Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2019, 12(1), 489-498
ISSN: 1307-9085, e-ISSN: 2149-4584
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2019, 12(1), 489-498
DOI: 10.18185/erzifbed.508746
Research Article

On the Matrix Representations of Operators on the Classical Sequence Spaces

Erdal BAYRAM

Department of Mathematics, Tekirdağ Namık Kemal University, Tekirdağ, Turkey

Geliş / Received: 05/01/2019, Kabul / Accepted: 08/03/2019

Abctract
The present study provides the necessary and sufficient conditions for the matrix characterizations of L- and M-weakly compact operators which are defined on certain classical sequence spaces as Banach lattices. It is known that these operators may coincide with both weakly compact and compact operators on Banach lattices. Our study offers a different alternative to some known results for the matrix characterizations of compact and weakly compact operators which are presented in terms of L- and M-weakly compactness.

Keywords: Matrix transformation, Weakly compact operator, Compact operator, L-weakly compact operator, M-weakly compact operator.

Klasik Dizi Uzayları Üzerinde Tanımlı Operatörlerin Matris Temsilleri Üzerine

Öz
Sunulan çalışma, Banach örgüsü yapısına sahip bazı klasik dizi uzayları üzerinde tanımlı olan L-zayıf ve M-zayıf kompakt operatörlerin matris temsilleri için gerekli ve yeterli koşullar sağlar. Bu operatör sınıflarının, Banach örgüleri üzerinde tanımlı zayıf kompakt ve kompakt operatörlerle çakışabildiği bilinmektedir. Böylece kompakt ve zayıf kompakt operatörler için bilinen bazı sonuçlar L-zayıf ve M-zayıf kompaktlık açısından farklı bir alternatif olarak sunulmuş oldu.

Anahtar Kelimeler: Matris dönüşümleri, Zayıf kompakt operatör, Kompakt operatör, L-zayıf kompakt operatör, M-zayıf kompakt operatör.

1. Introduction

A bounded linear operator which is defined between classical sequence spaces has an infinite matrix representation. It is hence important to find necessary and sufficient conditions of entries of this matrix representation. There is a huge number of studies in the literature on this subject. Matrix maps of the bounded operators can be found in the survey paper Stieglitz and Tietz (1977). Besides, it is also interesting to characterize the special subclasses of the bounded operators, such as compact operators; see (Sargent, 1966; Djolovic I. 2003; Jarrah and Malkowsky, 2003). A good tool used for matrix representations of compact operators is the Hausdorff measures of noncompactness. Further results concerning compact operators and matrix representations can be found in Djolovic (2003), Djolovic and Malkowsky, (2008), Malkowsky (2013), İlkhan and Kara (2018), as well as many other studies. In our study, the main objective is to reveal the matrix characterizations for M- and L-weakly compact operators, which are subclasses of weakly compact operators, and are defined between certain classical real sequence spaces. Since we only use characteristics of these operators, our approach here is different.

A sequence space will be used to mean a linear subspace of the space $\mathbb{R}^{\mathbb{N}}$ and the classical sequence space ℓ_∞, c, c_0 and φ consists of all bounded, convergent, null and finitely non-zero sequences, respectively. For $1 < p < \infty$, ℓ_p denotes the set $\{(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$. For $k \in \mathbb{N}$, e_k denotes the sequence $(0,\ldots 0,1,0,\ldots)$ with 1 in the k th place.

*Sorumlu Yazar: ebayram@nku.edu.tr

In the rest of this article, an "operator" means a "linear map" between two real vector spaces and the notations $\mathcal{L}(X,Y)$, $\mathcal{W}(X,Y)$, $\mathcal{W}_L(X,Y)$, $\mathcal{W}_M(X,Y)$ and $\mathcal{K}(X,Y)$ are used to show all the bounded, weakly compact, L-weakly compact, M-weakly compact, and compact operators, respectively. Also, whenever $X = Y$ hold, for the sake of simplicity, we use $A(X)$ instead of $A(X,X)$ for any operator class A(X).

2. Premilinaries

2.1. Operators on Banach Lattices

Our terminology and notations are standard and we refer to (Aliprantis and Burkinshaw, 1985; Meyer-Nieberg, 1991) for unexplained definitions and properties about Banach lattices and operators on them.

The classical sequence spaces $c_0, c, \ell_\infty, \ell_p$ for $1 \leq p < \infty$ are Riesz spaces with the ordering that $(u_n)_{n\in\mathbb{N}} \leq (v_n)_{n\in\mathbb{N}}$ if and only if $u_n \leq v_n$ for every $n \in \mathbb{N}$ and they are Banach lattices with their usual supremum norm. A subset A of a Riesz space E is called solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$. Recall that the solid hull of any subset A of E is the smallest solid set containing A and is exactly the set $\mathrm{sol}(A) = \{x \in E: \exists y \in A \text{ with } |x| \leq |y|\}$.

Let X and Y be normed spaces. It is well known that $\mathcal{L}(X,Y)$ is a normed space with the operator norm defined by $\|T\| = \sup\{\|Tx\|: x \in X \text{ and } \|x\| \leq 1\}$ and is a Banach space whenever Y is a Banach space. $T \in \mathcal{L}(X,Y)$ is said to be a compact operator whenever for every norm bounded sequence $(x_n)_{n\in\mathbb{N}}$ of X the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a norm convergent subsequence in Y. The collection $\mathcal{K}(X,Y)$ of all compact operators from X into Y forms a norm closed vector subspace of $\mathcal{L}(X,Y)$. Recall that $T \in \mathcal{L}(X,Y)$ is said to be weakly compact if for every norm bounded sequence $(x_n)_{n\in\mathbb{N}}$ of X the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in Y. If X and Y are Banach spaces, then the collection $\mathcal{W}(X,Y)$ of all compact operators from X into Y forms a norm closed vector subspace of $\mathcal{L}(X,Y)$. Clearly, every compact operator is weakly compact.

Let E and F be Banach lattices. A bounded subset A of E is said to be L-weakly compact if $\|x_n\| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in the solid hull of A. A bounded linear operator $T: X \to E$ is called L-weakly compact if $T(B_X)$ is L-weakly compact in E, where B_X denotes the closed unit ball of the Banach space X. A bounded linear operator $T: E \to X$ is M-weakly compact if $\|Tx_n\| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in B_E. $\mathcal{W}_M(E,F)$ and $\mathcal{W}_L(E,F)$ under the operator norm are closed subspaces of $\mathcal{L}(E,F)$ (Aliprantis and Burkinshaw, 1985).

It is worth to remember that L- and M-weakly compact operators are subclasses of $\mathcal{W}(E,F)$, whereas weakly compact operators need no L-weakly or M-weakly compactness property. For example, the identity operator $I: \ell_2 \to \ell_2$ is weakly compact which is not L- or M-weakly compact. However, if F is an AL-space (resp. E is an AM-space), then $\mathcal{W}_L(E,F) = \mathcal{W}(E,F)$ $\big(\text{resp.}\, \mathcal{W}_M(E,F) = \mathcal{W}(E,F)\big)$. In general, our operators and compact operators are of different classes. For instance, the operator $T: \ell_1 \to \ell_\infty$, defined by $T(\alpha_n) = (\sum_{n=1}^{\infty} \alpha_n, \sum_{n=1}^{\infty} \alpha_n, \dots)$, is clearly compact, which is not L- or M-weakly compact. Similarly, for $2 > p > 1$, the natural embedding $i: L^2[0,1] \to L^p[0,1]$ is an L- and M-weakly compact, which is not compact.

A Banach lattice E has an order continuous norm if $x_\alpha \downarrow 0$ in E implies $\|x_\alpha\| \downarrow 0$. For example, c_0 and ℓ_p $(1 \leq p < \infty)$ have order continuous norm, whereas ℓ_∞ and c (with their usual norm) do not. The order continuous part of a Banach lattice E is defined

$$E^a = \{x \in E: |x| \geq x_\alpha \downarrow 0 \ \| x_\alpha \| \to 0\}$$

For example, $(\ell_\infty)^a = c^a = c_0$. Order continuous part is very important for our operators. E^a is a closed order ideal and contains all L-weakly compact subsets of E; see Proposition 2.4.10, Proposition 3.6.2 of (Meyer-Nieberg, 1991). Thus, L-weakly compact operators take value in E^a.

An element u in the Riesz space E is called a discrete provided that the order ideal generated by u coincides with the vector space generated by u and if all discrete elements are order dense in E, then E is called discrete. For instance, c_0, c and ℓ_p for $1 \leq p \leq \infty$ are discrete.

2.2. Matrix Transformations

In this part, we give some basic notations and well–known results about matrix transformations. We refer to *(Maddox, 1971 and 1980; Wilansky, 1984; Mursaleen, 2014)* for unexplained definitions and properties about matrix classes.

Let X and Y be sequence spaces and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of real numbers. By $A_n = (a_{nk})_{k=0}^{\infty}$ we denote the sequence in the n-th row of A, and for $x = (x_k)_{k=0}^{\infty} \in X$ we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ for $n = 0,1,\ldots$ and $A(x) = \left(A_n(x)\right)_{n=0}^{\infty}$ provided that for each $n \in \mathbb{N}$, the series converges. Hence, we map the sequence $x \in X$ into the sequence $A(x) \in Y$. Then, it is said that A defines a matrix mapping from X into Y if $A(x)$ exists and is in Y for every $x \in X$. (X, Y) denotes the class of all matrices A that map X into Y.

On the other hand, every linear bounded operator does not need to have a matrix representation as follows:

Example 2.2.1. (Wilansky, 1985) Define the linear operator $T: c \to c$ by $T(x_n) = (\lim x_n, 0,0,\ldots)$. Suppose that T determines a matrix $(a_{nk})_{n,k=1}^{\infty}$. Hence, for each $k \in \mathbb{N}$

$$T(e_k) = \left(\sum_{i=1}^{\infty} a_{ni}\delta_i^k\right)_{n\in\mathbb{N}} = (a_{nk})_{n\in\mathbb{N}} = 0.$$

This shows that $(a_{nk})_{n,k=1}^{\infty}$ is a zero matrix since $(a_{nk})_{n\in\mathbb{N}}$ is k. column in $(a_{nk})_{n,k=1}^{\infty}$. But, $T(1,1,\ldots) = e_1 \neq 0$.

BK spaces are the most effective theory in the characterization of matrix mappings between sequence spaces. X is called a BK-space if X is a Banach sequence space with continuous coordinates $P_k: X \to \mathbb{C}$, $P_k(x) = x_k$ where $k = 0,1,2,\ldots$ and $x = (x_k) \in X$. The best known examples for the classical sequence spaces are ℓ_∞, c, c_0, ℓ_p $(1 \leq p < \infty)$.

Theorem 2.2.2. (Maddox, 1980) Any matrix map between BK spaces is continuous.

The above theorem shows that if X and Y are BK spaces, then every matrix $A \in (X, Y)$, where (X, Y) consists of all infinite matrices T that maps X into Y, defines an operator $T_A \in \mathcal{L}(X, Y)$ by $T_A(x) = A(x)$ for all $x \in X$, namely $(X, Y) \subset \mathcal{L}(X, Y)$. Conversely, it is natural to ask whether an operator $T \in \mathcal{L}(X, Y)$ can be given by an infinite matrix A, in which case we write $\mathcal{L}(X, Y) \subset (X, Y)$. But first we need to give a definition of AK property.

A BK space $X \supset \varphi$ is said to have AK if we have

$$x = \sum_{k=0}^{\infty} x_k e_k = \lim_{n\to\infty} \sum_{k=0}^{n} x_k e_k$$

for every sequence $x = (x_k) \in X$. The spaces c_0 and ℓ_p $(1 \leq p < \infty)$ have AK.

Theorem 2.2.3. (Jarrah and Malkowsky, 2003) If X and Y are BK spaces and X have AK, then $\mathcal{L}(X, Y) \subset (X, Y)$.

Definition 2.2.4. The β-dual or ordinary Köthe-Toeplitz dual of the sequence space X is defined by

$$X^{\beta}=\left\{\begin{array}{c} a=(a_k)\in\mathbb{R}^{\mathbb{N}}: \\ \sum_{k=0}^{\infty} a_k x_k \text{ converges for all } x=(x_k)\in X\end{array}\right\}.$$

Note that $(\ell_\infty)^\beta = c^\beta = c_0^\beta = \ell_1$, $(\ell_1)^\beta = \ell_\infty$ and $(\ell_p)^\beta = \ell_q$ such that $1 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$.

Theorem 2.2.5. (Wilansky, 1984) Let X, Z be BK spaces with AK, $Y = Z^\beta$. Then $(X, Y) = (X^{\beta\beta}, Y)$ and $A \in (X, Y)$ if and only if $A^T \in (Z, X^\beta)$.

The result of the previous theorem is sustained if X is any one of ℓ_∞, c, c_0, ℓ_p $(1 \le p < \infty)$. The theorem given covers the cases when X is an AK space. Also, by previous theorem,

$$(\ell_\infty, Y) = (c, Y) = (c_0, Y)$$

and $A \in (c_0, Y)$ if and only if $A^T \in (Z, \ell_1)$; thus, Theorem 2.2.5. holds for $X = \ell_\infty$, c; see Theorem 8.3.10 of (Wilansky, 1984).

3. Main Results

This section presents the matrix representations of our operators together with their relations with the weakly compact and compact operators. The property, which is used very often, is duality relation between our operators and is expressed as follows: an operator is M-weakly (L-weakly) compact if and only if its dual operator is L-weakly (M-weakly) compact. Another important property of L-weakly compact operators is that they take values in order continuous part. Hence, $\mathcal{W}_L(E, F) = \mathcal{W}_L(E, F^a)$ holds.

The next lemma, which is obtained from the proof of Theorem 2.6 in (Chen and Wickstead, 1999), has a key role for our results.

Lemma 3.1. If E is a Banach lattice and $T: \ell_1 \to E$ is a bounded operator such that $\|Te_k\| \to 0$ as $k \to \infty$ where $\{e_k : k \in \mathbb{N}\}$ is the natural basis of ℓ_1, then $T \in W_M(\ell_1, E)$.

Proof: For each disjoint sequence (x_n) where $x_n = (\lambda_{nk})_{k=1}^\infty \in \text{ball}(\ell_1)$ and each $\varepsilon > 0$, there exists $K > 0$ such that $\|Te_k\| < \varepsilon$ for all $k > K$. Also the disjointness of (x_n) implies that there exists $N > 0$ such that $\lambda_{nk} = 0$ for all $n > N$ and $1 \le k \le K$. So if $n > N$ (noting that $|T|(\lambda_k) = \sum_{k=1}^\infty \lambda_{nk}|Te_k|$ for all $(\lambda_k) \in \ell_1$), then we have

$$\begin{aligned}\||T|x_n\| &= \|\textstyle\sum_{k=1}^\infty \lambda_{nk}|Te_k|\| \\ &= \|\textstyle\sum_{k=K+1}^\infty \lambda_{nk}|Te_k|\| \\ &< \varepsilon \textstyle\sum_{k=K+1}^\infty |\lambda_{nk}| \le \varepsilon\|x_n\|_{\ell_1} \le \varepsilon\end{aligned}$$

which imlies that $|T|x_n \to 0$ as $n \to \infty$, that is $|T|$ is M-weakly compact, so does T.

Note also that, if $T \in W_M(\ell_1, E)$, then $\|Te_k\| \to 0$ always holds since $(e_k)_{k\in\mathbb{N}}$ is an disjoint bounded sequence in ℓ_1. Thus, we see that $T \in W_M(\ell_1, E)$ if and only if $\|Te_k\| \to 0$ for the natural basis of $(e_k)_{k\in\mathbb{N}}$. On the other hand, for every Banach lattice E, every M-weakly compact operator from ℓ_1 into E is compact via Theorem 2.7 in (Chen and Wickstead, 1999). However, a compact operator needs no M-weakly compact as $(\ell_1)'$ has no order continuous norm. For example, the operator $T: \ell_1 \to E$, defined by $T(x) = (f \otimes u)(x) = f(x)u$, is compact but it is not M-weakly compact where $0 \ne f \in (\ell_1)' \backslash c_0$ and $0 \ne u \in E$. Moreover, if $E = \ell_1$, then, as a result of that ℓ_1 is an AL-space (with Schur property), it can be seen that an operator $T: \ell_1 \to \ell_1$ is compact if and only if it is weakly compact, so it is L-weakly compact by Theorem 18.11 of (Aliprantis and Burkinshaw, 1985). Then, we have

$$\mathcal{W}_M(\ell_1, \ell_1) \subset \mathcal{C}(\ell_1, \ell_1) = \mathcal{W}_L(\ell_1, \ell_1) = \mathcal{W}(\ell_1, \ell_1).$$

According to (Stieglitz and Tietz, 1977), $T \in \mathcal{L}(\ell_1, \ell_1)$ if and only if $\sup_{k\in\mathbb{N}} \sum_{n=1}^\infty |a_{nk}| <$

∞. However, this condition is not sufficient for the bounded operator T to be an M-weakly compact. For example, the identity operator $I: \ell_1 \to \ell_1$ is not M-weakly compact and $\lim_{k\to\infty}(\sum_{n=1}^{\infty} |a_{nk}|) = 1$ for the matrix representation of I.

Theorem 3.2. $T \in \mathcal{W}_M(\ell_1, \ell_1)$ if and only if $\sum_{n=1}^{\infty} |a_{nk}| \to 0$ for $k \to \infty$.

Proof: By Theorem 2.13ii of (Maddox, 1980), $T \in \mathcal{L}(\ell_1, \ell_1)$ if and only if $\sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} |a_{nk}| < \infty$ holds. Also from Lemma 3.1, $T \in \mathcal{W}_M(\ell_1, \ell_1)$ if and only if $\|Te_k\|_{\ell_1} = \|(a_{nk})_{n\in\mathbb{N}}\|_{\ell_1} \to 0$, so if and only if $\lim_{k\to\infty}(\sum_{n=1}^{\infty} |a_{nk}|) = 0$.

A compact operator from c_0 to ℓ_∞ does not need to be an L-weakly compact. Indeed, since the norm of ℓ_∞ is not order continuous, the operator $x \to f \otimes y(x) = f(x)y$ where $f \in (c_0)'_+$ and $0 \le y \in \ell_\infty \backslash c_0$ are positive compact operator which is not L-weakly compact. So, we have

$$\mathcal{W}_L(c_0, \ell_\infty) \subset \mathcal{K}(c_0, \ell_\infty) = \mathcal{W}_M(c_0, \ell_\infty)$$
$$= \mathcal{W}(c_0, \ell_\infty).$$

Remark 3.3. If $F = c_0$ or $F = c$, then we can not use Theorem 8.3.9 of (Wilansky, 1984). However, there is a nice property of L-weakly compact operators. As we mentioned before, L-weakly compact operators take values in order continuous part, that is $\mathcal{W}_L(c, c_0) = \mathcal{W}_L(c, c) = \mathcal{W}_L(c, \ell_\infty)$ hold. Therefore, this property will help us for the next two cases.

Theorem 3.4. $T \in \mathcal{W}_L(c_0, \ell_\infty) = \mathcal{W}_L(c_0, c) = \mathcal{W}_L(c_0, c_0)$ if and only if $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0$.

Proof: From the equalities $(\ell_\infty)^a = c^a = c_0$, we have

$$\mathcal{W}_L(c_0, c) = \mathcal{W}_L(c_0, \ell_\infty) = \mathcal{W}_L(c_0, c_0)$$

By (Stieglitz and Tietz, 1977), $T \in \mathcal{L}(c_0, \ell_\infty)$if and only if $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$. As the adjoint of an L-weakly compact operator is M-weakly compact, by the help of Theorem 8.3.9 of (Wilansky, 1984) we have

$$(a_{nk}) \in \mathcal{W}_L(c_0, \ell_\infty) \Leftrightarrow (a_{nk})^T \in \mathcal{W}_M(\ell_1, \ell_1).$$

Thus, the claim can be seen from the case (ℓ_1, ℓ_1).

Let $T: c_0 \to c_0$ be any bounded linear operator. Then, T determines a matrix $(a_{nk})_{n,k=1}^{\infty}$ if and only if $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and $a_{nk} \to 0$ $(n \to \infty, k \text{ is fixed})$. However, these conditions are not sufficient for the bounded operators from c_0 to c_0 to be an M-weakly compact. Because, in that case, $\|T\| = \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}|$is satisfied; see Theorem 7.1 and 7.2 of (Maddox, 1971) and for the identity operator $I: c_0 \to c_0$, which is not M-weakly compact, $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 1$ holds. Moreover, an L-weakly (resp. M-weakly) compact operator is compact whenever E (resp. E') is a discrete Banach lattice; see Theorem 3.1 and 4.1 in (Aqzzouz et all, 2011).

Theorem 3.5. $T \in \mathcal{W}_M(c_0, c_0)$ if and only if $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0$.

Proof: Discreteness and order continuity of c_0 and $(c_0)'$ imply that

$$\mathcal{K}(c_0, c_0) = \mathcal{W}_M(c_0, c_0) = \mathcal{W}_L(c_0, c_0)$$
$$= \mathcal{W}(c_0, c_0)$$

Remain of proof can be obtain from previous theorem.

An operator from ℓ_1 into ℓ_p, $1 < p < \infty$ is L-weakly compact if and only if it is compact since ℓ_p is discrete Banach lattice with order continuous norm. So, the following inclusions hold.

$$\mathcal{W}_{\mathrm{M}}(\ell_1,\ell_{\mathrm{p}}) \subset \mathcal{K}(\ell_1,\ell_{\mathrm{p}}) = \mathcal{W}_{\mathrm{L}}(\ell_1,\ell_{\mathrm{p}}) \subset \mathcal{W}(\ell_1,\ell_{\mathrm{p}})$$

For the inclusion operator $i: \ell_1 \to \ell_p$, $p > 1$, which is not M-weakly compact, the equality $\lim_{k\to\infty} (\sum_{n=1}^{\infty} |a_{nk}|) = 1$ holds. This shows that the condition, $\mathrm{T} \in \mathcal{L}(\ell_1,\ell_{\mathrm{p}})$ where $\infty > \mathrm{p} > 1$ if and only if $\sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} |a_{nk}|^{p} < \infty$, is not sufficient for the bounded operator T to be an M-weakly compact.

Theorem 3.6. $\mathrm{T} \in \mathcal{W}_{\mathrm{M}}(\ell_1,\ell_{\mathrm{p}})$ where $1 < p < \infty$ if and only if $\sum_{n=1}^{\infty} |a_{nk}|^{p} \to 0$ for $k \to \infty$.

Proof: By Theorem 2.13(ii) in (Maddox, 1980), $\mathrm{T} \in \mathcal{L}(\ell_1,\ell_{\mathrm{p}})$ if and only if $\sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} |a_{nk}|^{p} < \infty$ holds. By Lemma 3.1, $\mathrm{T} \in \mathcal{W}_{\mathrm{M}}(\ell_1,\ell_{\mathrm{p}})$ if and only if $\|Te_k\|_{\ell_p} = \|(a_{nk})_{n\in\mathbb{N}}\|_{\ell_p} \to 0$, so if and only if $\lim_{k\to\infty} (\sum_{n=1}^{\infty} |a_{nk}|^{p}) = 0$.

A compact operator from ℓ_1 to ℓ_∞ do not need to be L-weakly compact or M-weakly compact as $(\ell_1)' = \ell_\infty \supset (\ell_\infty)^{a} = c_0$. However, since ℓ_1 is an AL-space (resp. c and ℓ_∞ are AM-spaces) every M-weakly compact (resp. L-weakly compact) operator is compact. On the other hand, L-weakly and M-weakly compact operators are different classes, that is $\mathcal{W}_{\mathrm{M}}(\ell_1,\ell_\infty) \not\subset \mathcal{W}_{\mathrm{L}}(\ell_1,\ell_\infty)$ and $\mathcal{W}_{\mathrm{L}}(\ell_1,\ell_\infty) \not\subset \mathcal{W}_{\mathrm{M}}(\ell_1,\ell_\infty)$. As a result, we have

$$\mathcal{W}_{\mathrm{M}}(\ell_1,\ell_\infty) \subset \mathcal{K}(\ell_1,\ell_\infty) \subset \mathcal{W}(\ell_1,\ell_\infty),$$

and

$$\mathcal{W}_{\mathrm{L}}(\ell_1,\ell_\infty) \subset \mathcal{K}(\ell_1,\ell_\infty) \subset \mathcal{W}(\ell_1,\ell_\infty).$$

Considering the bounded compact operator $\mathrm{T}: \ell_1 \to \ell_\infty$, defined by $\mathrm{T}(\alpha_n) = (\sum_{n=1}^{\infty} \alpha_n, \sum_{n=1}^{\infty} \alpha_n, \dots)$, which is not M-weakly compact, it can be seen $\lim_{k\to\infty} \left(\sup_{n\in\mathbb{N}} |a_{nk}|\right) = 1$. This shows that the condition $\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty$ indicated in (Stieglitz and Tietz, 1977) is not sufficient for the operator $\mathrm{T} \in \mathcal{L}(\ell_1,\ell_\infty)$ to be M-weakly compact.

Theorem 3.7. $\mathrm{T} \in \mathcal{W}_{\mathrm{M}}(\ell_1,\ell_\infty)$ if and only if $\lim_{k\to\infty} \left(\sup_{n\in\mathbb{N}} |a_{nk}|\right) = 0$.

Proof: By Theorem 2.13(i) in (Maddox, 1980), $\mathrm{T} \in \mathcal{L}(\ell_1,\ell_\infty)$ if and only if $\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty$. By the Lemma 3.1, $\mathrm{T} \in \mathcal{W}_{\mathrm{M}}(\ell_1,\ell_\infty)$ if and only if

$$\|Te_k\|_{\ell^\infty} = \|(a_{nk})_{n\in\mathbb{N}}\|_{\ell^\infty} \to 0$$

that is $\lim_{k\to\infty} \left(\sup_{n\in\mathbb{N}} |a_{nk}|\right) = 0$.

A compact operator from ℓ_1 to c need not to be L-weakly compact or M-weakly compact as $(\ell_1)' \neq (\ell_\infty)^{a} = c_0 = c^{a} \neq c$. However, since ℓ_1 is an AL-space (resp. c is an AM-space) every M-weakly compact (resp. L-weakly compact) operator is compact. On the other hand, L-weakly and M-weakly compact operators are different classes for the case (ℓ_1, c). Thus, we have

$$\mathcal{W}_{\mathrm{M}}(\ell_1,c) \subset \mathcal{K}(\ell_1,c) \subset \mathcal{W}(\ell_1,c) \text{ and}$$
$$\mathcal{W}_{\mathrm{L}}(\ell_1,c) \subset \mathcal{K}(\ell_1,c) \subset \mathcal{W}(\ell_1,c).$$

According to (Stieglitz and Tietz, 1977), $\mathrm{T} \in \mathcal{L}(\ell_1,c)$ if and only if $\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty$ and $\lim_{n\to\infty} a_{nk}$ exists for all $k \in \mathbb{N}$. However, the previous example also shows that these conditions are not sufficient for the bounded operator to be an M-weakly compact.

Theorem 3.8. $\mathrm{T} \in \mathcal{W}_{\mathrm{M}}(\ell_1,c)$ if and only if the following statements hold

1. $\lim_{n\to\infty} a_{nk}$ exists for all $k \in \mathbb{N}$,

2. $\lim_{k\to\infty} \left(\sup_{n\in\mathbb{N}} |a_{nk}|\right) = 0$.

Proof: Proof is similar with the previous case since the norms of c, ℓ_∞ are same.

L-weakly compactness and compactness coincide for the operator from ℓ_1 into c_0 since c_0 is a discrete Banach lattice with order continuous norm. In this way, the following inclusions hold.

$$\mathcal{W}_M(\ell_1, c_0) \subset \mathcal{K}(\ell_1, c_0) = \mathcal{W}_L(\ell_1, c_0) \subset \mathcal{W}(\ell_1, c_0).$$

Theorem 3.9. $T \in \mathcal{W}_M(\ell_1, c_0)$ if and only if the following statements hold.

1. $\lim_{n\to\infty} a_{nk} = 0$ for all $k \in \mathbb{N}$.
2. $\lim_{k\to\infty} \left(\sup_{n\in\mathbb{N}} |a_{nk}|\right) = 0.$

Proof. According to (Stieglitz and Tietz, 1977), it is well known that $T \in \mathcal{L}(\ell_1, c_0)$ if and only if $\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty$ and $\lim_{n\to\infty} a_{nk} = 0$ for all $k \in \mathbb{N}$. Other condition can be seen from previous case since the norms of c_0 and ℓ_∞ are same.

It can be easily seen that $\mathcal{W}_L(\ell_1, c_0) = \mathcal{K}(\ell_1, c_0) \subset \mathcal{W}(\ell_1, c_0)$ and $\mathcal{W}_L(\ell_1, F) \subset \mathcal{K}(\ell_1, F) \subset \mathcal{W}(\ell_1, F)$ whenever $F \in \{c, \ell_\infty\}$. Moreover, again for L-weakly compact operators, we use the fact that all L-weakly compact operators take values in the order continuous part. Hence, we obtain $\mathcal{W}_L(\ell_1, c) = \mathcal{W}_L(\ell_1, \ell_\infty) = \mathcal{W}_L(\ell_1, c_0)$ since $(\ell_\infty)^a = c^a = c_0$.

Theorem 3.10. $T \in \mathcal{W}_L(\ell_1, \ell_\infty) = \mathcal{W}_L(\ell_1, c) = \mathcal{W}_L(\ell_1, c_0)$ if and only if $\lim_{n\to\infty} \left(\sup_{k\in\mathbb{N}} |a_{nk}|\right) = 0.$

Proof: It follows that $(a_{ij}) \in \mathcal{W}_L(\ell_1, \ell_\infty) \Leftrightarrow (a_{ij})^T \in \mathcal{W}_M(\ell_1, \ell_\infty)$ by Theorem 8.3.9 in (Wilansky, 1984).

As it is well known, for every Banach space F, a bounded operator from ℓ_p to F is weakly compact as ℓ_p is reflexive. Since $(\ell_p)'$ is a discrete Banach lattice with order continuous norm an operator from ℓ_p into F is compact if and only if it is M-weakly compact. Furthermore, ℓ_p $(1 < p < \infty)$ space is BK space with AK, that is

$$\mathcal{L}(\ell_p, Y) = (\ell_p, Y)$$

where $Y \in \{c_0, c, \ell_1, \ell_p, \ell_\infty\}$. However, for $F \in \{c_0, c, \ell_\infty\}$, the inclusion operator $i: \ell_p \to F$ is a bounded (so weakly compact) which is not compact.

There is a positive compact operator from ℓ_p to ℓ_∞, which is not L-weakly compact, whereas every L-weakly compact operator is compact since $c^a = (\ell_\infty)^a = c_0$ is discrete. Then, we have:

$$\mathcal{W}_L(\ell_p, \ell_\infty) \subset \mathcal{W}_M(\ell_p, \ell_\infty) = \mathcal{K}(\ell_p, \ell_\infty) \subset \mathcal{W}(\ell_p, \ell_\infty) = \mathcal{L}(\ell_p, \ell_\infty).$$

In (Stieglitz and Tietz, 1977), it is described (ℓ_p, Y) such that $T \in \mathcal{L}(\ell_p, \ell_\infty)$ if and only if $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}|^q < \infty$ where $q = \frac{p}{p-1}$. However, this condition is not enough for M-weakly or L-weakly compact operators as follows:

Example 3.11. Inclusion map $i: \ell_p \to Y$, where $Y \in \{c_0, c, \ell_\infty\}$ is a bounded operator, which is not a M-weakly and L-weakly compact operator. On the other hand, if $(a_{nk})_{n=1}^{\infty}$ is a matrix representation of i then $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}|^q = 1$ and $\lim_{n\to\infty} a_{nk} = 0$ but $\lim_{k\to\infty} \left(\sup_{n\in\mathbb{N}} |a_{nk}|\right) = 1.$

Theorem 3.12. $T \in \mathcal{W}_L(\ell_p, \ell_\infty)$ if and only if $\sum_{k=1}^{\infty} |a_{nk}|^q \to 0$ for $n \to \infty$, where $q = \frac{p}{p-1}$.

Proof: It follows that $(a_{nk}) \in \mathcal{W}_L(\ell_p, \ell_\infty) \Leftrightarrow (a_{nk})^T \in \mathcal{W}_M(\ell_1, \ell_q)$ where $q = \frac{p}{p-1}$ by Theorem 8.3.9 in (Wilansky, 1984).

L-weakly compact and compact operators from ℓ_p to c_0 coincide since c_0 is discrete Banach lattice with order continuous norm. Hence, we obtain that

$$\mathcal{W}_M(\ell_p, c_0) = \mathcal{K}(\ell_p, c_0) = \mathcal{W}_L(\ell_p, c_0) \subset \mathcal{W}(\ell_p, c_0) = \mathcal{L}(\ell_p, c_0).$$

Theorem 3.13. $T \in \mathcal{W}_L(\ell_p, c_0) = \mathcal{W}_M(\ell_p, c_0) = \mathcal{K}(\ell_p, c_0)$ if and only $\sum_{k=1}^{\infty} |a_{nk}|^q \to 0$ for $n \to \infty$ where $q = \frac{p}{p-1}$

Proof. It follows that $\mathcal{W}_L(\ell_p, c_0) = \mathcal{W}_L(\ell_p, \ell_\infty)$ since $(\ell_\infty)^a = c_0$.

There is a positive compact operator from ℓ_p to c which is not L-weakly compact, whereas every L-weakly compact operator is compact since $c^a = (\ell_\infty)^a = c_0$ is discrete. Then,

$$\mathcal{W}_L(\ell_p, c) \subset \mathcal{W}_M(\ell_p, c) = \mathcal{K}(\ell_p, c) \subset \mathcal{W}(\ell_p, c) = \mathcal{L}(\ell_p, c).$$

Theorem 3.14. $T \in \mathcal{W}_L(\ell_p, c)$ if and only if the following statements hold.

1. $\lim_{n\to\infty} a_{nk}$ exists for all $k \in \mathbb{N}$;
2. $\sum_{k=1}^{\infty} |a_{nk}|^q \to 0$ for $n \to \infty$where $q = \frac{p}{p-1}$.

Proof: It follows that from (Stieglitz and Tietz, 1977) and the equality $\mathcal{W}_L(\ell_p, c) = \mathcal{W}_L(\ell_p, c_0)$ as $c^a = c_0$.

Theorem 3.15. $T \in \mathcal{W}_L(\ell_\infty, \ell_\infty) = \mathcal{W}_L(\ell_\infty, c_0) = \mathcal{W}_L(\ell_\infty, c)$ if and only if $\sum_{k=1}^{\infty} |a_{nk}| \to 0$ for $n \to \infty$.

Proof: It follows that $(a_{nk}) \in \mathcal{W}_L(\ell_\infty, \ell_\infty)$ if and only if $(a_{nk})^T \in \mathcal{W}_M(\ell_1, \ell_1)$.

As a result of c_0 being an AM-space with order continuous norm, the class of compact operators from ℓ_∞ into c_0 coincide with the class of L-weakly compact operators. Besides, the dual space of ℓ_∞ is an AL-space, which is not discrete, so it has positive Schur property but it has not Schur property. Also, we can find a sequence (g_n) in the dual of ℓ_∞ to be weakly null which is not norm null. Then, by Theorem 17.5 in (Aliprantis and Burkinshaw, 1985), the operator $T: \ell_\infty \to c_0$, defined by $T(x) = (g_1(x), g_2(x), \dots)$, is weakly compact, which is not compact. So, we have

$$\mathcal{W}_L(\ell_\infty, c_0) = \mathcal{K}(\ell_\infty, c_0) \subset \mathcal{W}_M(\ell_\infty, c_0) = \mathcal{W}(\ell_\infty, c_0) = \mathcal{L}(\ell_\infty, c_0).$$

On the other hand, in terms of regularity of operators, since the dual $(\ell_\infty)'$ and c_0 have order continuous norm by Theorem 18.16 of (Aliprantis and Burkinshaw, 1985), the following is mostly true:

$$\mathcal{W}_L^r(\ell_\infty, c_0) = \mathcal{K}^r(\ell_\infty, c_0) = \mathcal{W}_M^r(\ell_\infty, c_0) = \mathcal{W}^r(\ell_\infty, c_0) = \mathcal{L}^r(\ell_\infty, c_0).$$

Corollary 3.16. $T \in \mathcal{W}_L(c, \ell_\infty)$ if and only if $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0$.

Proof: By Theorem 8.3.10 in (Wilansky, 1984), it follows the fact that $T \in \mathcal{W}_L(c, \ell_\infty) \Leftrightarrow T \in \mathcal{W}_M(\ell_1, \ell_1)$.

Since F is a discrete Banach lattice with order continuous norm we can say that an operator from c to c_0 is L-weakly compact if and only if it is compact. Then, the following inclusions hold:

$$\mathcal{K}(c, c_0) = \mathcal{W}_M(c, c_0) = \mathcal{W}_L(c, c_0) =$$

$\mathcal{W}(c, c_0)$.

Corollary 3.17. $T \in \mathcal{W}_L(c, c_0) = \mathcal{W}_M(c, c_0)$ if and only if $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0$.

Proof: It can be seen from the equality $\mathcal{W}_L(c, c_0) = \mathcal{W}_L(c, \ell_\infty)$.

Since c has not order continuous norm, we can find a positive compact operator which is not L-weakly compact. Hence, we have

$$\mathcal{W}_L(c, c) \subset \mathcal{K}(c, c) = \mathcal{W}_M(c, c) = \mathcal{W}(c, c).$$

Corollary 3.18. $T \in \mathcal{W}_L(c, c)$ if and only if $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = 0$.

Proof: It can be seen from the equality $\mathcal{W}_L(c, c) = \mathcal{W}_L(c, \ell_\infty) = \mathcal{W}_L(c, c_0)$.

There are, of course, certain trivial cases that our operator classes equal the class of all bounded operators. This happens, for example, for (c_0, ℓ_p), (ℓ_∞, ℓ_1), (ℓ_p, ℓ_1) whenever $1 < p < \infty$.

Acknowledgement: We would like to thank A.W. Wickstead for offering me of this work and for his valuable comments.

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