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Research Article

On The Properties of The Complex Fibonacci and Lucas Numbers with Rational Subscript via Roots of the Fibonacci Matrix

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Abstract

In this study, we exploit general techniques from matrix theory to establish some identities for the complex Fibonacci and Lucas numbers with rational subscripts of the forms $n/2$ and n/s. For this purpose, we establish matrix functions $R \to R^{n/2}$ and $R \to R^{n/s}$ of the Fibonacci matrix R of order 3×3 for integer odd n and discuss some relations between two special matrices functions $R^{n/2}$ and $R^{n/s}$, respectively. Also, some identities related to the complex Fibonacci and Lucas numbers with rational subscripts of the forms $n/2$ and n/s are given for every integer odd n and $\gcd(n,s)=1,\ s\in \square$, respectively.

Keywords: Fibonacci sequences, Lucas sequences, Matrix functions, Root matrices

Fibonacci Matrisinin Kökleri Aracılığıyla Fibonacci ve Lucas Sayılarının Özellikleri Üzerine

Öz

Bu çalışmada, $n/2$ ve n/s formlarındaki rasyonel indisli kompleks Fibonacci ve Lucas sayıları için bazı eşitlikler oluşturmak için matris teorisinden genel tekniklerden faydalanırız. Bu amaçla, tek n tam sayıları için 3×3 mertebeden Fibonacci R matrisinin $R \to R^{n/2}$ and $R \to R^{n/s}$ matris fonksiyonlarını kurar ve sırasıyla $R^{n/2}$ and $R^{n/s}$ iki özel matris fonksiyonu arasındaki bazı ilişkileri ele alırız. Sırasıyla, tek n tam sayıları ve $ebob(n,s)=1,\ s\in N$ için $n/2$ ve n/s formlarındaki rasyonel indisli kompleks Fibonacci ve Lucas sayıları ile ilgili bazı eşitlikler veririz.

Anahtar Kelimeler: Fibonacci dizileri, Lucas dizileri, Matris fonksiyonları, Kök matrisleri

1. Introduction

The Fibonacci $\{F_n\}_{n=0}^{\infty}$ and Lucas $\{L_n\}_{n=0}^{\infty}$ sequences possess similar recurrence relations of order two, $X_n = X_{n-1} + X_{n-2}$, defined with the initial conditions $F_0 = 0$, $F_1 = 1$, and $L_0 = 2$, $L_1 = 1$. And also, any n^{th} entry for these sequences can be given with the Binet's formulas $F_n = \left(\alpha^n - \beta^n\right)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$, $\alpha = \left(1+\sqrt{5}\right)/2$, $\beta = -\alpha^{-1}$. These numbers are generalized with respect to the initial conditions, the coefficients of recurrence relation and the subscripts elements, etc. (Koshy, 2001); Debnath and Saha (2014); Deveci and Karaduman (2009); Taşyurdu, Çobanoğlu, and Dilmen (2016). For example, $F_{-n} = (-1)^{-n+1} F_n$ and $L_{-n} = (-1)^{-n} L_n$, $n \in Z^+$ are negative subscripts of these numbers (Koshy, 2001). Later, in Halsey (1965); Horadam and Shannon (1988); Parker, (1968); Richard, (1991) studies, subscripts of these numbers are tried to generalize any rational, reel and complex number by various functions expressed as the generalized Binet's formula:

$$F_x = \frac{\alpha^x - e^{x\pi i}\alpha^{-x}}{\alpha + \alpha^{-1}},\ L_x = \alpha^x + e^{x\pi i}\alpha^{-x} \quad . \quad (1)$$

Furthermore, the matrix methods generate the Fibonacci and the Lucas matrices, the entries of the powers of which are related to the Fibonacci and/or Lucas numbers. To make this procedure have been used the 2×2 Fibonacci matrices Q and Q_R Bicknell and Hoggatt (1963), the 3×3 matrix R Hoggatt and Bicknell (1964); Karakaya, Özdemir, and Petik (2018), and a number of $k\times k$ matrices Gould (1981). For example, in Hoggatt and Bicknell, (1964), the author established that for positive n integers, the n^{th} power of the matrix R is shown with the Fibonacci R^n matrix

$$R = \begin{bmatrix} 0\,0\,1 \\ 0\,1\,2 \\ 1\,1\,1 \end{bmatrix},$$

$$R^n = \begin{bmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{bmatrix} \quad (2)$$

In addition, many mathematicians have studied the negative powers of the Fibonacci matrices Q^n and Q_R^n (Koshy 2001; Halici and Akyüz 2016). Afterward, in Bicknell (1965); Witula (2011); Arslan and Köken (2017) studies, the authors have described the matrix functions of the Fibonacci matrices $Q^{n/2}$ and $Q_R^{n/2}$, the entries of these functions are related to the Fibonacci and Lucas numbers with rational subscripts and they generated a number of identities the Fibonacci and Lucas numbers with rational numbers by using these matrix functions $Q^{r/s}$ and $Q_R^{r/s}$.

Firstly, we give again the matrix R^n defining the matrix equation of order 2 with the Fibonacci number coefficient according to I, R and R^2. Relations between the matrix R^n given as matrix equation and matrix expression in (2) are obtained with techniques of matrix theory. We apply the matrix functions to the 3×3 matrix R to obtain new identities involving squares of terms from the Fibonacci and the Lucas numbers with rational subscripts. Also, the roots of the Fibonacci matrix R^n are evaluated by the functions $F^{(n,2)}(R)\equiv R^{n/2}$, $n\in Z-\{0\}$, and $F^{(n,s)}(R)\equiv R^{n/s}$, $n\in Z-\{0\}$, $\gcd(n,s)=1$, $s\in N$.

There are many equivalent ways for attainment of the R^n for any n integer values. By specializing $F^{(n)}(z)=z^n$, $n\in Z$, one of these is the matrix function $F^{(n)}(R)=R^n$ expressed under the polynomial expressions,

$$F^{(n)}(R) = \sum_{i=1}^{3}\prod_{\substack{j=1\\ i\neq j}}^{3} \frac{F^{(n)}(\lambda_i)}{\lambda_i - \lambda_j}\left[R - \lambda_j I\right] \quad , \quad (3)$$

where the eigenvalues λ_i, $i=1,2,3$ of the eigenvalues of the R are α^2, $\alpha\beta$ and β^2, lying within the circle of convergence of the function $F^{(n)}(z)=z^n$ and the I is a order 3×3 identity matrix (Gantmacher, 1960; Higham, 2008). The matrix function $F^{(n)}(R)$ given in (3) establishes a matrix relation of order 2 with the Fibonacci number coefficient according to the matrices I, R and R^2;

Lemma 1.1. Let F_n be the n^{th} Fibonacci number for every n integer, then

$$F^{(n)}(R) = F_nF_{n-1}R^2 + F_nF_{n-2}R - F_{n-1}F_{n-2}I \quad . \quad (4)$$

Proof. Follows directly from the equation (3). ■

Corollary 1.2 If the corresponding elements of matrices in (2) and (4) are equalized, then it is seen that

$$i)\ F_{n+1}^2 = 4F_nF_{n-1} + F_nF_{n-2} - F_{n-1}F_{n-2},$$
$$ii)\ F_nF_{n+1} = 2F_nF_{n-1} + F_nF_{n-2},$$
$$iii)\ F_{n+1}^2 - F_{n-1}F_n = 3F_nF_{n-1} + F_nF_{n-2} - F_{n-1}F_{n-2}.$$

A large number of the Fibonacci-type identities can be obtained by based on a polynomial representation of the function itself (Hoggatt and Bicknell, 1964).

2. Relations Between The Fibonacci and Lucas Numbers With Square Roots Of The Matrix R^n

In the theory of functions of matrices, the domain of an analytic function F is extended to include the matrix R of order 3 defining the $F(R)$ as a polynomial in the R of degree less than or equal to 2 provided that the $F(R)$ is defined on the spectrum of the matrix R (Gantmacher, 1960; Higham, 2008). In the case of the R and the scalar complex function $F^{(n,2)}(z)\equiv z^{n/2}$, for an odd integer n, the existence of a negative eigenvalue implies that the entries of any square root matrix are complex. It is well known that for every nonzero complex number $z=|z|\exp\left[i\arg(z)\right]$, $-\pi<\arg(z)\le\pi$, the function $F^{(n,2)}(z)\equiv z^{n/2}$ is a double-valued function, giving rise to 2 branches: $F^{(n,2)}(z)\equiv z^{n/2}=f_k^{(n,2)}(z)$, $k\in\{0,1\}$. These branches can be characterized for every nonzero z in as follow

$$f_k^{(n,2)}(z)=|z|^{n/2}\exp\left[i\left(\frac{n}{2}\arg(z)+nk\pi\right)\right]=\exp(ink\pi)z^{n/2},\ k\in\{0,1\}, \tag{5}$$

where $z^{n/2}$ denotes the principal branch of the $F^{(n,2)}(z)$, i.e. $f_0^{(n,2)}(z)=z^{n/2}$.

Therefore, there exist 2^3 matrix functions $F^{(n,2)}(R)\equiv R^{n/2}$ which can be derived from the two branches of the scalar function $f_k^{(n,2)}(z)$, $k\in\{0,1\}$ defined by (5). To emphasize, all these matrix functions are primary matrix functions, and can be specified by the Lagrange-Sylvester interpolation polynomial through the polynomial expression

$$F_{(k_1,k_2,k_3)}^{(n,2)}(R)=\sum_{i=1}^{3}f_{k_i}^{(n,2)}(\lambda_i)\prod_{\substack{j=1\\ i\neq j}}^{3}\frac{R-\lambda_j I}{\lambda_i-\lambda_j},\ (k_1,k_2,k_3)\in\{0,1\}^3 \tag{6}$$

where $\lambda_i\in\{\alpha^2,\ \beta^2,\ \alpha\beta\}$ are the eigenvalues of the R (Higham 2008; Gantmacher, 1960).

Theorem 2.1. *For every odd number $n\in Z$,*

$$F_{(0,0,0)}^{(n,2)}(R)=-F_{(1,1,1)}^{(n,2)}(R)=\begin{pmatrix} F_{\frac{n}{2}-1}^2 & F_{\frac{n}{2}-1}F_{\frac{n}{2}} & F_{\frac{n}{2}}^2 \\ 2F_{\frac{n}{2}-1}F_{\frac{n}{2}} & F_{\frac{n}{2}}^2+F_{\frac{n}{2}-1}F_{\frac{n}{2}+1} & 2F_{\frac{n}{2}}F_{\frac{n}{2}+1} \\ F_{\frac{n}{2}}^2 & F_{\frac{n}{2}}F_{\frac{n}{2}+1} & F_{\frac{n}{2}+1}^2 \end{pmatrix} \tag{7}$$

$$F_{(0,1,1)}^{(n,2)}(R)=-F_{(1,0,0)}^{(n,2)}(R)=\frac{1}{5}\begin{pmatrix} L_{\frac{n}{2}-1}^2 & L_{\frac{n}{2}-1}L_{\frac{n}{2}} & L_{\frac{n}{2}}^2 \\ 2L_{\frac{n}{2}-1}L_{\frac{n}{2}} & L_{\frac{n}{2}}^2+L_{\frac{n}{2}-1}L_{\frac{n}{2}+1} & 2L_{\frac{n}{2}}L_{\frac{n}{2}+1} \\ L_{\frac{n}{2}}^2 & L_{\frac{n}{2}}L_{\frac{n}{2}+1} & L_{\frac{n}{2}+1}^2 \end{pmatrix}, \tag{8}$$

$$F_{(0,0,1)}^{(n,2)}(R)=-F_{(1,1,0)}^{(n,2)}(R)=\frac{1}{\sqrt{5}}\begin{pmatrix} F_{\frac{n}{2}-1}L_{\frac{n}{2}-1}+\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n-1}{2}}L_{\frac{n-1}{2}}+\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n}{2}}L_{\frac{n}{2}}-\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} \\ 2\left(F_{\frac{n-1}{2}}L_{\frac{n-1}{2}}+\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}}\right) & 2F_{\frac{n}{2}}L_{\frac{n}{2}}+\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & 2\left(F_{\frac{n+1}{2}}L_{\frac{n+1}{2}}-\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}}\right) \\ F_{\frac{n}{2}}L_{\frac{n}{2}}-\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n+1}{2}}L_{\frac{n+1}{2}}-\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n}{2}+1}L_{\frac{n}{2}+1}+\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} \end{pmatrix},$$

$$F_{(0,1,0)}^{(n,2)}(R)=-F_{(1,0,1)}^{(n,2)}(R)=\frac{-1}{\sqrt{5}}\begin{pmatrix} F_{\frac{n}{2}-1}L_{\frac{n}{2}-1}-\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n-1}{2}}L_{\frac{n-1}{2}}-\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n}{2}}L_{\frac{n}{2}}+\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} \\ 2\left(F_{\frac{n-1}{2}}L_{\frac{n-1}{2}}-\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}}\right) & 2F_{\frac{n}{2}}L_{\frac{n}{2}}-\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & 2\left(F_{\frac{n+1}{2}}L_{\frac{n+1}{2}}+\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}}\right) \\ F_{\frac{n}{2}}L_{\frac{n}{2}}+\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n+1}{2}}L_{\frac{n+1}{2}}+\frac{(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} & F_{\frac{n}{2}+1}L_{\frac{n}{2}+1}-\frac{2(\alpha\beta)^{\frac{n}{2}}}{\sqrt{5}} \end{pmatrix}.$$

Proof. If the matrix functions $F_{(k_1,k_2,k_3)}^{(n,2)}(R)=\left[f_{ij}\right]_{3\times 3}$, $k_i\in\{0,1\}$ are given with formula (6), we rewrite considering the eigenvalues $\lambda_i\in\{\alpha\beta,\ \alpha^2,\ \beta^2\}$ of the R;

$$\left[f_{ij}\right]_{(k_1,k_2,k_3)}=\begin{cases} f_{11}=\frac{1}{5}\left(2(\alpha\beta)^{\frac{n}{2}}e^{nk_1\pi i}+\beta^2\left(\alpha^2\right)^{\frac{n}{2}}e^{nk_2\pi i}+\alpha^2\left(\beta^2\right)^{\frac{n}{2}}e^{nk_3\pi i}\right), \\ f_{22}=\frac{1}{5}\left((\alpha\beta)^{\frac{n}{2}}e^{nk_1\pi i}+2\left(\alpha^2\right)^{\frac{n}{2}}e^{nk_2\pi i}+2\left(\beta^2\right)^{\frac{n}{2}}e^{nk_3\pi i}\right), \\ f_{33}=\frac{1}{5}\left(2(\alpha\beta)^{\frac{n}{2}}e^{nk_1\pi i}+\alpha^2\left(\alpha^2\right)^{\frac{n}{2}}e^{nk_2\pi i}+\beta^2\left(\beta^2\right)^{\frac{n}{2}}e^{nk_3\pi i}\right), \\ 2f_{12}=f_{21}=\frac{2}{5}\left((\alpha\beta)^{\frac{n}{2}}e^{nk_1\pi i}-\beta\left(\alpha^2\right)^{\frac{n}{2}}e^{nk_2\pi i}-\alpha\left(\beta^2\right)^{\frac{n}{2}}e^{nk_3\pi i}\right), \\ f_{13}=f_{31}=\frac{1}{5}\left(-2(\alpha\beta)^{\frac{n}{2}}e^{nk_1\pi i}+\left(\alpha^2\right)^{\frac{n}{2}}e^{nk_2\pi i}+\left(\beta^2\right)^{\frac{n}{2}}e^{nk_3\pi i}\right), \\ f_{23}=2f_{32}=\frac{2}{5}\left(-(\alpha\beta)^{\frac{n}{2}}e^{nk_1\pi i}+\alpha\left(\alpha^2\right)^{\frac{n}{2}}e^{nk_2\pi i}+\beta\left(\beta^2\right)^{\frac{n}{2}}e^{nk_3\pi i}\right), \end{cases} \quad (9)$$

The matrix functions $F_{(k_1,k_2,k_3)}^{(n,2)}(R)$ are computed on the proof for the some special cases, such that $(k_1,k_2,k_3)\in\{0,1\}^3$, using the generalized Binet's formula (1). The $k_1=k_2=k_3=0$ values are rewritten in the equation (9), $F_{(0,0,0)}^{(n,2)}(R)=\left[a_{ij}\right]_{3\times 3}$ are obtained as

$$\left[a_{ij}\right]_{3\times 3}=\begin{cases} a_{11}=\frac{1}{5}\left(\left(\alpha^{\frac{n}{2}-1}\right)^2-2(\alpha\beta)^{\frac{n}{2}-1}+\left(\beta^{\frac{n}{2}-1}\right)^2\right)=F_{\frac{n}{2}-1}^2, \\ a_{22}=\frac{1}{5}\left(2\left(\alpha^{\frac{n}{2}}\right)^2+(\alpha\beta)^{\frac{n}{2}}+2\left(\beta^{\frac{n}{2}}\right)^2\right)=F_{\frac{n}{2}}^2+F_{\frac{n}{2}-1}F_{\frac{n}{2}+1}, \\ a_{33}=\frac{1}{5}\left(\left(\alpha^{\frac{n}{2}+1}\right)^2-2(\alpha\beta)^{\frac{n}{2}+1}+\left(\beta^{\frac{n}{2}+1}\right)^2\right)=F_{\frac{n}{2}+1}^2, \\ a_{21}=2a_{12}=\frac{1}{5}\left(-2\beta\left(\alpha^{\frac{n}{2}}\right)^2+2(\alpha\beta)^{\frac{n}{2}}-2\alpha\left(\beta^{\frac{n}{2}}\right)^2\right)=2F_{\frac{n}{2}-1}F_{\frac{n}{2}}, \\ a_{13}=a_{31}=\frac{1}{5}\left(\left(\alpha^{\frac{n}{2}}\right)^2-2(\alpha\beta)^{\frac{n}{2}}+\left(\beta^{\frac{n}{2}}\right)^2\right)=F_{\frac{n}{2}}^2, \\ a_{23}=2a_{32}=\frac{1}{5}\left(2\alpha\left(\alpha^{\frac{n}{2}}\right)^2-2(\alpha\beta)^{\frac{n}{2}}+2\beta\left(\beta^{\frac{n}{2}}\right)^2\right)=2F_{\frac{n}{2}}F_{\frac{n}{2}+1}, \end{cases}.$$

If the $k_1=k_2=k_3=1$ values are considered, then $F_{(1,1,1)}^{(n,2)}(R)=-F_{(0,0,0)}^{(n,2)}(R)$. The $k_1=0$, $k_2=k_3=1$ values are written in the equation (9), $F_{(0,1,1)}^{(n,2)}(R)=\left[b_{ij}\right]_{3\times 3}$ are obtained as

$$\left[b_{ij}\right]=\begin{cases} b_{11}=\frac{1}{5}\left(-\left(\alpha^{\frac{n}{2}-1}\right)^2+2\left(\alpha\beta\right)^{\frac{n}{2}}-\left(\beta^{\frac{n}{2}-1}\right)^2\right)=\frac{-1}{5}L^2_{\frac{n}{2}-1}, \\ b_{22}=\frac{1}{5}\left(-2\left(\alpha^{\frac{n}{2}}\right)^2+\left(\alpha\beta\right)^{\frac{n}{2}}-2\left(\beta^{\frac{n}{2}}\right)^2\right)=\frac{-1}{5}\left(L^2_{\frac{n}{2}}+L_{\frac{n}{2}-1}L_{\frac{n}{2}+1}\right), \\ b_{33}=\frac{1}{5}\left(-\left(\alpha^{\frac{n}{2}+1}\right)^2+2\left(\alpha\beta\right)^{\frac{n}{2}}-\left(\beta^{\frac{n}{2}+1}\right)^2\right)=\frac{-1}{5}L^2_{\frac{n}{2}+1}, \\ b_{21}=2b_{12}=\frac{1}{5}\left(-2\left(\left(\alpha^{\frac{n-1}{2}}\right)^2-\left(\alpha\beta\right)^{\frac{n}{2}}+\left(\beta^{\frac{n-1}{2}}\right)^2\right)\right)=\frac{-2}{5}L_{\frac{n}{2}-1}L_{\frac{n}{2}}, \\ b_{13}=b_{31}=\frac{1}{5}\left(-\left(\alpha^{\frac{n}{2}}\right)^2-2\left(\alpha\beta\right)^{\frac{n}{2}}-\left(\beta^{\frac{n}{2}}\right)^2\right)=\frac{-1}{5}L^2_{\frac{n}{2}}, \\ b_{23}=2b_{32}=\frac{1}{5}\left(-2\left(\left(\alpha^{\frac{n+1}{2}}\right)^2+\left(\alpha\beta\right)^{\frac{n}{2}}+\left(\beta^{\frac{n+1}{2}}\right)^2\right)\right)=\frac{-2}{5}L_{\frac{n}{2}}L_{\frac{n}{2}+1}, \end{cases}.$$

If the $k_1=1$, $k_2=k_3=0$ values are considered, then $F^{(n,2)}_{(0,1,1)}(R)=-F^{(n,2)}_{(1,0,0)}(R)$. To choice other values $(k_1,k_2,k_3)\in\{0,1\}^3$ gives desired other results. ■

For example, the Theorem 2.1 states that if the $k_1=k_2=k_3=0$ values or the $k_1=0$, $k_2=k_3=1$ values are rewritten in the equation (7) or (8), respectively, then the square root matrix of the R is equal following matrix;

$$\begin{pmatrix} F^2_{\frac{-1}{2}} & F_{\frac{-1}{2}}F_{\frac{1}{2}} & F^2_{\frac{1}{2}} \\ 2F_{\frac{-1}{2}}F_{\frac{1}{2}} & F^2_{\frac{1}{2}}+F_{\frac{-1}{2}}F_{\frac{3}{2}} & 2F_{\frac{1}{2}}F_{\frac{3}{2}} \\ F^2_{\frac{1}{2}} & F_{\frac{1}{2}}F_{\frac{3}{2}} & F^2_{\frac{3}{2}} \end{pmatrix}=\frac{1}{5}\begin{pmatrix} -1+2i & 2+i & 1-2i \\ 4+2i & 2+i & 6-2i \\ 1-2i & 3-i & 4+2i \end{pmatrix}$$

and

$$\frac{1}{5}\begin{pmatrix} L^2_{\frac{-1}{2}} & L_{\frac{-1}{2}}L_{\frac{1}{2}} & L^2_{\frac{1}{2}} \\ 2L_{\frac{-1}{2}}L_{\frac{1}{2}} & L^2_{\frac{1}{2}}+L_{\frac{-1}{2}}L_{\frac{3}{2}} & 2L_{\frac{1}{2}}L_{\frac{3}{2}} \\ L^2_{\frac{1}{2}} & L_{\frac{1}{2}}L_{\frac{3}{2}} & L^2_{\frac{3}{2}} \end{pmatrix}=\frac{1}{5}\begin{pmatrix} -1-2i & 2-i & 1+2i \\ 4-2i & 2-i & 6+2i \\ 1+2i & 3+i & 4-2i \end{pmatrix}.$$

Now, we desirable to obtain again these square root matrices specializing to the square root matrices $R^{n/2}$ from $F^{(n,2)}(R)\equiv c_0I+c_1R+c_2R^2$, where I denotes the 3×3 identity matrix, because we want to classify all possible $F^{(n,2)}(R)$ and to identify any distinguished the complex Fibonacci and Lucas equalities.

Lemma 2.2. The following two cases are valid for the matrix functions $F^{(n,2)}_{(k_1,k_2,k_3)}(R)$, $k_i\in\{0,1\}$ as

$$F^{(n,2)}_{(0,0,0)}(R)=F_{\frac{n}{2}-1}F_{\frac{n}{2}}R^2+F_{\frac{n}{2}-2}F_{\frac{n}{2}}R-F_{\frac{n}{2}-2}F_{\frac{n}{2}-1}I \quad , \quad (10)$$

$$F^{(n,2)}_{(1,0,0)}(R)=\frac{1}{5}\left(L_{\frac{n}{2}-1}L_{\frac{n}{2}}R^2+L_{\frac{n}{2}-2}L_{\frac{n}{2}}R-L_{\frac{n}{2}-2}L_{\frac{n}{2}-1}I\right). \quad (11)$$

Proof. Since the matrix R has distinct eigenvalues $\lambda_i\in\{\alpha^2,\ \beta^2,\ \alpha\beta\}$, coefficients c_i, $i=1,2,3$ of the polynomial representation $F^{(n,2)}(R)\equiv c_0I+c_1R+c_2R^2$ are given by the solution of the system

$$F^{(n,2)}(\lambda_i)=c_0+c_1\lambda_i+c_2\lambda_i^2,\ i=1,2,3.$$

In fact, three scalars c_0,c_1,c_2 for all branches $(k_1,k_2,k_3)\in\{0,1\}^3$ are given as

$$c_0 = \frac{1}{5}\left[(\alpha\beta)^{\frac{n}{2}} e^{nk_1\pi i} - \alpha^{-3}\left(\alpha^2\right)^{\frac{n}{2}} e^{nk_2\pi i} - \beta^{-3}\left(\beta^2\right)^{\frac{n}{2}} e^{nk_3\pi i}\right],$$
$$c_1 = \frac{1}{5}\left[3(\alpha\beta)^{\frac{n}{2}} e^{nk_1\pi i} + \alpha^{-2}\left(\alpha^2\right)^{\frac{n}{2}} e^{nk_2\pi i} + \beta^{-2}\left(\beta^2\right)^{\frac{n}{2}} e^{nk_3\pi i}\right], \quad (12)$$
$$c_2 = \frac{1}{5}\left[(\alpha\beta)^{\frac{n}{2}} e^{nk_1\pi i} + \alpha^{-1}\left(\alpha^2\right)^{\frac{n}{2}} e^{nk_2\pi i} + \beta^{-1}\left(\beta^2\right)^{\frac{n}{2}} e^{nk_3\pi i}\right].$$

From (12), a square root of the matrix R is obtained for the values $(k_1,k_2,k_3)=(0,0,0)$ with

$$c_0 = -\frac{\left(\alpha^{\frac{n}{2}-2} - \beta^{\frac{n}{2}-2}\right)\left(\alpha^{\frac{n}{2}-1} - \beta^{\frac{n}{2}-1}\right)}{5},$$

$$c_1 = \frac{\left(\alpha^{\frac{n}{2}-2} - \beta^{\frac{n}{2}-2}\right)\left(\alpha^{\frac{n}{2}} - \beta^{\frac{n}{2}}\right)}{5},$$

$$c_2 = \frac{\left(\alpha^{\frac{n}{2}-1} - \beta^{\frac{n}{2}-1}\right)\left(\alpha^{\frac{n}{2}} - \beta^{\frac{n}{2}}\right)}{5}.$$

Other particular cases for all possible square roots emerge for other choices of the $(k_1,k_2,k_3)\in\{0,1\}^3$ values. For simplicity's sake, we omit the details which will appear in a similar argument below. ■

Now, let $F_{\frac{n}{2}}$ denotes the complex Fibonacci number in branches $F^{(n,2)}_{(0,0,0)}(R)$, and $L_{\frac{n}{2}}$ denotes the complex Lucas number in branches $F^{(n,2)}_{(1,0,0)}(R)$. By equating corresponding elements in matrix equalities (9) and (12) for appropriate $(k_1,k_2,k_3)\in\{0,1\}^3$ values, we have obtained a numbers of identities.

Corollary 2.3. Let us consider that equating of the matrices (7) and (10), then the following identities are obtained:

$i)\ F^2_{\frac{n}{2}-1} = F_{\frac{n}{2}-1}\left(F_{\frac{n}{2}} - F_{\frac{n}{2}-2}\right),$

$ii)\ F^2_{\frac{n}{2}} = \left(F_{\frac{n}{2}-1} + F_{\frac{n}{2}-2}\right)F_{\frac{n}{2}},$

$iii)\ F_{\frac{n}{2}+1} = 2F_{\frac{n}{2}-1} + F_{\frac{n}{2}-2},$

$iv)\ F^2_{\frac{n}{2}+1} = 4F_{\frac{n}{2}-1}F_{\frac{n}{2}} + F_{\frac{n}{2}-2}F_{\frac{n}{2}} - F_{\frac{n}{2}-2}F_{\frac{n}{2}-1},$

$v)\ F^2_{\frac{n}{2}} + F_{\frac{n}{2}-1}F_{\frac{n}{2}+1} = 3F_{\frac{n}{2}-1}F_{\frac{n}{2}} + F_{\frac{n}{2}-2}F_{\frac{n}{2}} - F_{\frac{n}{2}-2}F_{\frac{n}{2}-1}$

By equating of the matrices (8) and (11), we have

$i)\ L^2_{\frac{n}{2}-1} = L_{\frac{1}{2}n}L_{\frac{1}{2}n-1} - L_{\frac{1}{2}n-1}L_{\frac{1}{2}n-2},$

$ii)\ L_{\frac{n}{2}}L_{\frac{n}{2}+1} = 2L_{\frac{1}{2}n}L_{\frac{1}{2}n-1} + L_{\frac{1}{2}n}L_{\frac{1}{2}n-2},$

$iii)\ L^2_{\frac{n}{2}} = L_{\frac{1}{2}n}L_{\frac{1}{2}n-1} + L_{\frac{1}{2}n}L_{\frac{1}{2}n-2},$

$iv)\ L^2_{\frac{n}{2}} + L_{\frac{n}{2}-1}L_{\frac{n}{2}+1} = 3L_{\frac{1}{2}n}L_{\frac{1}{2}n-1} + L_{\frac{1}{2}n}L_{\frac{1}{2}n-2} - L_{\frac{1}{2}n-1}L_{\frac{1}{2}n-2},$

$v)\ L^2_{\frac{n}{2}+1} = 4L_{\frac{1}{2}n}L_{\frac{1}{2}n-1} + L_{\frac{1}{2}n}L_{\frac{1}{2}n-2} - L_{\frac{1}{2}n-1}L_{\frac{1}{2}n-2}.$

By the definition, an alternative way to obtain a square root of the matrix R is to solve the matrix equation $F^{(n,2)}(R)\times F^{(n,2)}(R)=R^n$, that is, the square roots of the 3×3 matrix R^n are those 3×3 matrices $F^{(n,2)}(R)$. Then, for each square matrix $F^{(n,2)}_{(k_1,k_2,k_3)}(R)$ in given Theorem 2.1., this definition states that $F^{(n,2)}_{(k_1,k_2,k_3)}(R)\times F^{(n,2)}_{(k_1,k_2,k_3)}(R)=R^n$. Therefore, by equating corresponding elements for the matrix $F^{(n,2)}_{(0,0,0)}(R)\times F^{(n,2)}_{(0,0,0)}(R)=R^n$, we achieve some complex Fibonacci identities, and from matrix equation $F^{(n,2)}_{(1,0,0)}(R)\times F^{(n,2)}_{(1,0,0)}(R)=R^n$, some complex Lucas identities are achieved;

Corollary 2.4. The following identities are valid:

$$i)\ F^2_{n+1} = F^4_{\frac{n}{2}} + 2F^2_{\frac{n}{2}}F^2_{\frac{n}{2}+1} + F^4_{\frac{n}{2}+1} = \left(F^2_{\frac{n}{2}} + F^2_{\frac{n}{2}+1}\right)^2,$$

$$ii)\ F_nF_{n+1} = F_{\frac{n}{2}}F^3_{\frac{n}{2}+1} + F^3_{\frac{n}{2}}F_{\frac{n}{2}-1} + \left(F^2_{\frac{n}{2}} + F_{\frac{n}{2}-1}F_{\frac{n}{2}+1}\right)F_{\frac{n}{2}}F_{\frac{n}{2}+1},$$

$$iii)\ F^2_{n+1} - F_{n-1}F_n = \left(F^2_{\frac{n}{2}} + F_{\frac{n}{2}-1}F_{\frac{n}{2}+1}\right)^2 + 2F^2_{\frac{n}{2}}F^2_{\frac{n}{2}-1} + 2F^2_{\frac{n}{2}}F^2_{\frac{n}{2}+1},$$

$$iv)\ 25F^2_{n+1} = L^4_{\frac{n}{2}} + 2L^2_{\frac{n}{2}}L^2_{\frac{n}{2}+1} + L^4_{\frac{n}{2}+1} = \left(L^2_{\frac{n}{2}} + L^2_{\frac{n}{2}+1}\right)^2,$$

$$v)\ 25F_nF_{n+1} = L_{\frac{n}{2}}L^3_{\frac{n}{2}+1} + L^3_{\frac{n}{2}}L_{\frac{n}{2}-1} + \left(L^2_{\frac{n}{2}} + L_{\frac{n}{2}-1}L_{\frac{n}{2}+1}\right)L_{\frac{n}{2}}L_{\frac{n}{2}+1},$$

$$vi)\ 25\left(F^2_{n+1} - F_{n-1}F_n\right) = \left(L^2_{\frac{n}{2}} + L_{\frac{n}{2}-1}L_{\frac{n}{2}+1}\right)^2 + 2L^2_{\frac{n}{2}}L^2_{\frac{n}{2}-1} + 2L^2_{\frac{n}{2}}L^2_{\frac{n}{2}+1}.$$

Also, let us consider the matrix functions defined as

$$\mathcal{F}_I^{(n,2)}(R) = \begin{cases} F^{(t)}(R), & if\ n = 2t \\ F^{(n,2)}_{(0,0,0)}(R), & if\ n\ is\ odd \end{cases},$$

and

$$\mathcal{F}_{II}^{(n,2)}(R) = \begin{cases} F^{(t)}(R), & if\ n = 2t \\ F^{(n,2)}_{(1,0,0)}(R), & if\ n\ is\ odd \end{cases}$$

For any integer number $n,m \in Z$, by performing the Jordan form associated to the $F^{(n,2)}_{(k_1,k_2,k_3)}(R)$ and $F^{(m,2)}_{(k_1,k_2,k_3)}(R)$ matrices, we may write

$$\mathcal{F}_j^{(n,2)}(R) \times \mathcal{F}_j^{(m,2)}(R) = \mathcal{F}_j^{(n+m,2)}(R),\ j = I, II.$$

We have to consider two pertinent cases;

- If n is odd and $m = 2t$ is even, then $m+n$ is odd, thus

$$F^{(n,2)}_{(0,0,0)}(R) \times F^{(t)}(R) = F^{(n+m,2)}_{(0,0,0)}(R),$$
$$F^{(n,2)}_{(1,0,0)}(R) \times F^{(t)}(R) = F^{(n+m,2)}_{(1,0,0)}(R).$$

- If n and m are odd, then $m+n$ is even, thus

$$F^{(n,2)}_{(0,0,0)}(R) \times F^{(m,2)}_{(0,0,0)}(R) = F^{(n+m,2)}_{(0,0,0)}(R),$$
$$F^{(n,2)}_{(1,0,0)}(R) \times F^{(m,2)}_{(0,0,0)}(R) = F^{(n+m,2)}_{(1,0,0)}(R).$$

- If n and m are even, then $m+n$ is even, thus some existing results in literature occur (Koshy, 2001).

Finally, we underline that if the matrix function defined in Theorem 2.1. and Lemma 2.2 are evaluated by substituting $F^{(n,2)}_{(k_1,k_2,k_3)}(R)$, $k_i \in \{0,1\}$, many different equations can be obtained. But, for simplicity, we omit the details.

3. Properties Of n/s Roots Of The Fibonacci Matrix Of Order 3×3

Now, let us think the scalar complex function $F^{(n,s)}(z) \equiv z^{n/s} = f_k^{(n,s)}(z)$, $\quad k \in \{0,1,...,s-1\}$, where $(n,s) \in Z - \{0\} \times N^+$, such that $\frac{n}{s}$ is an irreducible fraction, i.e., $\gcd(n,s) = 1$. As it are mentioned before, the function $F^{(n,s)}(z)$ is s-valued function,

$$f_k^{(n,s)}(z) = |z|^{n/s} \exp\left[i\tfrac{n}{s}(\arg(z) + 2k\pi)\right],$$
$$k \in \{0,1,...,s-1\}$$

these branches can be characterized as

$$f_k^{(n,s)}(z) = \exp\left(\frac{2nk\pi}{s}i\right)z^{n/s},$$
$$f_0^{(n,s)}(z) = z^{n/s} = |z|^{n/s} \exp\left[\frac{n}{s}\arg(z)i\right].$$

There exist s^3 primary matrix functions $F^{(n,s)}(R) \equiv R^{n/s}$, that may be determined for $\gcd(n,s) = 1$ and $\lambda_i \in \{\alpha\beta,\ \alpha^2,\ \beta^2\}$ by the expression

$$F_{(k_1,k_2,k_3)}^{(n,s)}(R)=\sum_{i=1}^{3} f_{k_i}^{(n,s)}(\lambda_i)\prod_{\substack{j=1\\ i\neq j}}^{3}\frac{R-\lambda_j I}{\lambda_i-\lambda_j},\ k_i\in\{0,1,\dots,s-1\}\cdot$$

The matrix functions $F_{(k_1,k_2,k_3)}^{(n,s)}(R)=\left[h_{ij}\right]_{3\times 3}$, $k_i\in\{0,1,\dots,s-1\}$ are rewritten via;

$$\left[h_{ij}\right]=\begin{cases} h_{11}=\frac{1}{5}\left(2(\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}+\beta^2\left(\alpha^2\right)^{\frac{n}{s}}e^{\frac{2nk_2\pi i}{s}}+\alpha^2\left(\beta^2\right)^{\frac{n}{s}}e^{\frac{2nk_3\pi i}{s}}\right),\\ h_{22}=\frac{1}{5}\left((\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}+2\left(\alpha^2\right)^{\frac{n}{s}}e^{\frac{2nk_2\pi i}{s}}+2\left(\beta^2\right)^{\frac{n}{s}}e^{\frac{2nk_3\pi i}{s}}\right),\\ h_{33}=\frac{1}{5}\left(2(\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}+\alpha^2\left(\alpha^2\right)^{\frac{n}{s}}e^{\frac{2nk_2\pi i}{s}}+\beta^2\left(\beta^2\right)^{\frac{n}{s}}e^{\frac{2nk_3\pi i}{s}}\right),\\ 2h_{12}=h_{21}=\frac{2}{5}\left((\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}-\beta\left(\alpha^2\right)^{\frac{n}{s}}e^{\frac{2nk_2\pi i}{s}}-\alpha\left(\beta^2\right)^{\frac{n}{s}}e^{\frac{2nk_3\pi i}{s}}\right),\\ h_{13}=h_{31}=\frac{1}{5}\left(-2(\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}+\left(\alpha^2\right)^{\frac{n}{s}}e^{\frac{2nk_2\pi i}{s}}+\left(\beta^2\right)^{\frac{n}{s}}e^{\frac{2nk_3\pi i}{s}}\right),\\ h_{23}=2h_{32}=\frac{2}{5}\left(-(\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}+\alpha\left(\alpha^2\right)^{\frac{n}{s}}e^{\frac{2nk_2\pi i}{s}}+\beta\left(\beta^2\right)^{\frac{n}{s}}e^{\frac{2nk_3\pi i}{s}}\right), \end{cases}. \tag{13}$$

The matrix functions $F_{(k_1,k_2,k_3)}^{(n,s)}(R)$ can be calculated for the some special cases, such that $k_i\in\{0,1,\dots,s-1\}$, for example the $k_1=k_2=k_3=k$, $(0\le k\le s-1)$ values are calculated in the equation (13), the matrix functions $F_{(k,k,k)}^{(n,s)}(R)=R^{n/s}=\left[k_{ij}\right]_{3\times 3}$ are matrices involving the Fibonacci numbers with rational indices;

$$\left[k_{ij}\right]=\begin{cases} k_{11}=e^{\frac{2kn\pi i}{s}}\left(\frac{\alpha^{\frac{n}{s}-1}-\beta^{\frac{n}{s}-1}}{\sqrt{5}}\right)^2=e^{\frac{2kn\pi i}{s}}F_{\frac{n}{s}-1}^2,\\ k_{22}=e^{\frac{2kn\pi i}{s}}\frac{2\alpha^{\frac{2n}{s}}+q^{\frac{n}{s}}+2\beta^{\frac{2n}{s}}}{5}=e^{\frac{2kn\pi i}{s}}\left(F_{\frac{n}{s}}^2+F_{\frac{n}{s}-1}F_{\frac{n}{s}+1}\right),\\ k_{33}=e^{\frac{2kn\pi i}{s}}\left(\frac{\alpha^{\frac{n}{s}+1}-\beta^{\frac{n}{s}+1}}{\sqrt{5}}\right)^2=e^{\frac{2kn\pi i}{s}}F_{\frac{n}{s}+1}^2,\\ 2k_{12}=k_{21}=\frac{2}{5}e^{\frac{2kn\pi i}{s}}\left(\alpha^{\frac{2n}{s}-1}+q^{\frac{n}{s}}-\beta^{\frac{2n}{s}-1}\right)=2e^{\frac{2kn\pi i}{s}}F_{\frac{n}{s}-1}F_{\frac{n}{s}},\\ k_{13}=k_{31}=e^{\frac{2kn\pi i}{s}}\left(\frac{\alpha^{\frac{n}{s}}-\beta^{\frac{n}{s}}}{\sqrt{5}}\right)^2=e^{\frac{2kn\pi i}{s}}F_{\frac{n}{s}}^2,\\ k_{23}=2k_{32}=\frac{2}{5}e^{\frac{2kn\pi i}{s}}\left(\alpha^{\frac{2p}{q}+1}+2q^{\frac{n}{s}}+\beta^{\frac{2p}{q}+1}\right)=2e^{\frac{2kn\pi i}{s}}F_{\frac{n}{s}}F_{\frac{n}{s}+1}, \end{cases}. \tag{14}$$

As it was stated before, the matrix functions $f^{(n,s)}(R)\equiv R^{n/s}$ are the primary matrix functions. Hence, there a unique polynomial q of degree 2 such that $F^{(n,s)}(R)\equiv q(R)$ (Gantmacher, 1960; Higham, 2008). Therefore, there exist three scalars c_0,c_1,c_2 such that, $F^{(n,s)}(R)=c_0I_3+c_1R+c_2R^2$, where the scalars c_0, c_1 and c_2 are given by

$$c_0=\frac{1}{5}\left((\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}+\beta\left(\alpha^2\right)^{\frac{n}{s}-1}e^{\frac{2nk_2\pi i}{s}}+\alpha\left(\beta^2\right)^{\frac{n}{s}-1}e^{\frac{2nk_3\pi i}{s}}\right),$$

$$c_1=-\frac{1}{5}\left(3(\alpha\beta)^{\frac{n}{s}}e^{\frac{2nk_1\pi i}{s}}-\left(\alpha^2\right)^{\frac{n}{s}-1}e^{\frac{2nk_2\pi i}{s}}-\left(\beta^2\right)^{\frac{n}{s}-1}e^{\frac{2nk_3\pi i}{s}}\right), \tag{15}$$

$$c_2 = \tfrac{1}{5}\left(\left(\alpha\beta\right)^{\frac{n}{s}} e^{\frac{2nk_1\pi i}{s}} - \beta\left(\alpha^2\right)^{\frac{n}{s}} e^{\frac{2nk_2\pi i}{s}} - \alpha\left(\beta^2\right)^{\frac{n}{s}} e^{\frac{2nk_3\pi i}{s}}\right).$$

The $k_1 = k_2 = k_3 = k$ values are rewritten in the equation (15), it follows that

$$c_0 = -e^{\frac{2nk\pi i}{s}} F_{\frac{n}{s}-1} F_{\frac{n}{s}-2},\ c_1 = e^{\frac{2nk\pi i}{s}} F_{\frac{n}{s}-2} F_{\frac{n}{s}},\ c_2 = e^{\frac{2nk\pi i}{s}} F_{\frac{n}{s}} F_{\frac{n}{s}-1}.$$

The matrix functions $F^{(n,s)}_{(k,k,k)}(R) = \left[k_{ij}\right]_{3\times 3}$ are again obtained matrix recurrence equations with

$$F^{(n,s)}_{(k,k,k)}(R) = e^{\frac{2nk\pi i}{s}}\left(F_{\frac{n}{s}} F_{\frac{n}{s}-1} R^2 + F_{\frac{n}{s}-2} F_{\frac{n}{s}} R - F_{\frac{n}{s}-1} F_{\frac{n}{s}-2} I_3\right)$$

$$\left[k_{ij}\right] = \begin{cases} k_{11} = e^{\frac{2kn\pi i}{s}} F_{\frac{n}{s}} F_{\frac{n}{s}-1} - F_{\frac{n}{s}-1} F_{\frac{n}{s}-2}, \\ k_{22} = e^{\frac{2kn\pi i}{s}} 3F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}} - F_{\frac{n}{s}-1} F_{\frac{n}{s}-2}, \\ k_{33} = e^{\frac{2kn\pi i}{s}} 4F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}} - F_{\frac{n}{s}-1} F_{\frac{n}{s}-2}, \\ 2k_{12} = k_{21} = 2e^{\frac{2kn\pi i}{s}} 2F_{\frac{n}{s}-1} F_{\frac{n}{s}}, \\ k_{13} = k_{31} = e^{\frac{2kn\pi i}{s}} F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}}, \\ k_{23} = 2k_{32} = 2e^{\frac{2kn\pi i}{s}} \left(2F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}}\right), \end{cases} \quad (16)$$

The following identities are immediately found by equating corresponding elements of matrices (14) and (16);

$i)$ $F^2_{\frac{n}{s}-1} = F_{\frac{n}{s}} F_{\frac{n}{s}-1} - F_{\frac{n}{s}-1} F_{\frac{n}{s}-2}$,

$ii)$ $F^2_{\frac{n}{s}} = F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}}$,

$iii)$ $F^2_{\frac{n}{s}} + F_{\frac{n}{s}-1} F_{\frac{n}{s}+1} = 3F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}} - F_{\frac{n}{s}-1} F_{\frac{n}{s}-2}$,

$iv)$ $F_{\frac{n}{s}} F_{\frac{n}{s}+1} = 2F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}}$,

$v)$ $F^2_{\frac{n}{s}+1} = 4F_{\frac{n}{s}} F_{\frac{n}{s}-1} + F_{\frac{n}{s}-2} F_{\frac{n}{s}} - F_{\frac{n}{s}-1} F_{\frac{n}{s}-2}$.

The matrix functions in the $F^{(n,s)}_{(k_1,k_2,k_3)}(R)$ gives a number of equations in terms of the Fibonacci and Lucas numbers with rational subscripts by doing similar calculation evaluated for the matrix $F^{(n,2)}_{(k_1,k_2,k_3)}(R)$.

4. Conclusion

The matrix functions

$$F^{(n,s)}(R) = c_0 I_3 + c_1 R + c_2 R^2$$

are obtained by the theory of matrix functions. The functions gives all root matrices of the Fibonacci matrix. Some fundamental properties are acquired by matrix methods made of using these matrices. It is seen that identities for the Fibonacci and Lucas numbers with rational subscripts hold for analogous identities of the classical Fibonacci and Lucas numbers.

5. References

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