) = a\) and if for example, \(u(t) > v(t)\) on the interval \((0,b)\), then we obtain

\[u(b) - v(b) = \int_0^b \frac{1}{p(s)} \int_0^s p(x)q(x)[g(u(x)) - g(v(x))]dxds > 0\]

and therefore \(u(t) > v(t)\) on the whole interval \([0,T]\). The same holds for all points of intersections \(t_0 : u(t_0) = v(t_0)\). That is if \(u(t_0) = v(t_0)\), then for any \(\epsilon > 0\) there are infinitely many points \(t_n \in [t_0,t_0 + \epsilon]\) such that \(u(t_n) = v(t_n)\). Therefore, \(u(t) > v(t)\) (or <) on \([0,T]\). Without loss of generality let us suppose \(u(t) > v(t)\) on \([0,T]\)

and consider the operator \(N : C[0,T] \to C[0,T]\) defined by

\[Ny(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y(x))dxds.\]

Next let
K = \{ y \in C[0, T]: v(t) \leq y(t) \leq u(t) \text{ for } t \in [0, T]\}.

Clearly K is closed, convex, bounded subset of C[0, T] and N : K \to K. Let us show that N : K \to K is continuous and compact operator. Continuity follows from Lebesgue dominated convergence theorem: if \( y_n(t) \to y(t) \), then \( Ny_n(t) \to Ny(t) \). To show that N is completely continuous let \( y(t) \in K \), then

\[
|Ny(t) - Ny(r)| \leq M \left| \int_r^t \frac{1}{p(x)} \int_0^x p(z)q(z)dzds \right| \text{ for } t, r \in [0, T],
\]

that is N completely continuous on [0, T].

The Schauder-Tychonoff theorem guarantees that N has a fixed point \( w \in K \), i.e. \( w \) is a solution of (1.1). It follows from

\[
w'(t) = -\frac{1}{p(t)} \int_0^t p(x)q(x)g(w(x))dx,
\]

that if \( p(0) \neq 0 \) then \( w'(0) = 0 \). Now if \( p(0) = 0 \) but \( \lim_{t \to 0^+} \frac{p(t)q(t)}{p'(t)} = 0 \) we have from (??) that

\[
w'(0^+) = -\lim_{t \to 0^+} \int_0^t \frac{p(x)q(x)}{p(t)} g(w(x))dx = 0,
\]

that is \( w \) is a solution of (1.2).

The proof is completed.

Conflict of interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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