Approximation by Cheney-Sharma Chlodovsky operators

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Abstract

The main purpose of this paper is to construct Cheney-Sharma Chlodovsky operators. We study approximation properties of the new operators with the help of weighted Korovkin-type theorem and universal Korovkin-type theorem. We also give the rate of convergence by means of the modulus of continuity. Furthermore, we give A-statistical convergence property of these operators.

Mathematics Subject Classification (2010). 41A36, 40A35

Keywords. Cheney-Sharma Chlodovsky operators, weighted approximation, rate of approximation, $A-$statistical approximation

1. Introduction

Approximation theory is concerned with how functions can best be approximated with simpler functions. Since Korovkin theorem was obtained, the studies of the linear methods of approximation given by sequences of positive and linear operators have become an important area in approximation theory. The study of the statistical convergence in approximation theory for sequences of linear positive operators was attempted by Gadjiev and Orhan [23]. They proved a Korovkin type theorem by considering statistical convergence instead of ordinary convergence. Later, many authors gave several approximation results via summability methods, for example, we refer the readers to [1, 2, 4, 16, 28, 29, 31, 32, 34].

In 1932, Chlodovsky [14], introduced the classical Bernstein-Chlodovsky polynomials as a generalization of Bernstein polynomials on unbounded set.
For every $n \in \mathbb{N}$, $f \in C[0, \infty)$ and $x \in [0, \infty)$, these polynomials $C_n : C[0, \infty) \to C[0, \infty)$ defined by

$$C_n(f, x) := \begin{cases} \sum_{k=0}^{n} f\left(\frac{k}{n} b_n\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} & , \quad \text{if } 0 \le x \le b_n \\ f(x) & , \quad \text{if } x > b_n \end{cases},$$

where $0 \le x \le b_n$ and $\{b_n\}$ is a positive sequence with the properties;
$\lim_{n \to \infty} b_n = \infty$, $\lim_{n \to \infty} b_n/n = 0$.

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Recently some authors have studied some Chlodovsky type polynomials which may be found in [6, 12, 24–27, 30].

It is known that the Abel-Jensen equalities are given by the following formulas (see [5], p.326)

$$(u + v + n\beta)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} u (u + k\beta)^{k-1} v [v + (n - k)\beta]^{n-k-1} \qquad (1.1)$$

and

$$(u + v)(u + v + m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u (u + k\beta)^{k-1} v [v + (m - k)\beta]^{m-k-1}, \qquad (1.2)$$

where u, v and $\beta \in \mathbb{R}$. By means of these equalities Cheney-Sharma [13] introduced the following Bernstein type operators for $f \in C[0, 1]$, $x \in [0, 1]$ and $n \in \mathbb{N}$

$$Q_n(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x (x + k\beta)^{k-1} \qquad (1.3)$$
$$\times [1 - x + (n - k)\beta]^{n-k}$$

and

$$G_n(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x (x + k\beta)^{k-1} \qquad (1.4)$$
$$\times (1 - x)[1 - x + (n - k)\beta]^{n-k-1},$$

where β is a nonnegative real parameter. It is obvious that for $\beta = 0$ these operators reduce to the classical Bernstein operators. Cheney-Sharma proved that if $n\beta_n \to \infty$ as $n \to \infty$, then for $f \in C[0, 1]$, these operators uniformly convergence to f on $[0, 1]$. In [7], the authors showed that Cheney-Sharma operators preserve the Lipschitz constant and order of a Lipschitz continuous function as well as the properties of the function of modulus of continuity. They also gave a result of $G_n(f; x)$ under the convexity of f. Kantorovich type generalization of the Cheney-Sharma operators was studied in [33]. For these operators, we refer the readers to [3, 8, 15, 35, 36].

In the present paper, we construct the Chlodovsky-type generalization of the Cheney-Sharma operators is given by (1.4) and we prove that the weighted uniform convergence of $G_n^* f$ to f. We also give approximation results using universal Korovkin-type theorem and obtain rate of approximation in terms of the usual modulus of continuity. Finally, we study A-statistical convergence behaviours of new positive linear operators expressed as below.

We introduce the Cheney-Sharma Chlodovsky operators as

$$G_n^*(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right) \binom{n}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \qquad (1.5)$$
$$\times \left(1 - \frac{x}{b_n}\right) \left[1 - \frac{x}{b_n} + (n - k)\beta\right]^{n-k-1}$$

for $0 \leq x \leq b_n$ and $f(x)$ for $x > b_n$. Here $\{b_n\}$ is a positive sequence with the properties; $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} b_n/n = 0$. If we take $b_n = 1$, then we obtain the Cheney-Sharma operators (1.4).

Now, let us recall the concept of A-statistical convergence. Let $A = (a_{nk})$ be a summability matrix and let $x = (x_k)$ be a sequence.

If the series

$$(Ax)_n := \sum_k a_{nk} x_k$$

is convergent for each n then, we say that $Ax := \{(Ax)_n\}$ is the A-transformation of x. If the sequence Ax converges to a number L then, we say that x is A-summable to L. A summability matrix A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_k x_k = L$ [9].

Let $A = (a_{nk})$ be a nonnegative regular summability matrix and let K be a subset of positive integer. Then K is said to have A-density $\delta_A(K)$ if the limit

$$\delta_A(K) := \lim_n \sum_{k \in K} a_{nk}$$

exists [10, 11, 19]. The sequence $x = (x_k)$ is said to be A-statistically convergent to real number α if for any $\varepsilon > 0$

$$\lim_k \sum_{k : |x_k - \alpha| \geq \varepsilon} a_{nk} = 0.$$

In this case, we write $st_A - \lim x = \alpha$ [18, 20]. Note that x is A-statistically convergent to α if and only if for any $\varepsilon > 0$, $\delta_A(K_\varepsilon) = 0$, where $K_\varepsilon := \{k \in \mathbb{N} : |x_k - \alpha| \geq \varepsilon\}$. If A is the identity matrix I, then I-statistical convergence reduces to ordinary convergence, and, if $A = C_1$, the Cesáro matrix of order one, then it coincides with statistical convergence.

Throughout the paper, we will consider the following class of functions. Let $\rho(x) = 1 + x^2$,

$$B_\rho\left(\mathbb{R}^+\right) = \left\{f : \mathbb{R}^+ \to \mathbb{R}, \ |f(x)| \leq M_f \rho(x), \ x \geq 0\right\},$$

where M_f is a constant depending on f.

$$C_\rho\left(\mathbb{R}^+\right) = \left\{f \in B_\rho\left(\mathbb{R}^+\right); \ f \text{ is continuous on } \mathbb{R}^+\right\},$$

$$C_\rho^k\left(\mathbb{R}^+\right) = \left\{f \in C_\rho\left(\mathbb{R}^+\right); \ \lim_{x \to \infty} \frac{f(x)}{\rho(x)} = k_f\right\},$$

where k_f is a constant depending on f.

It is obvious that $C_\rho^k\left(\mathbb{R}^+\right) \subset C_\rho\left(\mathbb{R}^+\right) \subset B_\rho\left(\mathbb{R}^+\right)$. The space $B_\rho\left(\mathbb{R}^+\right)$ is a normed linear space with the norm $\|f\|_\rho = \sup_{x \in \mathbb{R}^+} \frac{|f(x)|}{\rho(x)}$.

2. Approximation by (G_n^*)

Korovkin theorem was extended to unbounded intervals and a weighted Korovkin type theorem in a subspace of continuous functions on the real axis $\mathbb{R}$ was proved in [21, 22]. Let us recall the weighted form of the Korovkin Theorem ([21, 22]).

Lemma 2.1. *The positive linear operators L_n, $n \geq 1$, act from $C_\rho\left(\mathbb{R}^+\right)$ to $B_\rho\left(\mathbb{R}^+\right)$ if and only if the inequality*

$$|L_n(\rho; x)| \leq K_n \rho(x), \ x \geq 0$$

holds, where K_n is a positive constant.

Theorem 2.2. *Let the sequence of linear positive operators $(L_n)_{n \geq 1}$ acting from $C_\rho\left(\mathbb{R}^+\right)$ to $B_\rho\left(\mathbb{R}^+\right)$ satisfy the three conditions*

$$\lim_{n \to \infty} \|L_n(t^v; x) - x^v\|_\rho = 0, \ v = 0, 1, 2. \tag{2.1}$$

Then for any function $f \in C_\rho^k\left(\mathbb{R}^+\right)$

$$\lim_{n \to \infty} \|L_n f - f\|_\rho = 0.$$

Lemma 2.3. *Let $e_i(t) = t^i$, $i = 0, 1, 2$. For the operators (1.5), we have*

$$G_n^*(e_0; x) = 1, \tag{2.2}$$

$$G_n^*(e_1; x) = x \tag{2.3}$$

and

$$G_n^* (e_2; x) \le x (x + 2b_n\beta) (1 + n\beta)$$
$$+ x \frac{b_n}{n} (n\beta)^2 (1 + n\beta) + x (x + 2b_n\beta) n\beta \tag{2.4}$$
$$+ x \frac{b_n}{n} (n\beta)^3 + x \frac{b_n}{n}.$$

Proof. Firstly, we show that $G_n^* (e_i; x)$ for $i = 0$,

$$G_n^* (e_0; x) = (1 + n\beta)^{1-n} \sum_{k=0}^{n} \binom{n}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta \right)^{k-1} \left(1 - \frac{x}{b_n} \right) \left[1 - \frac{x}{b_n} + (n - k) \beta \right]^{n-k-1}.$$

If we take $u = \frac{x}{b_n}, v = 1 - \frac{x}{b_n}$ in (1.2), we get

$$G_n^* (e_0; x) = 1. \tag{2.5}$$

Morever, taking $G_n^* (e_i; x)$ for $i = 1$, one can have

$$G_n^* (e_1; x) = (1 + n\beta)^{1-n} \sum_{k=1}^{n} \binom{n-1}{k-1} b_n \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta \right)^{k-1}$$
$$\times \left(1 - \frac{x}{b_n} \right) \left[1 - \frac{x}{b_n} + (n - k) \beta \right]^{n-k-1}. \tag{2.6}$$

If we take $k \to k + 1$ and considering the equality
$\frac{x}{b_n} + \beta + k\beta = 1 + n\beta - \left[1 - \frac{x}{b_n} + (n - k - 1) \beta \right]$, we reach to

$$G_n^* (e_1; x) = (1 + n\beta)^{1-n} b_n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + \beta + k\beta \right)^{k-1} (1 + n\beta)$$
$$\times \left(1 - \frac{x}{b_n} \right) \left[1 - \frac{x}{b_n} + (n - k - 1) \beta \right]^{n-k-2}$$
$$- (1 + n\beta)^{1-n} b_n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + \beta + k\beta \right)^{k-1}$$
$$\times \left(1 - \frac{x}{b_n} \right) \left[1 - \frac{x}{b_n} + (n - k - 1) \beta \right]^{n-k-1}. \tag{2.7}$$

In order to find the first sum, we replace in the identity (1.2), $u = \frac{x}{b_n} + \beta, v = 1 - \frac{x}{b_n}$, n by $n - 1$, we get

$$(1 + \beta) (1 + n\beta)^{n-2} = \left(\frac{x}{b_n} + \beta \right) \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{x}{b_n} + \beta + k\beta \right)^{k-1}$$
$$\times \left(1 - \frac{x}{b_n} \right) \left[1 - \frac{x}{b_n} + (n - k - 1) \beta \right]^{n-k-2}. \tag{2.8}$$

If we use (2.8) in (2.7), we have

$$G_n^* (e_1; x) = (1 + \beta) \frac{x b_n}{x + b_n\beta} - (1 + n\beta)^{1-n} b_n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + \beta + k\beta \right)^{k-1}$$
$$\times \left(1 - \frac{x}{b_n} \right) \left[1 - \frac{x}{b_n} + (n - k - 1) \beta \right]^{n-k-1}. \tag{2.9}$$

On the other hand, if we take u, v and n by $\frac{x}{b_n} + \beta$, $1 - \frac{x}{b_n}$ and $n-1$, respectively in (1.1), we get

$$(1 + n\beta)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{x}{b_n} + \beta\right) \left(\frac{x}{b_n} + \beta + k\beta\right)^{k-1}$$
$$\times \left[1 - \frac{x}{b_n} + (n-k-1)\beta\right]^{n-k-1}. \qquad (2.10)$$

By considering (2.10) in (2.9), so we have

$$G_n^*(e_1; x) = x. \qquad (2.11)$$

Finally, we estimate $G_n^*(e_i; x)$ for $i = 2$,

$$G_n^*(e_2; x) = (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta + 2\beta\right)^{k+1}$$
$$\times \left(1 - \frac{x}{b_n}\right) \left[1 - \frac{x}{b_n} + (n-k-2)\beta\right]^{n-k-3}$$
$$+ (1 + n\beta)^{1-n} \frac{b_n}{n} \sum_{k=1}^{n} \frac{k}{n} b_n \binom{n}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1}$$
$$\times \left(1 - \frac{x}{b_n}\right) \left[1 - \frac{x}{b_n} + (n-k)\beta\right]^{n-k-1}$$
$$= (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 \frac{x}{b_n} \sum_{k=0}^{n-2} \binom{n-2}{k} \left(\frac{x}{b_n} + k\beta + 2\beta\right)^{k+1}$$
$$\times \left[1 - \frac{x}{b_n} + (n-k-2)\beta\right]^{n-k-2}$$
$$- (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 \frac{x}{b_n} \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta + 2\beta\right)^{k+1}$$
$$\times (n-k-2)\beta \left[1 - \frac{x}{b_n} + (n-k-2)\beta\right]^{n-k-3}$$
$$+ \frac{b_n}{n} G_n^*(e_1; x).$$

In order to find an upper estimate for the $G_n^*(e_2; x)$, we need to give the following equality, as in [13]

$$L\left(j, n, \frac{x}{b_n}, 1 - \frac{x}{b_n}\right) := \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{b_n} + k\beta\right)^{k+j-1} \left(1 - \frac{x}{b_n} + (n-k)\beta\right)^{n-k}, \qquad (2.12)$$

where $n \in \mathbb{N}$, $0 \leq x \leq b_n$ and b_n is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{n} = 0$.

It is clear that the function $L\left(j, n, \frac{x}{b_n}, 1 - \frac{x}{b_n}\right)$ satisfies the following reduction formula

$$L\left(j, n, \frac{x}{b_n}, 1 - \frac{x}{b_n}\right) = \frac{x}{b_n} L\left(j-1, n, \frac{x}{b_n}, 1 - \frac{x}{b_n}\right) + n\beta L\left(j, n-1, \frac{x}{b_n} + \beta, 1 - \frac{x}{b_n}\right).$$

Therefore from (2.12), we get

$$
\begin{aligned}
G_n^* (e_2; x) &= \frac{x}{b_n} (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 L\left(2, n-2, \frac{x}{b_n} + 2\beta, 1 - \frac{x}{b_n}\right) \\
&\quad - \frac{x}{b_n} (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 (n-2) \beta L\left(2, n-3, \frac{x}{b_n} + 2\beta, 1 - \frac{x}{b_n} + \beta\right) \\
&\quad + \frac{b_n}{n} G_n^* (e_1; x) \\
&= K_1(n, x) + K_2(n, x) + \frac{b_n}{n} x.
\end{aligned}
$$

From Lemma 1 in [13], we know that

$$
\begin{aligned}
L(2, n, x, y) &= \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} \left[x \left(x + y + n\beta + s\beta + u\beta \right)^n \right. \\
&\quad \left. + n\beta^2 u \left(x + y + n\beta + s\beta + u\beta \right)^{n-1} \right] du
\end{aligned}
\tag{2.13}
$$

and so we have,

$$
\begin{aligned}
L&\left(2, n-2, \frac{x}{b_n} + 2\beta, 1 - \frac{x}{b_n}\right) \\
&= (1 + n\beta)^{n-2} \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} \left[\left(\frac{x}{b_n} + 2\beta \right) \left(1 + \frac{s\beta + u\beta}{1 + n\beta} \right)^{n-2} \right] du \\
&\quad + (1 + n\beta)^{n-3} \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} (n-2) \beta^2 u \left(1 + \frac{s\beta + u\beta}{1 + n\beta} \right)^{n-3} du \\
&= \alpha(n, x) + \beta(n, x).
\end{aligned}
\tag{2.14}
$$

Taking into account this inequality $(1 + v)^{n-2} \leq \exp(nv)$, we have

$$
\begin{aligned}
\alpha(n, x) &\leq (1 + n\beta)^{n-2} \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} \left[\left(\frac{x}{b_n} + 2\beta \right) \exp\left(n \left(\frac{s\beta + u\beta}{1 + n\beta} \right) \right) \right] du \\
&= \left(\frac{x}{b_n} + 2\beta \right) (1 + n\beta)^{n-2} \int_0^\infty e^{-s\left(\frac{1}{1+n\beta}\right)} ds \int_0^\infty e^{-u\left(\frac{1}{1+n\beta}\right)} du \\
&= \left(\frac{x}{b_n} + 2\beta \right) (1 + n\beta)^{n-2} (1 + n\beta)^2
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
\beta(n, x) &\leq (1 + n\beta)^{n-3} \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} u (n-2) \beta^2 \exp\left(n \left(\frac{s\beta + u\beta}{1 + n\beta} \right) \right) du \\
&= (n-2) \beta^2 (1 + n\beta)^{n-3} \int_0^\infty e^{-s\left(\frac{1}{1+n\beta}\right)} ds \int_0^\infty u e^{-u\left(\frac{1}{1+n\beta}\right)} du \\
&= (1 + n\beta)^n (n-2) \beta^2.
\end{aligned}
\tag{2.16}
$$

Taking (2.15) and (2.16) in (2.14), we get

$$K_1(n, x) \leq \left(\frac{x}{b_n} + 2\beta\right) \frac{x}{b_n} (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 (1 + n\beta)^n$$
$$+ \frac{x}{b_n} (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 (1 + n\beta)^n (n-2) \beta^2$$
$$\leq x (x + 2b_n\beta) (1 + n\beta) + x \frac{b_n}{n} (n\beta)^2 (1 + n\beta). \tag{2.17}$$

On the other hand

$$K_2(n, x) \leq \frac{x}{b_n} (1 + n\beta)^{1-n} \frac{n-1}{n} (b_n)^2 (n-2) \beta$$
$$\times L\left(2, n-3, \frac{x}{b_n} + 2\beta, 1 - \frac{x}{b_n} + \beta\right) \tag{2.18}$$

and

$$L\left(2, n-3, \frac{x}{b_n} + 2\beta, 1 - \frac{x}{b_n} + \beta\right)$$
$$= \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} \left(\frac{x}{b_n} + 2\beta\right) (1 + n\beta + s\beta + u\beta)^{n-3} du$$
$$+ \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} (n-2) \beta^2 (1 + n\beta + s\beta + u\beta)^{n-4} du$$
$$= \gamma(n, x) + \xi(n, x). \tag{2.19}$$

Here

$$\gamma(n, x) \leq (1 + n\beta)^{n-3} \left(\frac{x}{b_n} + 2\beta\right) \int_0^\infty e^{-s} ds \int_0^\infty e^{-u} \exp\left(n\left(\frac{s\beta + u\beta}{1 + n\beta}\right)\right) du$$
$$= (1 + n\beta)^{n-3} \left(\frac{x}{b_n} + 2\beta\right) \int_0^\infty e^{-s\left(\frac{1}{1+n\beta}\right)} ds \int_0^\infty e^{-u\left(\frac{1}{1+n\beta}\right)} du$$
$$= (1 + n\beta)^{n-3} \left(\frac{x}{b_n} + 2\beta\right) (1 + n\beta)^2 \tag{2.20}$$

and

$$\xi(n, x) = \int_0^\infty e^{-s} ds \int_0^\infty u e^{-u} (n-2) \beta^2 (1 + n\beta + s\beta + u\beta)^{n-4} du$$
$$\leq (1 + n\beta)^{n-4} (n-2) \beta^2 (1 + n\beta) (1 + n\beta)^2. \tag{2.21}$$

So, taking (2.19), (2.20) and (2.21) in (2.18), we have

$$K_2(n, x) \leq x (x + 2b_n\beta) n\beta + x \frac{b_n}{n} (n\beta)^3.$$

Hence, we can write

$$G_n^*(e_2; x) \leq x (x + 2b_n\beta) (1 + n\beta)$$
$$+ x \frac{b_n}{n} (n\beta)^2 (1 + n\beta) + x (x + 2b_n\beta) n\beta$$
$$+ x \frac{b_n}{n} (n\beta)^3 + x \frac{b_n}{n}. \tag{2.22}$$

So, we obtain the desired results of Lemma 2.1.

$\square$

Now, in order to get an approximation result, we consider β as a sequence of positive real numbers such that $\beta = \beta_n$ and $lim_{n\to\infty}n\beta_n = 0$.

We show here that the operators $G_n^*(f)$ satisfy the required conditions for the weighted Korovkin type theorem.

Theorem 2.4. *Suppose that $n \in \mathbb{N}$, (b_n) and (β_n) are sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} b_n/n = 0$ and $lim_{n\to\infty}n\beta_n = 0$. Then for each $f \in C_\rho^k(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty} \|G_n^* f - f\|_\rho = 0.$$

Proof. By Lemma 2.2, (G_n^*) are linear operators acting from $C_\rho(\mathbb{R}^+)$ to $B_\rho(\mathbb{R}^+)$. Indeed, from (2.5) and (2.22), we easily obtain that

$$|G_n^*(\rho; x)| \le \left(1 + x^2\right) K_n.$$

On the other hand, using (2.5), (2.11) and (2.22), one can write

$$\|G_n^*(e_0) - 1\|_\rho = 0,$$

$$\|G_n^*(e_1) - x\|_\rho = 0,$$

and

$$\left\|G_n^*(e_2) - x^2\right\|_\rho \le \sup\left\{\left|\frac{x^2}{1+x^2}(2n\beta_n)\right|\right.$$
$$+ \left|\frac{2x}{1+x^2}b_n\beta_n(1+n\beta_n)\right|$$
$$\left.+ \frac{x}{1+x^2}\frac{b_n}{n}(n\beta_n)^2(1+2n\beta_n) + \frac{x}{1+x^2}\frac{b_n}{n}\right\},$$

by the hypothesis, we have

$$\lim_{n\to\infty}\left\|G_n^*(e_2) - x^2\right\|_\rho = 0.$$

So the proof is completed. $\square$

In the following theorem, we give approximation results of the operators (G_n^*) with the help of universal Korovkin-type theorem.

Theorem 2.5. *Let (G_n^*) be the operators given by (1.5). Then, for any $f \in C[0,\infty) \cap E$, the following relation holds*

$$\lim_{n\to\infty} G_n^*(f; x) = f(x)$$

uniformly on each compact subset of $[0,\infty)$, where

$$E := \left\{f : x \in [0,\infty), \ \frac{f(x)}{1+x^2} \ is \ convergent \ as \ x \to \infty\right\}.$$

Proof. Using universal Korovkin-type theorem [5], it is sufficient to prove that the operators (G_n^*) verify the conditions

$$\lim_{n\to\infty} G_n^*(e_i; x) = x^i, \ i = 0, 1, 2,$$

uniformly on each compact subset of $[0,\infty)$. From $(2.2) - (2.4)$, we obtain that the mentioned conditions are supplied. So the proof is completed. $\square$

3. Rate of convergence

In this section, we compute rate of convergence of the operators (G_n^*) by means of the modulus of continuity.

Let $f \in \tilde{C}[0,\infty)$ such that f is uniformly continuous for any $\delta > 0$. The usual modulus of continuity for f is defined as

$$\omega(f,\delta) = \sup_{\substack{|x-y| \leq \delta \\ x,y \in [0,\infty)}} |f(x) - f(y)|.$$

For $f \in \tilde{C}[0,\infty)$ and any $x, y \in [0,\infty)$, we have

$$|f(x) - f(y)| \leq \omega(f; |y - x|),$$

and for any $\delta > 0$

$$|f(x) - f(y)| \leq \omega(f,\delta) \left(\frac{|x-y|}{\delta} + 1\right). \tag{3.1}$$

Now, we give the following result.

Theorem 3.1. *Suppose that $n \in \mathbb{N}$, (b_n) and (β_n) are sequences of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} b_n/n = 0$ and $\lim_{n\to\infty} n\beta_n = 0$. Then for each $f \in C[0,\infty) \cap E$, we have*

$$|G_n^*(f; x) - f(x)| \leq 2\omega(f; \delta_n(x)),$$

where $\delta_n(x) = x^2(2n\beta_n) + 2xb_n\beta_n(1 + n\beta_n) + x\frac{b_n}{n}(n\beta_n)^2(1 + 2n\beta_n) + x\frac{b_n}{n}$.

Proof. Using the linearity and positivity of $G_n^*(f; x)$ and then applying (3.1), we have

$$|G_n^*(f; x) - f(x)| \leq G_n^*(|f(t) - f(x)|; x)$$

$$\leq (1 + n\beta)^{1-n} \sum_{k=0}^{n} \omega(f,\delta) \left(1 + \frac{|t-x|}{\delta}\right) \binom{n}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1}$$

$$\times \left(1 - \frac{x}{b_n}\right) \left[1 - \frac{x}{b_n} + (n-k)\beta\right]^{n-k-1}$$

$$= \omega(f,\delta)(1 + n\beta)^{1-n} \sum_{k=0}^{n} \left(1 + \frac{|t-x|}{\delta}\right) \binom{n}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1}$$

$$\times \left(1 - \frac{x}{b_n}\right) \left[1 - \frac{x}{b_n} + (n-k)\beta\right]^{n-k-1},$$

from (2.2), one has

$$|G_n^*(f; x) - f(x)| \leq \omega(f,\delta) \left\{1 + \frac{1}{\delta}(1 + n\beta_n)^{1-n} \sum_{k=0}^{n} |t-x| \binom{n}{k} \frac{x}{b_n} \left(\frac{x}{b_n} + k\beta\right)^{k-1} \right.$$

$$\left. \times \left(1 - \frac{x}{b_n}\right) \left[1 - \frac{x}{b_n} + (n-k)\beta\right]^{n-k-1}\right\}.$$

By the Cauchy-Schwarz inequality, we reach to

$$|G_n^*(f; x) - f(x)| \leq \omega(f,\delta) \left\{1 + \frac{1}{\delta} \left\{G_n^*\left((t-x)^2; x\right)\right\}^{\frac{1}{2}}\right\}.$$

On the other hand, using (2.2), (2.3) and (2.4), we get

$$G_n^*(\varphi; x) = G_n^*(e_2; x) - 2xG_n^*(e_1; x) + x^2 G_n^*(e_0; x)$$

$$\leq x^2(2n\beta_n) + 2xb_n\beta_n(1 + n\beta_n) + x\frac{b_n}{n}(n\beta_n)^2(1 + 2n\beta_n) + x\frac{b_n}{n}.$$

Let choose $\delta = \delta_n(x) := x^2(2n\beta_n) + 2xb_n\beta_n(1 + n\beta_n) + x\frac{b_n}{n}(n\beta_n)^2(1 + 2n\beta_n) + x\frac{b_n}{n}$, then we have

$$|G_n^*(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\delta_n(x)}\right).$$

This concludes the proof. □

4. *A*-Statistical approximation by (G_n^*)

In this section, we obtain *A*-statistical convergence of the Cheney-Sharma-Chlodovsky operators to identity operator on the weighted spaces. Let us recall the weighted Korovkin type approximation theorem for the A-statistical convergence was given by Duman and Orhan in [17].

Theorem 4.1. *Let $A = (a_{nk})$ be a nonnegative regular summability matrix and let $\bar{\rho}_1, \bar{\rho}_2$ weight functions such that $\overline{\rho_1}$*

$$\lim_{|x|\to\infty} \frac{\overline{\rho_1}(x)}{\overline{\rho_2}(x)} = 0. \tag{4.1}$$

Assume that $(T_n)_{n\geq 1}$ is a sequence of positive linear operators from $C_{\overline{\rho_1}}(\mathbb{R})$ into $B_{\overline{\rho_2}}(\mathbb{R})$. One has

$$st_A - \lim_n \|T_n f - f\|_{\overline{\rho_2}} = 0,$$

for all $f \in C_{\overline{\rho_1}}(\mathbb{R})$ if and only if

$$st_A - \lim_n \|T_n F_v - F_v\|_{\overline{\rho_1}} = 0, \quad k = 0, 1, 2,$$

where

$$F_v(x) = \frac{x^v \overline{\rho_1}(x)}{1 + x^2}, \quad v = 0, 1, 2.$$

Corollary 4.2. [17]. *Let $A = (a_{nk})$ be a nonnegative regular summability matrix and let (L_n) be a sequence of positive linear operators acting from $C_{\rho_0}(R^+)$ into $B_{\rho_\lambda}(R^+)$, $\lambda > 0$ one has*

$$st_A - \lim_n \|L_n f - f\|_{\rho_\lambda} = 0, \quad f \in C_{\rho_0}\left(R^+\right),$$

if and only if

$$st_A - \lim_n \|L_n e_i - e_i\|_{\rho_0} = 0, \ i = 0, 1, 2, \tag{4.2}$$

where $\rho_0(x) = 1 + x^2$ and $\rho_\lambda(x) = 1 + x^{2+\lambda}$, $\lambda > 0$.

Using this theorem the following Korovkin type statistical theorem can be proved for (G_n^*):

Theorem 4.3. *Let $A = (a_{nk})$ be a nonnegative regular summability matrix and let (b_n) and (β_n) be sequences of positive numbers such that $st_A - \lim_{n\to\infty} b_n/n = 0$, $st_A - \lim_{n\to\infty} n\beta_n = 0$. Then for each $f \in C_\rho^k(\mathbb{R}^+)$, we have*

$$st_A - \lim_{n\to\infty} \|G_n^* f - f\|_{\rho_\lambda} = 0,$$

where $\rho_0(x) = 1 + x^2$ and $\rho_\lambda(x) = 1 + x^{2+\lambda}$, $\lambda > 0$.

Proof. Using Corollary 4.2, it is sufficient to prove that the operators (G_n^*) verify the conditions given in (4.1). Indeed, from (2.2) and (2.3), it is clear that

$$st_A - \lim_n \|G_n^*(e_0; .) - e_0\|_{\rho_0} = 0$$

and

$$st_A - \lim_n \|G_n^*(e_1; .) - e_1\|_{\rho_0} = 0.$$

Now using (2.4), one can have

$$\left\| G_n^* (e_2) - x^2 \right\|_{\rho_0} \leq \sup \left\{ \left| \frac{x^2}{1 + x^2} \left(2n\beta_n \right) \right| \right.$$

$$+ \left| \frac{2x}{1 + x^2} b_n \beta_n \left(1 + n\beta_n \right) \right|$$

$$\left. + \frac{x}{1 + x^2} \frac{b_n}{n} \left(n\beta_n \right)^2 \left(1 + 2n\beta_n \right) + \frac{x}{1 + x^2} \frac{b_n}{n} \right\},$$

which implies

$$\left\| G_n^* (e_2) - x^2 \right\|_{\rho_0} \leq 2n\beta_n + 2b_n\beta_n \left(1 + n\beta_n \right) + \frac{b_n}{n} \left(n\beta_n \right)^2 \left(1 + 2n\beta_n \right) + \frac{b_n}{n}$$

$$= K_n.$$

Now, for a given $\epsilon > 0$, let us define the following sets:

$$M := \left\{ k : \left\| G_n^* (e_2; .) - e_2 \right\|_{\rho_0} \geq \epsilon \right\},$$

$$M_1 := \left\{ k : 2n\beta_n \geq \frac{\epsilon}{5} \right\},$$

$$M_2 := \left\{ k : 2b_n\beta_n \left(1 + n\beta_n \right) \geq \frac{\epsilon}{5} \right\},$$

$$M_3 := \left\{ \frac{1}{n} \left(n\beta \right)^2 \left(1 + 2n\beta \right) \geq \frac{\epsilon}{5} \right\},$$

$$M_4 := \left\{ \frac{b_n}{n} \left(n\beta_n \right)^2 \left(1 + 2n\beta_n \right) \geq \frac{\epsilon}{5} \right\},$$

$$M_5 := \left\{ \frac{b_n}{n} \geq \frac{\epsilon}{5} \right\}.$$

Then, we see that $M \subseteq M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$. Therefore, we get

$$\sum_{n : \| G_n^* (e_2; .) - e_2 \|_{\rho_0} \geq \epsilon} a_{k,n} \leq \sum_{n \in M_1} a_{k,n} + \sum_{n \in M_2} a_{k,n} + \sum_{n \in M_3} a_{k,n} + \sum_{n \in M_4} a_{k,n} + \sum_{n \in M_5} a_{k,n} \quad (4.3)$$

and, taking the limit $k \to \infty$ in (4.3), we have

$$st_A - \lim_n \left\| G_n^* (e_2; .) - e_2 \right\|_{\rho_0} = 0.$$

This proves the theorem. $\square$

Remark 4.4. In Theorem 4.3, we have stronger results than approximation theorems given in section 2. Indeed, Theorem 4.3 may be useful when $n\beta_n$ and b_n/n does not convergent to zero as $n \to \infty$.

The following example shows that there exist some sequences (β_n) and (b_n) such that A-satistical convergence holds but ordinary convergence does not hold for $n\beta_n$ and b_n/n.

Example 4.5. Let (b_n) and (β_n) be the sequences defined by

$$b_n = \begin{cases} e^{n^2}, & \text{if } n \text{ is a perfect square} \\ \sqrt{n}, & \text{otherwise.} \end{cases}$$

and

$$\beta_n = \begin{cases} n, & \text{if } n \text{ is a perfect square} \\ \frac{1}{n^2}, & \text{otherwise.} \end{cases}$$

It is easy to see that b_n/n and $n\beta_n$ are not convergent but statistically convergent, i.e., C_1-statistically convergent.

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