INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 10 NO. 2 PAGE 73–81 (2017)

Doubly Warped Products in Locally Conformal Almost Cosymplectic Manifolds

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(Communicated by Ion Mihai)

ABSTRACT

Recently, the author established a general inequality for doubly warped products in arbitrary Riemannian manifolds [16]. In the present paper, we obtain similar inequalities for doubly warped products isometrically immersed in locally conformal almost cosymplectic manifolds. Some applications are derived.

Keywords: *Doubly warped product; minimal immersion; totally real submanifold; locally conformal almost cosymplectic manifold.*

AMS Subject Classification (2010): *Primary: 53C40 ; Secondary: 53C25; 53B25; 53C42.*

1. Introduction

Bishop and O'Neill [1] introduced the concept of warped products to study manifolds of negative sectional curvature. O'Neill discussed warped products and explored curvature formulas of warped products in terms of curvatures of components of warped products.

Doubly warped products can be considered as a generalization of singly warped products.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let $f_1 : M_1 \to (0, \infty)$ and $f_2 : M_2 \to (0, \infty)$ be differentiable functions.

The *doubly warped product* $M =_{f_2} M_1 \times_{f_1} M_2$ is the product manifold $M_1 \times M_2$ endowed with the metric

$$g = f_2^2 g_1 + f_1^2 g_2.$$

More precisely, if $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are natural projections, the metric g is defined by

$$g = (f_2 \circ \pi_2)^2 \, \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \, \pi_2^* g_2.$$

The functions f_1 and f_2 are called *warping functions*. If either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a Riemannian product manifold. If neither f_1 nor f_2 is constant, then we have a *non-trivial* doubly warped product [20].

Let $x :_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}$ be an isometric immersion of a doubly warped product $_{f_2}M_1 \times_{f_1} M_2$ into a Riemannian manifold $\widetilde{M}$. We denote by h the second fundamental form of x and by $H_i = \frac{1}{n_i} trace \; h_i$ the partial mean curvatures, where $trace \; h_i$ is the trace of h restricted to M_i and $n_i = \dim M_i$ $(i = 1, 2)$.

The immersion x is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any vector fields X and Z tangent to D_1 and D_2, respectively, where D_i are the distributions obtained from the vectors tangent to M_i (or more precisely, vectors tangent to the horizontal lifts of M_i).

In [4], B. Y. Chen studied the relationship between the Laplacian of the warping function f and the squared mean curvature of a warped product $M_1 \times_f M_2$ isometrically immersed in a Riemannian manifold $\widetilde{M}(c)$ of constant sectional curvature c given by

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} ||H||^2 + n_1 c, \tag{1.1}$$

where $n_i = \dim M_i$, $i = 1, 2$, and Δ is the Laplacian operator of M_1. Moreover, the equality case of (1.1) holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.

Received : *24-04-2017,* ***Accepted :*** *19-09-2017*
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Later, several Chen inequalities have been obtained for different submanifolds in ambient manifolds possessing different kind of structures (see [7], [8], [10], [11], [13], [14]).

In 2008, B. Y. Chen and F. Dillen extended inequality (1.1) to multiply warped product manifolds in arbitrary Riemannian manifolds (see [5]).

In [16], the present author established the following general inequality for arbitrary isometric immersions of doubly warped product manifolds in arbitrary Riemannian manifolds:

Theorem 1.1. *Let x be an isometric immersion of an n-dimensional doubly warped product $M = {}_{f_2} M_1 \times_{f_1} M_2$ into an m-dimensional arbitrary Riemannian manifold $\widetilde{M}^m$. Then:*

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} ||H||^2 + n_1 n_2 \max \widetilde{K}, \tag{1.2}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$, Δ_i is the Laplacian operator of M_i, $i = 1, 2$ and $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}^m$ restricted to 2-plane sections of the tangent space T_pM of M at each point p in M. Moreover, the equality case of (1.2) holds if and only if the following two statements hold:

1. *x is a mixed totally geodesic immersion satisfying $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors of M_i.*

2. *at each point $p = (p_1, p_2) \in M$, the sectional curvature function $\widetilde{K}$ of $\widetilde{M}^m$ satisfies $\widetilde{K}(u, v) = \max \widetilde{K}(p)$ for each unit vector $u \in T_{p_1} M_1$ and each unit vector $v \in T_{p_2} M_2$.*

Later, in [18] and [19], the present author studied doubly warped product submanifolds in generalized Sasakian space forms and S-space forms, respectively. In [6], M. Crasmareanu introduced and studied bi-warped 3-metrics from the point of view of Webster curvature. In [21], D. W.Yoon, K. S. Cho, S. G. Han investigated warped product submanifolds in locally conformal almost cosymplectic manifolds.

Motivated by the studies of the above authors, the aim of this paper is to obtain similar inequalities for doubly warped products in locally conformal almost cosymplectic manifolds of pointwise constant φ-sectional curvature c.

2. Preliminaries

In this section, we recall some general definitions and basic formulas which will be used later. For more background on locally conformal almost cosymplectic manifolds, we recommend the reference [15].

Let $\widetilde{M}$ be a $(2m+1)$-dimensional almost contact manifold (see [2]). Denote by (φ, ξ, η) the almost contact structure of $\widetilde{M}$. Thus, φ is a $(1,1)-$tensor field, ξ is a vector field and η a 1-form on $\widetilde{M}$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1.$$

Define an almost complex structure J on the product manifold $\widetilde{M} \times \mathbb{R}$ by

$$J(X, \lambda \frac{d}{ds}) = (\varphi X - \lambda \xi, \eta(X) \frac{d}{ds}), \tag{2.1}$$

where X and $\lambda \frac{d}{ds}$ are vectors tangent to $\widetilde{M}$ and $\mathbb{R}$, respectively, $\mathbb{R}$ being the real line with coordinate s.

The manifold $\widetilde{M}$ is said to be *normal* ([2]) if the almost complex structure J is integrable (i.e., J arises from a complex structure on $\widetilde{M} \times \mathbb{R}$). The necessary and sufficient condition for $\widetilde{M}$ to be normal is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$$

for any vector fields X and Y tangent to $\widetilde{M}$, $[\cdot, \cdot]$ being the Lie bracket of vector fields.

Let g be a Riemannian metric on $\widetilde{M}$ compatible with (φ, ξ, η), such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y tangent to $\widetilde{M}$. Thus, the manifold $\widetilde{M}$ is almost contact metric and the quadruple (φ, ξ, η, g) is its almost contact metric structure. Clearly, we have $\eta(X) = g(X, \xi)$ for any vector field X tangent to $\widetilde{M}$. Let Φ denote the fundamental 2-form of $\widetilde{M}$ defined by $\Phi(X, Y) = g(\varphi X, Y)$ for any vector fields X and Y tangent to $\widetilde{M}$. The manifold $\widetilde{M}$ is said to be *almost cosymplectic* if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$, where d is the operator of exterior differentiation. If $\widetilde{M}$ is almost cosymplectic and normal, then it is called *cosymplectic* ([2]). It is well known that the almost contact metric manifold is cosymplectic if and only if $\widetilde{\nabla}\varphi$ vanishes identically, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$. The products of almost Kähler manifolds and $\mathbb{R}$ are the simplest examples of almost cosymplectic manifolds.

An almost contact metric manifold $\widetilde{M}$ is called a *locally conformal almost cosymplectic manifold* ([15]) if there exists a 1-form ω such that $d\Phi = 2\omega \wedge \Phi$, $d\eta = \omega \wedge \eta$ and $d\omega = 0$.

A necessary and sufficient condition for a structure to be normal locally conformal almost cosymplectic is ([9])

$$(\widetilde{\nabla}_X \varphi)Y = u(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{2.2}$$

where $\omega = u\eta$. From (2.2) it follows that $\widetilde{\nabla}_X \xi = u(X - \eta(X)\xi)$.

A plane section σ in $T_p\widetilde{M}$ of an almost contact structure manifold $\widetilde{M}$ is called a *φ-section* if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. $\widetilde{M}$ is of pointwise constant φ- sectional curvature if at each point $p \in \widetilde{M}$, the section curvature $\widetilde{K}(\sigma)$ does not depend on the choice of the φ- section σ of $T_p\widetilde{M}$ and, in this case for $p \in \widetilde{M}$ and for any φ- section σ of $T_p\widetilde{M}$, the function c defined by $c(p) = \widetilde{K}(\sigma)$ is called the *φ- sectional curvature* of $\widetilde{M}$. A locally conformal almost cosymplectic manifold $\widetilde{M}$ of dimension ≥ 5 is of pointwise constant φ- sectional curvature if and only if its curvature tensor $\widetilde{R}$ is of the form ([15])

$$\begin{aligned}
\widetilde{R}(X, Y, Z, W) = {} & \frac{c - 3u^2}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\
& + \frac{c + u^2}{4}\{g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) \\
& - 2g(X, \varphi Y)g(Z, \varphi W)\} \\
& - (\frac{c + u^2}{4} + u')\{g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W)\} \\
& + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)\} \\
& + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),
\end{aligned} \tag{2.3}$$

where u is a function such that $\omega = u\eta$, $u' = \xi u$ and c is the pointwise constant φ- sectional curvature of $\widetilde{M}$.

Let M be an n-dimensional manifold isometrically immersed in a locally conformal almost cosymplectic manifold $\widetilde{M}$. Let ∇ be the induced Levi-Civita connection of M. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\widetilde{\nabla}_X V = -A_V X + D_X V$$

for vector fields X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, D the normal connection and A_V the shape operator in the direction of V. The second fundamental form and the shape operator are related as

$$g(h(X, Y), V) = g(A_V X, Y).$$

We also denote by g the induced Riemannian metric on M as well as on the locally conformal almost cosymplectic manifold $\widetilde{M}$.

Let X be tangent to M we put $\varphi X = PX + FX$, where PX and FX are the tangential and the normal components of φX, respectively. The squared norm of P for an othonormal basis $\{e_1, ..., e_n\}$ of M is defined by

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j)$$

and the mean curvature vector $H(p)$ at $p \in M$ is given by

$$H = \frac{1}{n} traceh = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i).$$

As is known, M is said to be *minimal* if H vanishes identically.
Also, for any $r \in \{n+1, ..., 2m+1\}$ we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), i, j \in \{1, ..., n\},$$

and

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$$

where $\{e_{n+1}, ..., e_{2m+1}\}$ is an orthonormal basis of $T_p^{\perp} M$.

A submanifold M is totally geodesic in $\widetilde{M}$ if $h = 0$ and totally real submanifold if P is identically zero, that is, $\varphi X \in T_p^{\perp} M$ for any $X \in T_p M$, $p \in M$.

For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M$, $p \in M$. For any orthonormal basis $e_1, ..., e_n$ of the tangent space $T_p M$, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

Let M be a Riemannian p-manifold and $\{e_1, ..., e_p\}$ be an orthonormal basis of M. For a differentiable function f on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{p} \{(\nabla_{e_j} e_j) f - e_j e_j f\}.$$

We recall the following general algebraic lemma of Chen for later use.

Lemma 2.1. *[3]Let $n \ge 2$ and $a_1, a_2, ..., a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1 a_2 \ge b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

3. Doubly warped product submanifolds in locally conformal almost cosymplectic manifolds

Next, we investigate doubly warped product submanifolds tangent to the structure vector field ξ in a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$.

Theorem 3.1. *Let x be an isometric immersion of an n-dimensional doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. Then:*

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \le \frac{n^2}{4}||H||^2 + n_1 n_2 \frac{c - 3u^2}{4} - (\frac{c + u^2}{4} + u')n_2 + \frac{3(c + u^2)}{4} n_2, \tag{3.1}$$

where $n_i = \dim M_i$ and Δ_i is the Laplacian operator of M_i, $i = 1, 2$.

Proof. Suppose that ${}_{f_2}M_1 \times_{f_1} M_2$ is a doubly warped product submanifold of a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with pointwise constant φ- sectional curvature c whose structure vector field ξ is tangent to M_1. Since ${}_{f_2}M_1 \times_{f_1} M_2$ is a doubly warped product, then

$$\begin{cases} \nabla_X Y = \nabla^1_X Y - \frac{f_2^2}{f_1^2} g_1(X,Y)\nabla^2(\ln f_2), \\ \nabla_X Z = Z(\ln f_2)X + X(\ln f_1)Z, \end{cases} \tag{3.2}$$

for any vector fields X, Z tangent to M_1 and M_2, respectively, where ∇^1 and ∇^2 are the Levi-Civita connections of the Riemannian metrics g_1 and g_2, respectively. Here, $\nabla^2(\ln f_2)$ denotes the gradient of $\ln f_2$ with respect to the metric g_2.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = \frac{1}{f_1}\{(\nabla^1_X X) f_1 - X^2 f_1\} + \frac{1}{f_2}\{(\nabla^2_Z Z) f_2 - Z^2 f_2\}. \tag{3.3}$$

We choose a local ortonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$ such that $e_1, ..., e_{n_1} = \xi$ are tangent to M_1, $e_{n_1+1}, ..., e_n$ are tangent to M_2 and e_{n+1} is parallel to the mean curvature vectore H.

Then from (3.3), it follows that

$$\sum_{\substack{1\le j\le n_1\\ n_1+1\le s\le n}} K(e_j \wedge e_s) = n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2}. \tag{3.4}$$

Using the equation of Gauss, we get

$$n^2||H||^2 = 2\tau + ||h||^2 - n(n-1)\frac{c-3u^2}{4} - \frac{3(c+u^2)}{4}||P||^2 + (\frac{c+u^2}{4} + u')(2n-2), \tag{3.5}$$

where τ denotes the scalar curvature of ${}_{f_2}M_1 \times_{f_1} M_2$, that is,

$$\tau = \sum_{1\le i<j\le n} K(e_i \wedge e_j).$$

Let us consider that

$$\delta = 2\tau - n(n-1)\frac{c-3u^2}{4} - \frac{3(c+u^2)}{4}||P||^2 + (\frac{c+u^2}{4} + u')(2n-2) - \frac{n^2}{2}||H||^2. \tag{3.6}$$

Then, (3.5) can be written as

$$n^2||H||^2 = 2\left(\delta + ||h||^2\right). \tag{3.7}$$

Thus with respect to the above orthonormal frame, the above equation takes the following form:

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2 = 2\left[\delta + \sum_{i=1}^n \left(h_{ii}^{n+1}\right)^2 + \sum_{i\ne j}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^n \left(h_{ij}^r\right)^2\right].$$

Assume that $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$. Then the above equation becomes

$$\left(\sum_{i=1}^3 a_i\right)^2 = 2[\delta + \sum_{i=1}^3 a_i^2 + \sum_{1\le i\ne j\le n}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^n \left(h_{ij}^r\right)^2$$
$$- \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}].$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.,

$$\left(\sum_{i=1}^3 a_i\right)^2 = 2\left(b + \sum_{i=1}^3 a_i^2\right),$$

with

$$b = \delta + \sum_{1\le i\ne j\le n} \left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} \left(h_{ij}^{r}\right)^2 - - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1} h_{tt}^{n+1}.$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$.
In the case under consideration, it follows that

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1} h_{tt}^{n+1} \ge \ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n} \left(h_{\alpha\beta}^{n+1}\right)^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{\alpha,\beta=1}^{n} \left(h_{\alpha\beta}^{r}\right)^2. \tag{3.8}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.9}$$

Using again the Gauss equation, we find that

$$\begin{aligned} n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} &= \tau - \sum_{1\le j<k\le n_1} K\left(e_j \wedge e_k\right) - \sum_{n_1+1\le s<t\le n} K\left(e_s \wedge e_t\right) = \\ &= \tau - \frac{n_1\left(n_1-1\right)\left(c-3u^2\right)}{8} - \sum_{1\le j<k\le n_1} g^2(Pe_j, e_k)\frac{3(c+u^2)}{4} + \\ &+ \left(\frac{c+u^2}{4} + u'\right)(n_1-1) - \sum_{r=n+1}^{2m+1} \sum_{1\le j<k\le n_1} \left(h_{jj}^r h_{kk}^r - \left(h_{jk}^r\right)^2\right) - \\ &- \frac{n_2\left(n_2-1\right)\left(c-3u^2\right)}{8} - \sum_{n_1+1\le s<t\le n} g^2(Pe_s, e_t)\frac{3(c+u^2)}{4} - \\ &- \sum_{r=n+1}^{2m+1} \sum_{n_1+1\le s<t\le n} \left(h_{ss}^r h_{tt}^r - \left(h_{st}^r\right)^2\right). \end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10) and taking account of (3.4), we obtain

$$\begin{aligned} n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} &\le \tau - \frac{n\left(n-1\right)\left(c-3u^2\right)}{8} + n_1 n_2\frac{c-3u^2}{4} - \\ &- \frac{\delta}{2} + \left(\frac{c+u^2}{4} + u'\right)(n_1-1) - \\ &- \sum_{1\le j<k\le n_1} g^2(Pe_j, e_k)\frac{3(c+u^2)}{4} - \sum_{n_1+1\le s<t\le n} g^2(Pe_s, e_t)\frac{3(c+u^2)}{4}. \end{aligned} \tag{3.11}$$

Hence, from (3.6), the inequality (3.11) reduces to

$$\begin{aligned} n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} &\le \frac{n^2}{4}||H||^2 + n_1 n_2\frac{c-3u^2}{4} - \left(\frac{c+u^2}{4} + u'\right)n_2 + \\ &+ \frac{3(c+u^2)}{4} \sum_{n_1+1\le j<t\le n} g^2(Pe_j, e_t) \le \\ &\le \frac{n^2}{4}||H||^2 + n_1 n_2\frac{c-3u^2}{4} - \left(\frac{c+u^2}{4} + u'\right)n_2 + \end{aligned}$$

$$+\frac{3(c+u^2)}{4}\min\{n_1,n_2\}. \tag{3.12}$$

We distinguish two cases:
(a) $n_1 \le n_2$, in this case the inequality (3.12) implies (3.1).
(b) $n_1 > n_2$, in this case the inequality (3.12) also becomes (3.1). It completes the proof. □

Remark 3.1. If $f_2 \equiv 1$, then the inequality (3.1) is exactly the inequality (6) from [21] for warped products.

Corollary 3.1. *Let x be an isometric immersion of an n-dimensional totally real doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. Then, we have:*

$$n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2} \le \frac{n^2}{4}||H||^2+n_1n_2\frac{c-3u^2}{4}-(\frac{c+u^2}{4}+u')n_2, \tag{3.13}$$

where $n_i = \dim M_i$ and Δ_i is the Laplacian operator of M_i, $i=1,2$.
Moreover, the equality case of (3.13) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where H_i, $i=1,2$, are the partial mean curvature vectors.

Proof. Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly warped product submanifold of a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$. Then we have $g(Pe_i,e_s)=0$ for $0 \le i \le n_1$, $n_1+1 \le s \le n$. Therefore, by (3.12) we can easily obtain the inequality (3.13). Also, we see that the equality of (3.12) holds if and only if

$$h^r_{jt}=0,\ 1 \le j \le n_1,\ n_1+1 \le t \le n,\ n+1 \le r \le 2m+1, \tag{3.14}$$

and

$$\sum_{i=1}^{n_1} h^r_{ii} = \sum_{t=n_1+1}^{n} h^r_{tt} = 0,\ n+2 \le r \le 2m+1. \tag{3.15}$$

Obviously (3.14) is equivalent to the mixed totally geodesic of the doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ and (3.9) and (3.15) imply $n_1H_1 = n_2H_2$. The converse statement is straightforward. □

As applications, we obtain certain obstructions to the existence of minimal totally real doubly warped product submanifolds in locally conformal almost cosymplectic manifolds.
By using the above corollary, we can obtain some important consequences:

Corollary 3.2. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a totally real doubly warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. If the warping functions are harmonic, then ${}_{f_2}M_1 \times_{f_1} M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c < \frac{1}{n_1-1}(u^2+3n_1u^2+4u')$.*

Proof. Assume f_1 is a harmonic function on M_1, f_2 is a harmonic function on M_2 and ${}_{f_2}M_1 \times_{f_1} M_2$ admits a minimal totally real immersion in a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$. Then, the inequality (3.13) becomes $c \ge \frac{1}{n_1-1}(u^2+3n_1u^2+4u')$. □

Corollary 3.3. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a totally real doubly warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. If the warping functions f_1 and f_2 of ${}_{f_2}M_1 \times_{f_1} M_2$ are eigenfunctions of the Laplacian on M_1 and M_2, respectively, with corresponding eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then ${}_{f_2}M_1 \times_{f_1} M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c \le \frac{1}{n_1-1}(u^2+3n_1u^2+4u')$.*

Corollary 3.4. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a totally real doubly warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. If one of the warping functions is harmonic and the other one is an eigenfunction of the Laplacian with corresponding eigenvalue $\lambda > 0$, then ${}_{f_2}M_1 \times_{f_1} M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c \le \frac{1}{n_1-1}(u^2+3n_1u^2+4u')$.*

Theorem 3.2. *Let x be an isometric immersion of an n-dimensional doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. Then:*

$$n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4}||H||^2 + n_1 n_2\frac{c-3u^2}{4} - (\frac{c+u^2}{4} + u')n_1 + \frac{3(c+u^2)}{4}n_2, \tag{3.16}$$

where $n_i = \dim M_i$ and Δ_i is the Laplacian operator of M_i, $i = 1, 2$.

Remark 3.2. If $f_2 \equiv 1$, then the inequality (3.16) is exactly the inequality from Theorem 8 from [21] for warped products.

Corollary 3.5. *Let x be an isometric immersion of an n-dimensional totally real doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. Then, we have:*

$$n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4}||H||^2 + n_1 n_2\frac{c-3u^2}{4} - (\frac{c+u^2}{4} + u')n_1, \tag{3.17}$$

where $n_i = \dim M_i$ and Δ_i is the Laplacian operator of M_i, $i = 1, 2$.

Moreover, the equality case of (3.17) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.

Corollary 3.6. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a totally real doubly warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. If the warping functions are harmonic, then ${}_{f_2}M_1 \times_{f_1} M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c < \frac{1}{n_2-1}(u^2 + 3n_2u^2 + 4u')$.*

Corollary 3.7. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a totally real doubly warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. If the warping functions f_1 and f_2 of ${}_{f_2}M_1 \times_{f_1} M_2$ are eigenfunctions of the Laplacian on M_1 and M_2, respectively, with corresponding eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then ${}_{f_2}M_1 \times_{f_1} M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c \leq \frac{1}{n_2-1}(u^2 + 3n_2u^2 + 4u')$.*

Corollary 3.8. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a totally real doubly warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. If one of the warping functions is harmonic and the other one is an eigenfunction of the Laplacian with corresponding eigenvalue $\lambda > 0$, then ${}_{f_2}M_1 \times_{f_1} M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c \leq \frac{1}{n_2-1}(u^2 + 3n_2u^2 + 4u')$.*

References

[1] Bishop, R. L. and O'Neill, B., Manifolds of negative curvature. *Trans. Amer. Math. Soc.* **145** (1969), 1-49.
[2] Blair, D. E., Contact Manifolds in Riemannian Geometry. Lecture Notes in Math. 509, Springer, Berlin, 1976.
[3] Chen, B. Y., Some pinching and classification theorems for minimal submanifolds. *Arch. Math.* **60** (1993), 568-578.
[4] Chen, B. Y., On isometric minimal immersions from warped products into real space forms. *Proc. Edinburgh Math. Soc.* **45** (2002), 579-587.
[5] Chen, B. Y. and Dillen, F., Optimal inequalities for multiply warped product submanifolds. *Int. Electron. J. Geom., Vol.* **1** (2008), 1-11.
[6] Crasmareanu, M., Adapted metrics and Webster curvature on three classes of 3-dimensional geometries. *Int. Electron. J. Geom.*, **7 (2)** (2014), 37-46.
[7] Malek, F. and Nejadakbary, V., Warped product submanifold in generalized Sasakian space form. *Acta Math. Acad. Paedagog. Nyhazi. (N.S.)* **27** no. 2 (2011), 325-338.
[8] Matsumoto, K. and Mihai, I., Warped product submanifolds in Sasakian space forms. *SUT Journal of Mathematics* **38** (2002), 135-144.
[9] Matsumoto, K., Mihai, I. and Rosca, R., A certain locally conformal almost cosymplectic manifold and its submanifolds. *Tensor N.S.* **51 (1)** (1992), 91-102.
[10] Mihai, A., Warped product submanifolds in complex space forms. *Acta Sci. Math. (Szeged)* **70** (2004), 419-427.
[11] Mihai, A., Warped product submanifolds in quaternion space forms. *Rev. Roumaine Math. Pures Appl.* **50** (2005), 283-291.
[12] Mihai, A., Mihai I. and Miron, R. (Eds.), Topics in Differential Geometry, Ed. Academiei Romane, Bucuresti, 2008.
[13] Mihai, I. and Presura, I., An improved Chen first inequality for Legendrian submanifolds in Sasakian space forms. *Period. Math. Hung.* **74 (2)** (2017), 220-226.
[14] Murathan, C., Arslan, K., Ezentas, R. and Mihai, I., Warped product submanifolds in Kenmotsu space forms. *Taiwanese J. Math.* **10** (2006), 1431-1441.

[15] Olszak, Z., Locally conformal almost cosymplectic manifolds. *Collq. Math.* **57(1)** (1989), 73-87.
[16] Olteanu, A., A general inequality for doubly warped product submanifolds. *Math. J. Okayama Univ.* **52** (2010), 133-142.
[17] Olteanu, A., Recent results in the geometry of warped product submanifolds, Matrix Rom, 2011.
[18] Olteanu, A., Doubly warped product submanifolds in generalized Sasakian space forms, *Proceedings RIGA 2014*, Ed. Univ. Bucuresti (2014), 174-184.
[19] Olteanu, A., Doubly warped products in S-space forms. *Rom. J. Math. Comput. Sci.* **4** Issue 1 (2014), 111-124.
[20] Ünal, B., Doubly warped products. *Differ. Geom. App.* **15(3)** (2001), 253-263.
[21] Yoon, D. W., Cho, K. S. and Han, S. G., Some inequalities for warped products in locally conformal almost cosymplectic manifolds. *Note Mat.* **23 (1)** (2004), 51-60.

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