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Research Article

On Telescopic Numerical Semigroup Families With Embedding Dimension 3

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Abstract
In this study, the set of all telescopic numerical semigroups families with embedding dimension three is obtained for some fixed multiplicity by some parameters. Also, some invariants of these families are calculated in term of their generators.

Keywords: Numerical semigroups, Frobenius number, Genus.

Gömme Boyutu 3 Olan Teleskopik Sayısal Yarıgrup Aileleri Üzerine

Öz
Bu çalışmada, bazı parametreli belirli katlılıkla elde gömme boyutu üç olan tüm teleskopik sayısal yarı grup ailelerini elde edeceğiz. Ayrıca bu ailelerin bazı değişmezlerini onların üreteçleri yardımı ile hesaplayacağız.

Anahtar Kelimeler: Sayısal yarıgruplar, Frobenius sayısı, Cins.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote, the sets of integers and non-negative integers, respectively. The subset S of $\mathbb{N}$ is called a numerical semigroup if it is closed under addition and $0 \in S$. We have $\mathbb{N} \setminus S$ with finite elements if and only if $gcd(S) = 1$ (gcd stands for greatest common divisor).

Let S be a numerical semigroup and let $A = \{u_1, u_2, \dots, u_e\} \subset \mathbb{N}$ such that $gcd(u_1, u_2, \dots, u_e) = 1$ and $e \geq 1$. The set A is a system of generators of S if there are $n_1, n_2, \dots, n_e \in \mathbb{N}$ for all $s \in S$ and $s = \sum_{i=1}^{p} n_i u_i$. In this case, we write $S = \langle A \rangle$. Let $u_1 < u_2 < \dots < u_e$ and $e \geq 1$. If there are no elements $n_1, \dots, n_{j-1}$ in form $u_j = \sum_{i=1}^{j-1} n_i u_i$ for all $j = 2, \dots, e$, then the set $A = \{u_1, u_2, \dots, u_e\}$ is a minimal generating system for S. Also, it is known that each numerical semigroup has a finite system of generators, and the set of minimal generators is unique.

Let $A = \{u_1, u_2, \dots, u_e\}$ be the set of minimal generators of the numerical semigroup S. Then the number u_1 is called multiplicity of S, denoted by $\mu(S)$, the cardinality of A is called embedding dimension of S, denoted by $e(S)$. In general, we have $e(S) \leq \mu(S)$ when $S \neq \mathbb{N}$. For a numerical semigroup S, the Frobenius number of S is the largest integer that is not in S, and denoted by $F(S)$. The cardinallity number of the set $\{0,1,2, \dots, F(S)\} \cap S$ is called determiner number of S, and denoted by $n(S)$. If $S = \langle u_1, u_2, \dots, u_e \rangle$ and $S \neq \mathbb{N}$, then we can write

$$S = \{0 = s_0, s_1, \dots, s_n = F(S) + 1, \rightarrow \cdots\}$$

where " $\rightarrow$ " means that every integer greater than s_n belongs to S, for $i = 1,2, \dots, n$, $s_i < s_{i+1}$ and $n = n(S)$.

If $a \in \mathbb{N}$ and $a \notin S$, then a is called gap of S. The set of all gaps of S is denoted by $G(S)$. Its cardinality is the genus of S, and denoted

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by $g(S)$. $F(S)$ is the largest gap of S, and $g(S) + n(S) = F(S) + 1$ when $S \neq \mathbb{N}$.

Let $(u_1, u_2 \dots, u_e)$ be a sequence of positive integers such that the greatest common divisor is 1. Define $d_i = gcd(u_1, u_2, \dots, u_i)$ and $A_i = \{u_1/d_i, u_2/d_i, \dots, u_i/d_i\}$ for $i = 1, \dots, e$. Let $d_0 = 0$. Let S_i be the semigroup generated by A_i. If $u_i/d_i \in S_{i-1}$ for $i = 2, \dots, e$, we call the sequence $(u_1, u_2 \dots, u_e)$ telescopic. The semigroup is generated by a telescopic sequence is called telescopic (Kirfel and Pellikaan, 1995).

In particular for $e = 3$, we have $S = \langle u_1, u_2, u_3 \rangle$. If $u_3 \in \langle u_1/d, u_2/d \rangle$, then S is called triply-generated telescopic semigroup, where $d = gcd(u_1, u_2)$ (Matthews, 2001).

A numerical semigroup is called irreducible if it can not be expressed as the intersection of two numerical semigroups properly containing it. It is known that telescopic numerical semigroups are irreducible.

Kirfel and Pellikaan (1995) showed that a proper subclass of complete intersection numerical semigroups is the class of telescopic numerical semigroups, and they worked on the Feng-Rao distance. Garcia-Sanchez, Heredia and Leamer (2016) have established a relationship between the second Feng-Rao number and the multiplicity of telescopic numerical semigroups. Nowadays, telescopic numerical semigroups continue to be updated with applications in algebraic error correcting codes. Ilhan (2006) showed the class of triply-generated telescopic numerical semigroups, and again Ilhan and Herzinger (2009) investigated principal ideals of this class. Deveci and Akuzum (2014) obtained some rules for the orders of the cyclic groups and semigroups. And in other study, they obtained the cyclic groups and the semigroups by using the generating some matrices, they gave their miscellaneous properties. (Deveci, Aküzüm and Campbell, 2018).

In this study, we will give some classes of triply generated telescopic semigroups for multiplicity four and six. Also, we will obtain formulas for Frobenius number, genus and determiner number.

2. Result and Discussion

In this section, we obtain the set of all telescopic numerical semigroups families with embedding dimension three and multiplicity four and six. We find formulas for some notable elements for families of numerical semigroups.

2.1. Proposition Let m and n be positive integer. Let $gcd(m,n) = 1$.

1. $F(\langle m, n \rangle) = mn - m - n$,
2. $g(\langle m, n \rangle) = \frac{mn-m-n+1}{2}$ (Sylvester, 1884).

2.2. Proposition Let S be a numerical semigroup with minimal system of generator $\{u_1, u_2, \dots, u_e\}$. Let $d = gcd(u_1, u_2, \dots, u_{e-1})$ and set $T = \langle u_1/d, \dots, u_{e-1}/d, u_e \rangle$. Then

1. $F(S) = dF(T) + (d-1)n_e$,
2. $g(S) = dg(T) + \frac{(d-1)(n_e-1)}{2}$ (Johnson, 1960).

2.3. Theorem The following conditions are equivalent:

1. S is an irreducible numerical semigroup, $F(S)$ is odd, $m(S) = 4$ and $e(S) = 3$,

2. S is a numerical semigroup generated by $\{4, 2a, a+2b\}$ with $b \in \mathbb{N} \backslash \{0\}$ and a is an odd integer greater than or equal to 3 (Rosales and Branco, 2003).

2.4. Theorem Let S be a numerical semigroup with embedding dimension three and multiplicity four. The numerical semigroup S is telescopic if and only if S is a member of the family

$\Gamma = \{\langle 4, 4k+2, x\rangle : k \in \mathbb{Z}^+, x \in \mathbb{Z}_o \ and \ x > 4k+2 \ \}$
(where $\mathbb{Z}^+ \ and \ \mathbb{Z}_o$ denote the set of positive integers and the set of positive odd integers, respectively).

Proof. *(Necessity)* If the numerical semigroup S is telescopic, then S is an irreducible numerical semigroup and S is generated by $\{4, 2a, a+2b\}$, from 2.3. Theorem. If we take $a = 2k+1$ and $b = \frac{x-2k-1}{2}$, then $S \in \Gamma$.

(Sufficiency) Let $S \in \Gamma$ be a numerical semigroups. Then $gcd(4, 4k+2) = 2$ and we write

$$F(\langle 4/2\, , 4k+2/2\rangle) = F(\langle 2, 2k+1\rangle) = 2k-1 < x$$

by 2.1. Proposition. In this case we have $x \in \langle 2, 2k+1\rangle$. Therefore S is telescopic.

2.5. Proposition If S is a telescopic numerical semigroup member of the family

$\Gamma = \{\langle 4, 4k+2, x\rangle : k \in \mathbb{Z}^+, x \in \mathbb{Z}_o \ and \ x > 4k+2 \ \}$,
then we have

i. $F(S) = 4k + x - 2$,
ii. $g(S) = 2k + \frac{x-1}{2}$,
iii. $n(S) = 2k + \frac{x-1}{2}$.

Proof: Let S be a telescopic numerical semigroup member of the family

$\Gamma = \{\langle 4, 4k+2, x\rangle : k \in \mathbb{Z}^+, x \in \mathbb{Z}_o \ and \ x > 4k+2 \ \}$.
Then $2 = gcd(4, 4k+2)$ and we obtain that $T = \langle 4/2\, , 4k+2/2, x\rangle = \langle 2, 2k+1\rangle$.
Thus, the following equalities hold by 2.1. Proposition and 2.2. Proposition:

i. $F(S) = 2(2k-1) + (2-1)x = 4k + x - 2$,
ii. $g(S) = 2k + \frac{(2-1)(x-1)}{2} = 2k + \frac{x-1}{2}$
and from $g(S) + n(S) = F(S) + 1$

$$n(S) = (4k + x - 2) + 1 - \left(2k + \frac{x-1}{2}\right) = 2k + \frac{x-1}{2}.$$

2.6. Example Let $S = \langle 4, 18, 21\rangle = \{0, 4, 8, 12, 16, 18, 20, 21, 22, 24, 25, 26, 28, 29, 30, 32, 33, 34, 36, \rightarrow \cdots\}$. S is a telescopic numerical semigroup since $gcd(4, 18) = 2$ and $21 \in \langle 2, 9\rangle$. For this numerical semigroup S;

$F(S) = 35$,

$H(S) = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 19, 23, 27, 31, 35\}$,

$g(S) = 18$,

$n(S) = \#(\{0, 1, 2, \dots, F(S)\} \cap S) = \#(\{0, 4, 8, 16, 18, 20, 21, 22, 24, 25, 26, 28, 29, 30, 32, 33, 34\}) = 18$

(Where $\#(A)$ is the number of elements in the set A).

In other way, if we take $k = 4$ and $x = 21$ in 2.4. Theorem, then $S = \langle 4, 18, 21\rangle \in \Gamma$. We can calculate the following values without seeing the open state of the numerical semigroup S by 2.5. Proposition,

$$F(S) = 4k + x - 1 = 4 \cdot 4 + 21 - 1 = 35,$$
$$g(S) = 2k + \frac{x-1}{2} = 2 \cdot 4 + \frac{21-1}{2} = 18,$$
$$n(S) = 2k + \frac{x-1}{2} = 2 \cdot 4 + \frac{21-1}{2} = 18.$$

2.7. Theorem Let S be a numerical semigroup with embedding dimension three and multiplicity six. The numerical semigroup S is telescopic if and only if S is a member of the following families:

i. $\Delta = \{\langle 6,6k+2,x\rangle : k \in \mathbb{Z}^+, x \in \mathbb{Z}_o \ and \ x > 6k+2\}$,
ii. $\Theta = \{\langle 6,6k+3,t\rangle : t \in \mathbb{Z}^+, t > 6k+3 \ and \ 3 \nmid t\}$,
iii. $\Lambda = \{\langle 6,6k+4,y\rangle : k \in \mathbb{Z}^+ \ and \ y \in \mathbb{Z}_o \ and \ y > 6k+4\}$.

Proof: *(Necessity)* Let $S = \langle 6, A, B\rangle$ be a telescopic numerical semigroup with multiplicity six and embedding dimension three. Then we can write $gcd(6, A) = d$ and $B \in \langle 6/d, A/d\rangle$ by definition of telescopic numerical semigroups. In this case $d = 1$ or $d = 2$ or $d = 3$.

a) If $d = 1$, then $B \in \langle 6/d, A/d\rangle$ contradicts embedding dimension three. Thus $d = 2$ or $d = 3$.
b) If $d = 2$, then $A = 6k + 2$ or $A = 6k + 4$ for $k \in \mathbb{Z}^+$.
If $A = 6k + 2$ for $k \in \mathbb{Z}^+$, then $6 < A < B$ and $gcd(6, A, B) = 1$ by the definition of the minimal generators. So, $B > 6k + 2$ and $gcd(2, B) = 1$. In this case, B must be an odd integer. Under these conditions, there is only one family of telescopic numerical semigroups that can be written as $\Delta = \{\langle 6,6k+2,x\rangle : k \in \mathbb{Z}^+, x \in \mathbb{Z}_o \ and \ x > 6k+2\}$. Thus, we obtain $S \in \Delta$.

If $A = 6k + 4$ for $k \in \mathbb{Z}^+$, then $6 < A < B$ and $gcd(6, A, B) = 1$ by the definition of the minimal generators. So, $B > 6k + 4$ and $gcd(2, B) = 1$. In this case, B must be an odd integer. Under these conditions, there is only one family of telescopic numerical semigroups that can be written as $\Lambda = \{\langle 6,6k+4,y\rangle : k \in \mathbb{Z}^+, y \in \mathbb{Z}_o \ and \ y > 6k+4\}$. Hence, $S \in \Lambda$.
c) If $d = 3$, then $A = 6k + 3$ for $k \in \mathbb{Z}^+$, then $6 < A < B$ and $gcd(6, A, B) = 1$ by the definition of the minimal generators. Thus, $B > 6k + 3$ and $gcd(3, B) = 1$. In this case, B is a positive integer that is not a multiple of 3 and $B > 6k + 3$. Under these conditions, there is only one family of telescopic numerical semigroup family that can be written as $\Theta = \{\langle 6,6k+3,t\rangle : k, t \in \mathbb{Z}^+, t > 6k+3 \ ve \ 3 \nmid t\}$. Thus, $S \in \Theta$.

(Sufficiency) Let numerical semigroup S be a member of Δ or Θ or Λ.

i. If the numerical semigroup S is a member of Δ, then $gcd(6,6k+2) = 2$ and $\langle 6/2, 6k+2/2\rangle = \langle 3,3k+1\rangle$. $\langle 3,3k+1\rangle$ is a numerical semigroup since $gcd(3,3k+1) = 1$. By 2.1. Proposition, $F(\langle 3,3k+1\rangle) = 6k - 1 < 6k + 2 < x$ and $x \in \langle 3,3k+1\rangle$. Thus, S is a telescopic numerical semigroup.
ii. If the numerical semigroup S is a member of Θ, then $gcd(6,6k+3) = 3$ and $\langle 6/3, 6k+3/3\rangle = \langle 2,2k+1\rangle$. $\langle 2,2k+1\rangle$ is a numerical semigroup since $gcd(2,2k+1) = 1$. By 2.1. Proposition, $F(\langle 2,2k+1\rangle) = 2k - 1 < 6k + 3 < t$ for $k \in \mathbb{Z}^+$ and $t \in \langle 2,2k+1\rangle$. Hence, S is a telescopic numerical semigroup.
iii. If the numerical semigroup S is a member of Λ, then $gcd(6,6k+4) = 2$ and $\langle 6/2, 6k+4/2\rangle = \langle 3,3k+2\rangle$. $\langle 3,3k+2\rangle$ is a numerical semigroup since $gcd(3,3k+2) = 1$. By 2.1. Proposition, $F(\langle 3,3k+2\rangle) = 6k + 1 < 6k + 4 < y$ for $k \in \mathbb{Z}^+$ and $y \in \langle 3,3k+2\rangle$. Therefore, S is a telescopic numerical semigroup.

2.8. Proposition If S is a telescopic numerical semigroup member of the family

$\Delta = \{\langle 6,6k+2,x\rangle : k \in \mathbb{Z}^+, x \in \mathbb{Z}_o \ and \ x > 6k+2\}$, then

i. $F(S) = 12k + x - 2$,
ii. $g(S) = 6k + \frac{(x-1)}{2}$,
iii. $n(S) = 6k + \frac{(x-1)}{2}$.

Proof: Let S be a telescopic numerical semigroup member of the family

$\Delta = \{\langle 6,6k+2,x\rangle : k \in \mathbb{Z}^+, x \in \mathbb{Z}_o \ and \ x > 6k+2\}$.
Then $d = gcd(6,6k+2) = 2$ and $T = \langle 6/2\,, 6k+2/2\,, x\rangle = \langle 3,3k+1\rangle$. In this case, we obtain following equalities by 2.1. Proposition and 2.2. Proposition,

i. $F(S) = 2(6k-1) + (2-1)x$
$= 12k + x - 2$,
ii. $g(S) = 2(3k) + \frac{(2-1)(x-1)}{2}$
$= 6k + \frac{(x-1)}{2}$,
iii. By $g(S) + n(S) = F(S) + 1$,
$n(S) = (12k + x - 2) + 1 - \left(6k + \frac{(x-1)}{2}\right)$
$= 6k + \frac{(x-1)}{2}$.

2.9. Proposition If S is a telescopic numerical semigroup member of the family $\Theta = \{\langle 6,6k+3,t\rangle : t \in \mathbb{Z}^+, \ t > 6k+3 \ and \ 3 \nmid t\}$, then

i. $F(S) = 6k + 2t - 3$,
ii. $g(S) = 3k + t - 1$,
iii. $n(S) = 3k + t - 1$.

Proof: Let S be a telescopic numerical semigroup member of the family $\Theta = \{\langle 6,6k+3,t\rangle : t \in \mathbb{Z}^+, t > 6k+3 \ and \ 3 \nmid t\}$. Then $d = gcd(6,6k+3) = 3$ and $T = \langle 6/3\,, 6k+3/3\,, x\rangle = \langle 2,2k+1\rangle$ since $x \in \langle 2,2k+1\rangle$. Thus, we have following equalities from 2.1. Proposition and 2.2. Proposition:

i. $F(S) = 3(2k-1) + (3-1)t$
$= 6k + 2t - 3$,
ii. $g(S) = 3k + \frac{(3-1)(t-1)}{2}$
$= 3k + t - 1$,
iii. By $g(S) + n(S) = F(S) + 1$,
$n(S) = (6k + 2t - 3) + 1 - (3k + t - 1)$
$= 3k + t - 1$.

2.10. Proposition If S is a telescopic numerical semigroup member of the family $\Lambda = \{\langle 6,6k+4,y\rangle : k \in \mathbb{Z}^+ \ and \ y \in \mathbb{Z}_o \ and \ y > 6k+4\}$, then

i. $F(S) = 12k + y + 2$,
ii. $g(S) = \frac{12k+y+3}{2}$,
iii. $n(S) = \frac{12k+y+3}{2}$.

Proof: Let S be a telescopic numerical semigroup member of the family $\Lambda = \{\langle 6,6k+4,y\rangle : k \in \mathbb{Z}^+ \ and \ y \in \mathbb{Z}_o \ and \ y > 6k+4\}$. Then $d = gcd(6,6k+4) = 2$ and $T = \langle 6/2\,, 6k+4/2\,, y\rangle = \langle 3,3k+2,y\rangle = \langle 3,3k+2\rangle$ since $y \in T$. So, we obtain that following equalities from 2.1. Proposition and 2.2. Proposition:

i. $F(S) = 2(6k+1) + (2-1)y$
$= 12k + y + 2$,
ii. $g(S) = 2(3k+1) + \frac{(2-1)(y-1)}{2}$
$= \frac{12k+y+3}{2}$,
iii. By $g(S) + n(S) = F(S) + 1$,
$n(S) = (12k + y + 2) + 1 - \left(\frac{12k+y+3}{2}\right)$
$= \frac{12k+y+3}{2}$.

2.11. Example Let $S = \langle 6,15,17\rangle = \{0,6,12,15,17,18,21,23,24,27,29,30,32,33$

$34,35,36,38,39,40,41,42,44, \rightarrow \cdots\}$. S is a telescopic numerical semigroup since

$gcd(6,15) = 3$ and $17 \in \langle 3,4 \rangle$. For this numerical semigroup S;

$F(S) = 43$,

$H(S) = \{1,2,3,4,5,7,8,9,10,11,13,14,16,19$

$20,22,25,26,28,31,37,43\}$,

$g(S) = 22$,

$n(S) = \#(\{0,1,2,\dots,F(S)\} \cap S) = \#(\{0,6$
$12,15,17,18,21,23,24,27,29,30,32,33,34,$
$35,36,38,39,40,41,42\}) = 22$.

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