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Extremities Involving B. Y. Chen's Invariants for Real Hypersurfaces in Complex Quadric

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ABSTRACT

The article is concerned with the study of real hypersurfaces of the complex quadric Q^m. We establish B. Y. Chen's inequalities for real hypersurfaces of the complex quadric Q^m and by considering the equality case, we obtain some consequences. Also, we establish an inequality in terms of the warping function and the scalar curvature for a warped product real hypersurface of Q^m and some obstructions have been given. Moreover, we investigate the expression of the curvature tensor of a real hypersurface in the complex quadric Q^m admitting semi-symmetric metric connection. Using this curvature, we derive inequalities involving Chen δ-invariant admitting a semi-symmetric metric connection. Furthermore, the equality case is considered.

Keywords: *real hypersurface; complex quadric; scalar curvature; Chen δ-invariant; semi-symmetric metric connection.*

AMS Subject Classification (2010): *Primary: 53C40; Secondary: 53C55; 53B05; 53B15.*

1. Introduction

In 1968, S. S. Chern raised a question involving minimal isometric immersion into Euclidean space [12]. Then, Chen found some obstructions to Chern's problem and proposed inequalities for submanifolds in Riemannian space form concerning the sectional curvature, the scalar curvature and the squared mean curvature [9]. Moreover, he proposed inequality concerning $\delta(n_1, n_2, ..., n_k)$ and the squared mean curvature for the submanifolds in real space form [10].

Afterwards, many papers have been appeared in submanifolds of space forms in the version of real and complex like, generalised complex space forms [11], (k, μ)-contact space forms [1] and Sasakian space forms [13]. Further, the geometry of the complex quadric has been studied by H. Reckziegel [16] in 1995 and Y. J. Suh, obtained some analyzing results on real hypersurfaces in the complex quadric by considering some geometric conditions like parallel Ricci tensor [17], Reeb parallel shape operator [18]. Also, the classifications of real hypersurface of the complex quadric with isometric Reeb flow were obtained by Berndt and Suh [5] and many more work have been studied by different authors considering the same ambient space ([2]-[4],[19]).

However, Hayden [14] originated the idea of a semi-symmetric metric connection on a Riemannian manifold. Yano [20] deliberated this connection and found some properties of a Riemannian manifold with the same connection. Also, A. Mihai and C. Özgür studied the Chen extremities for submanifolds of the real space forms with same connection [15].

Here, we first establish Chen's extremities for real hypersurfaces of the complex quadric Q^m and considering the equality case, we obtain some consequences. Also, we establish an inequality in terms of the warping function and the scalar curvature for warped product real hypersurface of Q^m and some obstructions have been given. Then, we study real hypersurface of Q^m admitting semi-symmetric metric connection and find the curvature tensor of a real hypersurface in Q^m with the semi-symmetric metric connection. Additionally, using this curvature we develop Chen's inequality for a real hypersurfaces of the complex quadric Q^m admitting semi symmetric metric connection.

As long as, by virtue of simpleness, throughout a paper we denote semi-symmetric metric connection, Levi-Civita connection and Warped product by SSMC, LC connection and WP, respectively.

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2. The complex quadric Q^m

For more details of the geometry of complex quadric we refer to ([5],[16],[17]). The complex hypersurface of $\mathbb{C}P^{m+1}$ is known as the complex quadric Q^m defined by the equation $z_1^2 + ... + z_{m+1}^2 = 0$, where $z_1, ..., z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$ equipped with the induced Riemannian metric g. Then, naturally the canonical Kähler structure (J, g) on Q^m is induced by Kähler structure on $\mathbb{C}P^{m+1}$ [18]. The 1-dimensional quadric Q^1 is congruent to the round 2-sphere S^2. The 2-dimensional quadric Q^2 is congruent to the Riemannian product $S^2 \times S^2$. For this, we will assume $m \geq 3$ throughout the paper.

Apart from J there is one more geometric structure on Q^m, known as the complex conjugation A on the tangent spaces of Q^m which is a parallel rank-two vector bundle $\mathcal{U}$ containing S^1-bundle of real structures. For $x \in Q^m$, let $A_{\overline{x}}$ be the shape operator of Q^m in $\mathbb{C}P^{m+1}$. Then we have $A_{\overline{x}}W = W$ for $W \in T_xQ^m$, that is, A is an involution or $A_{\overline{x}}$ is a complex conjugation restricted to T_xQ^m. Now, T_xQ^m is decomposed as [18]:

$$T_xQ^m = \mathcal{V}(A_{\overline{x}}) \oplus J\mathcal{V}(A_{\overline{x}}),$$

such that $\mathcal{V}(A_{\overline{x}})$ and $J\mathcal{V}(A_{\overline{x}})$, respectively denote the (+1)-eigenspace and (-1)-eigenspace of the involution $A_{\overline{x}}^2 = I$ on $T_xQ^m, x \in Q^m$.

Now, a tangent vector $W \neq 0 \in T_xQ^m$ is known as the *singular* if it is tangent to more than one maximal flat in Q^m. Classification of singular tangent vectors for Q^m are given as [19]:

1 If there exists $A \in \mathcal{U}$ such that W is an eigenvector corresponding to an eigenvalue (+1), then the singular tangent vector W is known as *$\mathcal{U}$-principal*.
2 If there exists $A \in \mathcal{U}$ and orthonormal vectors $U, V \in \mathcal{V}(A)$ such that $W/||W|| = (U + JV)/\sqrt{2}$, then the singular tangent vector W is known as *$\mathcal{U}$-isotropic*.

Let $\mathcal{M}^n$ be a real hypersurface of Q^m with a connection ∇ induced from the LC connection $\overline{\nabla}$ in Q^m. Then, the transform JU of the Kähler structure J on Q^m is defined by $JU = \phi U + \eta(U)N$ where ϕU is the tangential component of JU and $N \in T_p^{\perp}\mathcal{M}$, for $U \in T_p\mathcal{M}$. Here, $\mathcal{M}$ associates an induced *almost contact metric structure* (ϕ, ξ, η, g) satisfying the following relations [6]:

$$\xi = -JN, \eta(\xi) = 1, \eta(U) = g(\xi, U), \phi^2U + U = \eta(U)\xi, \phi\xi = 0,$$
$$\eta(\phi U) = 0, g(\phi U, \phi V) + \eta(U)\eta(V) = g(U, V), g(\phi U, V) = -g(U, \phi V).$$

Moreover, the real hypersurface $\mathcal{M}$ of Q^m satisfy

$$\nabla_U\xi = \phi SU,$$

where S is the shaper operator of $\mathcal{M}$.
On the other hand, the Gauss and the Weingarten formulas for M follows

$$\overline{\nabla}_UV = \nabla_UV + h(U, V) \quad \text{and} \quad \overline{\nabla}_UN = -SU,$$

respectively, for $U, V \in T_p\mathcal{M}$ and $N \in T_p^{\perp}\mathcal{M}$. The second fundamental form h and the shape operator S of $\mathcal{M}$ are related by

$$g(h(U, V), N) = g(S_NU, V) = g(SU, V).$$

Now, we take $A \in \mathcal{U}_x$ such that $N = \cos(t)Z_1 + \sin(t)JZ_2$, where Z_1, Z_2 are orthonormal vectors in $\mathcal{V}(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 [16]) which is a function on $\mathcal{M}$.
Since $\xi = -JN$, we have

$$\begin{aligned} N &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned}$$

from which it follows that $g(\xi,\ AN) = 0$.

3. B. Y. Chen inequality for a real hypersurface of Q^m

Here, we obtain the general inequality associated with the Chen δ-invariant for a real hypersurfaces $\mathcal{M}$ of the complex quadric Q^m.

Now, from the Gauss equation, the Riemannian curvature tensor R of connection ∇ in terms of J and $A \in \mathcal{U}$ is defined as [18]:

$$\begin{aligned} R(U,V)W &= g(V,W)U - g(U,W)V + g(\phi V,W)\phi U - g(\phi U,W)\phi V - 2g(\phi U,V)\phi W \\ &\quad + g(AV,W)AU - g(AU,W)AV + g(JAV,W)JAU - g(JAU,W)JAV \\ &\quad + g(SV,W)SU - g(SU,W)SV, \end{aligned} \tag{3.1}$$

where $U, V, W \in T_p\mathcal{M}$.
Then, we can see

$$g(R(U,V)W + R(V,W)U + R(W,U)V, W^{'}) = 0, \qquad for\ U,V,W,W^{'} \in T_pM \tag{3.2}$$

that is, the first Bianchi Identity holds for $\mathcal{M}$ of LC connection ∇.

Next, the curvature tensor R of the Hopf hypersurface $\mathcal{M}$ (i.e. $\alpha = g(S\xi, \xi)$), where α is a smooth function on $\mathcal{M}$ satisfies

$$\begin{aligned} R(U,\xi)V &= \eta(V)[U + \alpha SU] - [g(U,V) + \alpha g(SU,V)]\xi + g(A\xi,V)AU - g(AU,V)A\xi \\ &\quad - g(AN,V)JAU + g(JAU,V)AN, \\ R(U,V)\xi &= \eta(V)[U + \alpha SU] - \eta(U)[V + \alpha SV] + g(AV,\xi)AU - g(AU,\xi)AV \\ &\quad - g(AN,V)JAU + g(AU,N)JAV. \end{aligned}$$

Moreover, for a real hypersurface $\mathcal{M}$ and $U, V, W, W^{'} \in T_p\mathcal{M}$, the relation (3.1) produce

$$\begin{aligned} g(R(U,V)W,W^{'}) &= g(V,W)g(U,W^{'}) - g(U,W)g(V,W^{'}) + g(\phi V,W)g(\phi U,W^{'}) \\ &\quad - g(\phi U,W)g(\phi V,W^{'}) - 2g(\phi U,V)g(\phi W,W^{'}) + g(AV,W)g(AU,W^{'}) \\ &\quad - g(AU,W)g(AV,W^{'}) + g(JAV,W)g(JAU,W^{'}) - g(JAU,W)g(JAV,W^{'}) \\ &\quad + g(SV,W)g(SU,W^{'}) - g(SU,W)g(SV,W^{'}). \end{aligned} \tag{3.3}$$

By taking $U = W^{'} = e_i$ in (3.3), one can have [17]

$$\begin{aligned} \mathrm{Ric}(V,W) &= ng(V,W) - 3\eta(V)\eta(W) - g(AN,N)g(AV,W) + g(AW,N)g(AV,N) \\ &\quad + g(AW,\xi)g(AV,\xi) + \mathrm{tr}(S)g(SV,W) - g(S^2V,W), \end{aligned} \tag{3.4}$$

where the Ricci tensor of $\mathcal{M}$ with connection ∇ is symbolized by Ric which satisfy

$$\mathrm{Ric}(U,\xi) = (2n - 4 + \alpha h - \alpha^2)\eta(X) - 2g(AN,N)g(AU,\xi).$$

Consider an orthonormal basis $\{e_i\}_1^n$ and $\{e_{n+1} = N\}$ of $T_p\mathcal{M}$ and $T_p^{\perp}\mathcal{M}$ respectively, where $n+1 = 2m$. Conveniently, let $h_{ij}^{n+1} = g(h(e_i,e_j), e_{n+1}) = g(h(e_i,e_j), N)$ for $i,j \in \{1,...,n\}$. Now, one defines the squared mean curvature $||\mathcal{H}||^2$ of $\mathcal{M}$ in Q^m and the squared norm $||h||^2$ of h are given by:

$$||\mathcal{H}||^2 = \frac{1}{n^2}\Big(\sum_{i,j=1}^{n} h_{ij}^{n+1}\Big)^2, \; ||h||^2 = \sum_{i,j=1}^{n}(h_{ij}^{n+1})^2,$$

respectively.
Now, the scalar curvature τ has the expression

$$\tau = \sum_{1 \le i < j \le n} \mathcal{K}(e_i \wedge e_j),$$

where $\mathcal{K}(\pi)$ denotes the sectional curvature of $\mathcal{M}$ involved with a plane section $\pi \subset T_p\mathcal{M}$ and is spanned by tangent vectors $\{e_i, e_j\}$ and $\sum_{1\le i<j<n}\mathcal{K}(e_i \wedge e_j) = \sum_{1\le i<j<n} g(R(e_i,e_j)e_j,e_i)$.

Revoke that the *Chen first invariant* ([9],[10]) is defined by

$$\delta_m(p) = \tau(p) - \inf\{\mathcal{K}(\pi) \mid \pi \subset T_p\mathcal{M},\ dim\,\pi = 2\},$$

where $\tau(p)$ is the scalar curvature at p.

We give one algebraic result which we will use to proof our result.

Lemma 3.1. *[9] Let $a_1, a_2, ..., a_k, b$ be $(k+1)(k \geq 2)$ real numbers satisfying*

$$(\sum_{i=1}^{k} a_i)^2 = (k-1)(\sum_{i=1}^{k} a_i^2 + b)$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = ... = a_k$.

Theorem 3.1. *For a real hypersurface $\mathcal{M}$ of Q^m with 2-plane section $\pi \subset T_p\mathcal{M}$ spanned by tangent vectors e_1 and e_2, we have*

$$\tau(p) - \mathcal{K}(\pi) \leq \frac{n^2}{2}\left\{1 + \left(\frac{n-2}{n-1}\right)||\mathcal{H}||^2\right\} + g^2(Ae_1, e_2) + \frac{g^2(AN, N)}{2} + g^2(JAe_1, e_2). \tag{3.5}$$

Moreover, equality holds in (3.5) at $p \in \mathcal{M}$ if and only if there exist an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ and orthonormal normal frame $\{e_{n+1} = N\}$ of $T_p^{\perp}\mathcal{M}$, such that the matrix of the shape operator S takes the following form

$$S = \begin{pmatrix} p' & 0 & 0 \\ 0 & q' & 0 \\ 0 & 0 & M \end{pmatrix}, \tag{3.6}$$

where M is the diagonal matrix of order $n-2$ with diagonal entry $r = p' + q'$.

Proof. From (3.4), we deduce that

$$2\tau = n^2 + g^2(AN, N) - 1 + n^2||\mathcal{H}||^2 - ||h||^2 \tag{3.7}$$

where we have used

$$\begin{aligned} ||h||^2 &= g(h(e_i, e_j), h(e_i, e_j)) = g(g(Se_i, e_j)N, g(Se_i, e_j)N) \\ &= \text{tr}(S^2). \end{aligned}$$

Let us denote

$$\epsilon = 2\tau - n^2 - g^2(AN, N) + 1 - \frac{n^2(n-2)}{n-1}||\mathcal{H}||^2. \tag{3.8}$$

We obtain

$$\epsilon = n^2||\mathcal{H}||^2 - ||h||^2 - \frac{n^2(n-2)}{n-1}||\mathcal{H}||^2$$

which provide

$$n^2||\mathcal{H}||^2 = (n-1)\{\epsilon + ||h||^2\}. \tag{3.9}$$

or, equivalently

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\left\{\epsilon + \sum_{i=1}^{n}(h_{ij}^{n+1})^2 + \sum_{i \neq j}(h_{ij}^{n+1})^2\right\}. \tag{3.10}$$

Using lemma (3.1) together with equation (3.10), we obtain

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i \neq j}\left(h_{ij}^{n+1}\right)^2 + \epsilon. \tag{3.11}$$

Also, the Gauss equation implies that

$$\begin{aligned} \mathcal{K}(\pi) &= g(R(e_1, e_2)e_2, e_1) \\ &= 1 + 3g^2(\phi e_1, e_2) + g(Ae_2, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2) \\ &\quad + g(JAe_2, e_2)g(JAe_1, e_1) - g^2(JAe_1, e_2) - \left(h_{12}^{n+1}\right)^2 + h_{22}^{n+1}h_{11}^{n+1}. \end{aligned} \tag{3.12}$$

Incorporating (3.11) in (3.12) yields

$$\begin{aligned}\mathcal{K}(\pi) &\geq 1+3g^2(\phi e_1,e_2)+g(Ae_2,e_2)g(Ae_1,e_1)-g^2(Ae_1,e_2)+g(JAe_2,e_2)g(JAe_1,e_1)\\ &\quad -g^2(JAe_1,e_2)+\frac{1}{2}\{\sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2+\epsilon\}-\left(h_{12}^{n+1}\right)^2\\ &= 1+3g^2(\phi e_1,e_2)+g(Ae_2,e_2)g(Ae_1,e_1)-g^2(Ae_1,e_2)+g(JAe_2,e_2)g(JAe_1,e_1)\\ &\quad -g^2(JAe_1,e_2)+\tau-\frac{n^2}{2}-\frac{g^2(AN,N)}{2}+\frac{1}{2}-\frac{n^2(n-2)}{2(n-1)}||\mathcal{H}||^2+\frac{1}{2}\sum_{i\neq j,i,j\geq 2}\left(h_{ij}^{\alpha}\right)^2\\ &\geq 1+3g^2(\phi e_1,e_2)+g(Ae_2,e_2)g(Ae_1,e_1)-g^2(Ae_1,e_2)+g(JAe_2,e_2)g(JAe_1,e_1)\\ &\quad -g^2(JAe_1,e_2)+\tau-\frac{n^2}{2}-\frac{g^2(AN,N)}{2}+\frac{1}{2}-\frac{n^2(n-2)}{2(n-1)}||\mathcal{H}||^2.\end{aligned}$$

Thus, finally we have

$$\begin{aligned}\tau(p)-\mathcal{K}(\pi) &\leq (n-2)\{\frac{n^2}{2(n-1)}||\mathcal{H}||^2+\frac{3n-2}{2(n-2)}\}-3g^2(\phi e_1,e_2)-g(Ae_2,e_2)g(Ae_1,e_1)\\ &\quad +g^2(Ae_1,e_2)-g(JAe_2,e_2)g(JAe_1,e_1)+g^2(JAe_1,e_2)+g(\phi Se_2,e_2)\\ &\quad +g(\phi Se_1,e_1)-\eta(e_2)^2-\eta(e_1)^2+\frac{g^2(AN,N)}{2}+\frac{1}{2}-(n-1)g(\phi Se_i,e_i)\end{aligned}$$

or

$$\tau(p)-\mathcal{K}(\pi)\leq\frac{n^2}{2}\left\{1+\left(\frac{n-2}{n-1}\right)||\mathcal{H}||^2\right\}+g^2(Ae_1,e_2)+g^2(JAe_1,e_2)+\frac{g^2(AN,N)}{2}. \tag{3.13}$$

Now, finally we get the equality in (13) at $p\in\mathcal{M}$ if and only if we have the equality case of lemma i.e.,

$$\begin{aligned}h_{ij}^{n+1} &= 0 \quad \text{for all } i\neq j,\\ h_{11}^{n+1}+h_{22}^{n+1} &= h_{33}^{n+1}=h_{44}^{n+1}=\ldots=h_{n\,n}^{n+1}.\end{aligned}$$

Thus, we may have the choice for $\{e_1,e_2\}$ such that $h_{12}^{n+1}=0$. Hence, the matrix of the shape operator has the form (3.6). □

Corollary 3.1. *Let $\mathcal{M}$ be a real hypersurface of Q^m with 2-plane section $\pi\subset T_p\mathcal{M}$ spanned by tangent vectors e_1 and e_2 such that the normal vector field is $\mathcal{U}$-principal. Then, we have*

$$\tau(p)-\mathcal{K}(\pi)\leq\frac{n^2}{2}\left\{1+\left(\frac{n-2}{n-1}\right)||\mathcal{H}||^2\right\}+g^2(Ae_1,e_2)+g^2(JAe_1,e_2)+\frac{1}{2}. \tag{3.14}$$

Moreover, equality holds in (3.14) at $p\in\mathcal{M}$ if and only if there exist an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ and orthonormal normal frame $\{e_{n+1}=N\}$ of $T_p^{\perp}\mathcal{M}$, such that the matrix of the shape operator S takes the following form

$$S=\begin{pmatrix} p' & 0 & 0\\ 0 & q' & 0\\ 0 & 0 & M\end{pmatrix}, \tag{3.15}$$

where M is the diagonal matrix of order $n-2$ with diagonal entry $r=p'+q'$.

Corollary 3.2. *Let $\mathcal{M}$ be a real hypersurface of Q^m with 2-plane section $\pi\subset T_p\mathcal{M}$ spanned by tangent vectors e_1 and e_2 such that the normal vector field is $\mathcal{U}$-isotropic. Then, we have*

$$\tau(p)-\mathcal{K}(\pi)\leq\frac{n^2}{2}\left\{1+\left(\frac{n-2}{n-1}\right)||\mathcal{H}||^2\right\}+g^2(Ae_1,e_2)+g^2(JAe_1,e_2). \tag{3.16}$$

Moreover, equality holds in (3.16) at $p\in\mathcal{M}$ if and only if there exist an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ and orthonormal normal frame $\{e_{n+1}=N\}$ of $T_p^{\perp}\mathcal{M}$, such that the matrix of the shape operator S takes the following form

$$S=\begin{pmatrix} p' & 0 & 0\\ 0 & q' & 0\\ 0 & 0 & M\end{pmatrix}, \tag{3.17}$$

where M is the diagonal matrix of order $n-2$ with diagonal entry $r=p'+q'$.

4. WP real hypersurface of Q^m

In this section, we develop inequalities involving the warping function of a WP real hypersurface $\mathcal{M}$ of Q^m.

Next, we consider two Riemannian manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of dimensions n_1 and n_2 equipped with Riemannian metrics ς_1 and ς_2 respectively. Let ζ be a positive function on $\mathcal{M}_1$. The WP manifold $\mathcal{M}_1 \otimes_\zeta \mathcal{M}_2$ is defined to be the product manifold $\mathcal{M}_1 \otimes \mathcal{M}_2$ with the warped metric $g = \varsigma_1 + \zeta^2\varsigma_2$ [7].

Consider an isometric immersion $\Psi : \mathcal{M} = \mathcal{M}_1 \otimes_\zeta \mathcal{M}_2 \to Q^m$ of a WP manifold $\mathcal{M}_1 \otimes_\zeta \mathcal{M}_2$ into a Riemannian manifold Q^m. Let h be the second fundamental form of Ψ and the mean curvature vectors denoted by $\mathcal{H}_i = \frac{1}{n_i}\mathrm{tr}(h_i)$ where $\mathrm{tr}(h_i)$ is the trace of h restricted to $\mathcal{M}_i (i = 1, 2)$.

Theorem 4.1. *Let $\Psi : \mathcal{M}^n = \mathcal{M}_1 \otimes_\zeta \mathcal{M}_2 \to Q^m$ be an isometric immersion of a WP real hypersurface into Q^m with $\xi \in T_p\mathcal{M}_1$. Then*

$$n_2\frac{\Delta\zeta}{\zeta} \leq -\frac{n^2}{2} + \frac{1}{2}g^2(AN, N) - 2n + 2n_1n_2 + \frac{11}{2} + \frac{n^2}{4}||\mathcal{H}||^2 + 2\sum_{i=1}^{n} g(A^2e_i, e_i),$$

where $n_i = dim\mathcal{M}_i$ for $i = 1, 2$, Δ is the Laplacian operator of $\mathcal{M}_1$ and $\{e_i\}_1^n$ is an orthonormal basis of $T_p\mathcal{M}$.

Proof. Let us consider an isometric immersion $\Psi : \mathcal{M} = \mathcal{M}_1 \otimes_\zeta \mathcal{M}_2 \to \mathcal{N}(s)$ of a WP real hypersurface $\mathcal{M}_1 \otimes_\zeta \mathcal{M}_2$ into Q^m whose structure vector field $\xi \in T_p\mathcal{M}_1$. Then, one can easily have [8]

$$\mathcal{K}(X \wedge Z) = \frac{1}{\zeta}\{(\nabla_X X)\zeta - X^2\zeta\}.$$

Now we choose an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ such that $e_1, ..., e_{n_1}$ are tangent to $\mathcal{M}_1$ and $e_{n_1+1}, ..., e_n$ are tangent to $\mathcal{M}_2$. Then, with the virtue of above defined relation, we obtain

$$\frac{\Delta\zeta}{\zeta} = \sum_{1\leq i\leq n_1}\sum_{n_1+1\leq j\leq n} \mathcal{K}(e_i \wedge e_j). \tag{4.1}$$

By definition of scalar curvture τ and (4.1) yields

$$n_2\frac{\Delta\zeta}{\zeta} = \tau - \sum_{1\leq i\leq n_1} \mathcal{K}(e_i \wedge e_j) - \sum_{n_1+1\leq j\leq n} \mathcal{K}(e_i \wedge e_j) \tag{4.2}$$

From (3.7), we have

$$n^2||\mathcal{H}||^2 = 2(\delta + ||h||^2) \tag{4.3}$$

where

$$\delta = 2\tau - n^2 - g^2(AN, N) + 1 - \frac{n^2}{2}||\mathcal{H}||^2. \tag{4.4}$$

Moreover, in local coordinates (4.3) has the following expression

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = 2\Big(\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ii}^{n+1})^2\Big)$$

or, equivalently

$$\begin{aligned}\Big(h_{11}^{n+1} + \sum_{i=2}^{n_1} h_{ii}^{n+1} + \sum_{i=n_1+1}^{n} h_{ii}^{n+1}\Big)^2 &= 2\Big\{\delta + (h_{11}^{n+1})^2 + \sum_{i=2}^{n_1}(h_{ii}^{n+1})^2 + \sum_{i=n_1+1}^{n}(h_{ii}^{n+1})^2 \\ &\quad + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2\Big\} \\ &= 2\Big\{\delta + (h_{11}^{n+1})^2 + \Big(\sum_{i=2}^{n_1} h_{ii}^{n+1}\Big)^2 - \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n_1+1}h_{kk}^{n_1+1} \\ &\quad + \Big(\sum_{i=n_1+1}^{n} h_{ii}^{n+1}\Big)^2 - \sum_{n_1+1\leq j\neq k\leq n} h_{jj}h_{kk} + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2\Big\}\end{aligned}$$

Using lemma (3.1), we have

$$\sum_{1\le j\ne k\le n_1} h^{n+1}_{jj}h^{n+1}_{kk} + \sum_{n_1+1\le j\ne k\le n} h^{n+1}_{jj}h^{n+1}_{kk} \ \ge\ \frac{\delta}{2} + \sum_{1\le i<j\le n}(h^{n+1}_{ij})^2 \tag{4.5}$$

Furthermore, equality holds if and only if

$$\sum_{i=1}^{n_1} h^{n+1}_{ii} = \sum_{i=n_1+1}^{n} h^{n+1}_{ii} \tag{4.6}$$

We also know that

$$\begin{aligned}\tau &= \sum_{1\le i<j\le n}\mathcal{K}(e_i\wedge e_j)\\ &= \sum_{1\le i<j\le n_1}\mathcal{K}(e_i\wedge e_j) + \sum_{n_1+1\le i<j\le n}\mathcal{K}(e_i\wedge e_j) + \sum_{j=n_1+1}^{n}\sum_{i=1}^{n_1}\mathcal{K}(e_i\wedge e_j)\end{aligned}$$

So, from (4.1), we derive

$$\begin{aligned}n_2\frac{\Delta\zeta}{\zeta} &= \sum_{j=n_1+1}^{n}\sum_{i=1}^{n_1}\mathcal{K}(e_i\wedge e_j)\\ &= \tau - \sum_{1\le i<j\le n_1}\mathcal{K}(e_i\wedge e_j) - \sum_{n_1+1\le i<j\le n}\mathcal{K}(e_i\wedge e_j)\\ &= \tau - n_1(n_1-1) - 3(n_1-1) - n_2(n_2-1) - 3(n_2-1) + 2\sum_{i=1}^{n} g(A^2e_i,e_i)\\ &\quad -\left\{\sum_{1\le i<j\le n_1} g(Ae_i,e_i)g(Ae_j,e_j) + \sum_{n_1+1\le i<j\le n} g(Ae_i,e_i)g(Ae_j,e_j)\right\}\\ &\quad -\left\{\sum_{1\le i<j\le n_1} g(JAe_i,e_i)g(JAe_j,e_j) + \sum_{n_1+1\le i<j\le n} g(JAe_i,e_i)g(JAe_j,e_j)\right\}\\ &\quad -\left\{\sum_{1\le i<j\le n_1} h^{n+1}_{ii}h^{n+1}_{jj} + \sum_{n_1+1\le i<j\le n} h^{n+1}_{ii}h^{n+1}_{jj}\right\} + \left\{\sum_{1\le i<j\le n_1}(h^{n+1}_{ij})^2\right.\\ &\quad \left. + \sum_{n_1+1\le i<j\le n}(h^{n+1}_{ij})^2\right\}\end{aligned}$$

Using (4.5), we have

$$\begin{aligned}n_2\frac{\Delta\zeta}{\zeta} &\le \tau - n_1(n_1-1) - 3(n_1-1) - n_2(n_2-1) - 3(n_2-1) + 2\sum_{i=1}^{n} g(A^2e_i,e_i)\\ &\quad -\left\{\sum_{1\le i<j\le n_1} g(Ae_i,e_i)g(Ae_j,e_j) + \sum_{n_1+1\le i<j\le n} g(Ae_i,e_i)g(Ae_j,e_j)\right\}\\ &\quad -\left\{\sum_{1\le i<j\le n_1} g(JAe_i,e_i)g(JAe_j,e_j) + \sum_{n_1+1\le i<j\le n} g(JAe_i,e_i)g(JAe_j,e_j)\right\}\\ &\quad -\left\{\frac{1}{2}\delta + \sum_{1\le i<j\le n_1}(h^{n+1}_{ij})^2\right\} + \left\{\sum_{1\le i<j\le n_1}(h^{n+1}_{ij})^2 + \sum_{n_1+1\le i<j\le n}(h^{n+1}_{ij})^2\right\}\end{aligned}$$

which gives

$$n_2\frac{\Delta\zeta}{\zeta} \le \tau - n_1(n_1-1) - 3(n_1-1) - n_2(n_2-1) - 3(n_2-1) + 2\sum_{i=1}^{n} g(A^2e_i,e_i) - \frac{1}{2}\delta.$$

Incorporating (4.4) with the above relation, we derive

$$n_2\frac{\Delta\zeta}{\zeta} \leq -\frac{1}{2}n^2+\frac{1}{2}g^2(AN,N)-2n+2n_1n_2+\frac{11}{2}+\frac{n^2}{4}||\mathcal{H}||^2+2\sum_{i=1}^{n}g(A^2e_i,e_i)-\frac{1}{2}\delta$$

from which we conclude our result. □

Corollary 4.1. *Let $\Psi:\mathcal{M}^n=\mathcal{M}_1\otimes_\zeta\mathcal{M}_2\to Q^m$ be an isometric immersion of a WP real hypersurface into Q^m with $\xi\in T_p\mathcal{M}_1$. Then, for a $\mathcal{U}$-principal normal vector field, we have the inequality*

$$n_2\frac{\Delta\zeta}{\zeta} \leq -\frac{n^2}{2}-2n+2n_1n_2+6+\frac{n^2}{4}||\mathcal{H}||^2+2\sum_{i=1}^{n}g(A^2e_i,e_i)$$

where $n_i=dim\mathcal{M}_i$, for $i=1,2$, Δ is the Laplacian operator of $\mathcal{M}_1$ and $\{e_i\}_1^n$ is an orthonormal basis of $T_p\mathcal{M}$.

Corollary 4.2. *Let $\Psi:\mathcal{M}^n=\mathcal{M}_1\otimes_\zeta\mathcal{M}_2\to Q^m$ be an isometric immersion of a WP real hypersurface into Q^m with $\xi\in T_p\mathcal{M}_1$. Then, for a $\mathcal{U}$-isotropic normal vector field, we have the inequality*

$$n_2\frac{\Delta\zeta}{\zeta} \leq -\frac{n^2}{2}-2n+2n_1n_2+\frac{11}{2}+\frac{n^2}{4}||\mathcal{H}||^2+2\sum_{i=1}^{n}g(A^2e_i,e_i)$$

where $n_i=dim\mathcal{M}_i$, for $i=1,2$, Δ is the Laplacian operator of $\mathcal{M}_1$ and $\{e_i\}_1^n$ is an orthonormal basis of $T_p\mathcal{M}$.

5. Curvature tensor of real hypersurface $\mathcal{M}$ in Q^m admitting SSMC

In this section, we study SSMC and then we obtain the curvature tensor of a real hypersurface $\mathcal{M}$ in Q^m with respect to SSMC and then we find the intrinsic scalar curvature with respect to SSMC.

Consider a Riemannian manifold $(\mathcal{M}^n,g)$ with linear connection $\tilde{\nabla}$. Then, $\tilde{\nabla}$ is called *semi-symmetric connection* [20] if its torsion tensor $\tilde{\mathcal{T}}$, defined by

$$\tilde{\mathcal{T}}(U,V)=\tilde{\nabla}_UV-\tilde{\nabla}_VU-[U,V], \tag{5.1}$$

satisfy

$$\tilde{\mathcal{T}}(U,V)=\eta(V)U-\eta(U)V, \tag{5.2}$$

for $U,V\in T_p\mathcal{M}$ and a 1-form η. In addition, a semi-symmetric linear connection is said to be SSMC $\tilde{\nabla}$ if it holds

$$\tilde{\nabla}g=0, \tag{5.3}$$

for all $U,V\in T_p\mathcal{M}$, otherwise it is said to be a *semi-symmetric non-metric connection*.

A SSMC $\tilde{\nabla}$ in terms of the LC connection ∇ on $\mathcal{M}$ is defined by

$$\tilde{\nabla}_UV=\nabla_UV+\eta(V)U-g(U,V)\xi, \tag{5.4}$$

for $U,V\in T_p\mathcal{M}$.

Now, let us consider the complex quadric Q^m admitting SSMC $\tilde{\overline{\nabla}}$ and the LC connection $\overline{\nabla}$. Next, let $\mathcal{M}$ be a real hypersurface of Q^m with the induced SSMC $\tilde{\nabla}$ and the induced LC connection ∇. Let $\tilde{\overline{R}}$ and $\overline{R}$ be the curvature tensors of Q^m with respect to the connections $\tilde{\overline{\nabla}}$ and $\overline{\nabla}$ respectively. Put $\tilde{R}$ as the curvature tensor field of $\tilde{\nabla}$ and R as the curvature tensor field of ∇ on $\mathcal{M}$. Then the Gauss formulae with respect to $\tilde{\nabla}$ and ∇ has the expression

$$\tilde{\overline{\nabla}}_UV=\tilde{\nabla}_UV+\tilde{h}(U,V),\quad \overline{\nabla}_UV=\nabla_UV+h(U,V)$$

respectively, where $\tilde{h}$ is the (0,2)-tensor of $\mathcal{M}$ in Q^m and from these two relations, one can easily get $\tilde{h}(U,V)=h(U,V)$.

Furthermore, using (5.4) for $U, V \in T_p\mathcal{M}$, we have

$$\begin{aligned}(\tilde{\nabla}_U\eta)(V) &= (\nabla_U\eta)(V) + g(\phi U, \phi V) = g(\phi SU, V) + g(\phi U, \phi V),\\ (\tilde{\nabla}_U\phi)(V) &= (\nabla_U\phi)(V) - g(U, \phi V)\xi - \eta(V)\phi U\\ &= \eta(V)SU - \eta(V)\phi U - g(SU, V)\xi + g(\phi U, V)\xi,\end{aligned}$$

and the covariant derivative of torsion tensor of $\tilde{\nabla}$ with respect to SSMC follows

$$\begin{aligned}(\tilde{\nabla}_U\tilde{\mathcal{T}})(V, W) &= g(\phi SU, V)W - g(\phi SU, W)V + g(U, V)W - g(U, W)V\\ &\quad -\eta(U)[\eta(V)W - \eta(W)V],\end{aligned}$$

for $U, V, W \in T_p\mathcal{M}$.

Now, we know the curvature tensor $\tilde{R}$ can be calculated by

$$\tilde{R}(U, V)W = \tilde{\nabla}_U\tilde{\nabla}_V W - \tilde{\nabla}_V\tilde{\nabla}_U W - \tilde{\nabla}_{[U,V]}W.$$

Thus, using the relation (5.4), we obtain the relation between curvature tensor vector $\tilde{R}$ and R of $\mathcal{M}$ in Q^m admitting SSMC $\tilde{\nabla}$ and LC connection ∇ given by

$$\begin{aligned}\tilde{R}(U, V)W &= R(U, V)W + g(\phi SU, W)V - g(\phi SV, W)U + \eta(W)[\eta(V)U - \eta(U)V]\\ &\quad -g(V, W)[\phi SU + U - \eta(U)\xi] + g(U, W)[\phi SV + V - \eta(V)\xi]\end{aligned} \tag{5.5}$$

Then from (5.5), one can easily obtain

$$\begin{aligned}\tilde{R}(U, \xi)W &= R(U, \xi)W + g(\phi SU, W)\xi - \eta(W)\phi SU,\\ \tilde{R}(U, V)\xi &= R(U, V)\xi - \eta(V)\phi SU + \eta(U)\phi SV.\end{aligned}$$

Also for $U, V, W, W' \in T_p\mathcal{M}$, we have

$$\begin{aligned}g(\tilde{R}(V, U)W, W') &= -g(\tilde{R}(U, V)W, W'),\\ g(\tilde{R}(U, V)W', W) &= -g(\tilde{R}(U, V)W, W')\end{aligned}$$

Now, if we assume that $\mathcal{M}$ satisfies $\phi S + S\phi$=0, then we derive

$$\begin{aligned}g(\tilde{R}(W, W')U, V) = g(\tilde{R}(U, V)W, W'),\\ g(\tilde{R}(U, V)W + \tilde{R}(V, W)U + \tilde{R}(W, U)V, W') = 0.\end{aligned}$$

Thus, we are able to state the following results

Theorem 5.1. *Let M be a real hypersurface $\mathcal{M}$ in Q^m admitting SSMC. Then for $U, V, W, W' \in T_p\mathcal{M}$, we have*

(a) The curvature tensor of $\mathcal{M}$ with SSMC is given by (5.5)
(b) $g(\tilde{R}(V, U)W, W') + g(\tilde{R}(U, V)W, W') = 0$
(c) $g(\tilde{R}(U, V)W', W) + g(\tilde{R}(U, V)W, W') = 0$.

Proposition 5.1. *In a real hypersurface $\mathcal{M}$ of Q^m admitting SSMC together with $\phi S + S\phi = 0$, we have*

(a) $g(\tilde{R}(U, V)W, W') - g(\tilde{R}(W, W')U, V) = 0$ for $U, V, W, W' \in T_p\mathcal{M}$
(b) $\mathcal{M}$ holds first Bianchi identity with respect to SSMC.

Proof. By using the assumption, the result follows immediately. □

Relation (5.5) can be rewritten as

$$\begin{aligned}g(\tilde{R}(U, V)W, W') &= g(R(U, V)W, W') + \eta(U)[\eta(W)g(SV, W') - g(SV, W)\eta(W')]\\ &\quad -\eta(V)[\eta(W)g(SU, W') - g(SU, W)\eta(W')] - g(\phi SU, V)g(\phi W, W')\\ &\quad +g(\phi SV, U)g(\phi W, W').\end{aligned}$$

Now, on contracting U and W' in above defined relation, we derive

$$\begin{aligned}\tilde{Ric}(V,W) &= Ric(V,W) - (n-2)g(\phi SV,W) + (n-2)\eta(V)\eta(W) \\ &\quad -g(V,W)\big[\sum_{i=1}^{n} g(\phi Se_i,e_i) + (n-2)\big],\end{aligned} \tag{5.6}$$

where $\tilde{Ric}(V,W)$ and $Ric(V,W)$ are the Ricci tensors of the connection $\tilde{\nabla}$ and ∇ respectively.
Again, by applying contraction on V and W, the scalar curvature $\tilde{\tau}$ with SSMC has the following expression

$$2\tilde{\tau} = 3(n-1) - 2 + g^2(AN,N) + n^2||\tilde{\mathcal{H}}||^2 - ||h||^2 - 2(n-1)g(\phi Se_i,e_i). \tag{5.7}$$

Thus, we have

Lemma 5.1. *In a real hypersurface $\mathcal{M}$ of Q^m admitting SSMC such that $\phi S = S\phi$, we have*

(a) $\tilde{Ric}(V,W) = Ric(V,W) - (n-2)[g(\phi SV,W) + g(\phi V,\phi W)]$
(b) $\tilde{Q}V = QV - (n-2)[V - \eta(V)\xi - \phi SV]$
(c) $\tilde{Ric}(V,\xi)$ *coincides with* $Ric(V,\xi)$

for all $V,W \in T_p\mathcal{M}$.

Proof. Let us assume that $\phi S = S\phi$. Then, we have

$$\begin{aligned} g(\phi Se_i,e_i) &= g(S\phi e_i,e_i) \\ &= -g(\phi Se_i,e_i) \end{aligned}$$

This results $g(\phi Se_i,e_i) = 0$, which together with (5.6) follows (a) and hence (b). By using the assumption and inserting $W = \xi$ in (5.6), we get (c).
Also, we know that the Ricci operator $\tilde{Q}$ of SSMC is defined by

$$\tilde{Ric}(V,W) = g(\tilde{Q}V,W),\ \forall\ V,W \in T_p\mathcal{M}.$$

From this incorporating (5.6) together with the assumption, we have

$$\tilde{Q}V = QV - (n-2)[V - \eta(V)\xi - \phi SV].$$

□

6. Chen's inequality for a real hypersurface $\mathcal{M}$ of Q^m with SSMC

Here, we obtain inequality for the mean curvature, the scalar and the sectional curvature associated with the induced SSMC for a real hypersurfaces $\mathcal{M}$ of Q^m.
Here, we have the squared mean curvature $||\tilde{\mathcal{H}}||^2$ of M in Q^M and the squared norm $||h||^2$ of h as

$$||\tilde{\mathcal{H}}||^2 = \frac{1}{n^2}\big(\sum_{i,j=1}^{n} h_{ij}^{n+1}\big)^2 \ \ and\ \ ||h||^2 = \sum_{i,j=1}^{n}(h_{ij}^{n+1})^2$$

respectively, where $h_{ij}^{n+1} = g(h(e_i,e_j),\ N)$ and the mean curvature vector field $\tilde{\mathcal{H}}$ of $\tilde{\nabla}$ and $\mathcal{H}$ of ∇ are invariant.
Now, the scalar curvature $\tilde{\tau}$ for an orthonormal basis $\{e_i\}_1^n$ reads

$$\tilde{\tau} = \sum_{1\le i<j\le n} \mathcal{K}(e_i \wedge e_j).$$

Theorem 6.1. *Let $\mathcal{M}$ be a real hypersurface of Q^m admitting SSMC $\tilde{\nabla}$. Then, for 2-plane section $\pi \subset T_p\mathcal{M}$ spanned by tangent vectors e_1 and e_2, we have*

$$\begin{aligned}\tilde{\tau}(x) - \mathcal{K}(\pi) \le &(n-2)\Big\{\frac{n^2||\tilde{\mathcal{H}}||^2}{2(n-1)} + \frac{3(n-1)-2}{2(n-2)}\Big\} + g^2(Ae_1,e_2) + \frac{g^2(AN,N)}{2} \\ &+ g(\phi Se_1,e_1) + g(\phi Se_2,e_2) + g^2(JAe_1,e_2) + (n-1)g(\phi Se_i,e_i).\end{aligned} \tag{6.1}$$

Moreover, equality holds in (6.1) at $p \in \mathcal{M}$ if and only if there exist an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ and orthonormal normal frame $\{e_{n+1} = N\}$ of $T_p^{\perp}\mathcal{M}$, such that the matrix of the shape operator S takes the following form

$$S = \begin{pmatrix} p' & 0 & 0 \\ 0 & q' & 0 \\ 0 & 0 & M \end{pmatrix}, \tag{6.2}$$

where M is the diagonal matrix of oerder $n-2$ with diagonal entry $r = p' + q'$

Proof. First of all we put

$$\epsilon = 2\tilde{\tau} - 3(n-1) + 2 - g^2(AN, N) + 2(n-1)g(\phi Se_i, e_i) - \frac{n^2(n-2)}{n-1}||\tilde{\mathcal{H}}||^2 \tag{6.3}$$

Thus, we have

$$n^2||\tilde{\mathcal{H}}||^2 = (n-1)\{\epsilon + ||h||^2\}. \tag{6.4}$$

Moreover, we can write

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\Big\{\epsilon + \sum_{i=1}^{n}(h_{ij}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2\Big\} \tag{6.5}$$

Using lemma (3.1), we obtain

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \epsilon \tag{6.6}$$

The Gauss equation yields

$$\begin{aligned} \tilde{\mathcal{K}}(\pi) &= g(\tilde{R}(e_1,e_2)e_2,e_1) \\ &= 3g^2(\phi e_1,e_2) + g(Ae_2,e_2)g(Ae_1,e_1) - g^2(Ae_1,e_2) + g(JAe_2,e_2)g(JAe_1,e_1) - \left(h_{12}^{n+1}\right)^2 \\ &\quad + h_{22}^{n+1}h_{11}^{n+1} - g(\phi Se_2,e_2) - g(\phi Se_1,e_1) + \eta(e_2)^2 + \eta(e_1)^2 - g^2(JAe_1,e_2) \end{aligned} \tag{6.7}$$

Inserting (6.6) into (6.7) yields

$$\begin{aligned} \tilde{\mathcal{K}}(\pi) &\geq 3g^2(\phi e_1,e_2) + g(Ae_2,e_2)g(Ae_1,e_1) - g^2(Ae_1,e_2) + g(JAe_2,e_2)g(JAe_1,e_1) \\ &\quad - g^2(JAe_1,e_2) + \frac{1}{2}\{\sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \epsilon\} - \left(h_{12}^{n+1}\right)^2 - g(\phi Se_2,e_2) - g(\phi Se_1,e_1) \\ &\quad + \eta(e_2)^2 + \eta(e_1)^2 \\ &= 3g^2(\phi e_1,e_2) + g(Ae_2,e_2)g(Ae_1,e_1) - g^2(Ae_1,e_2) + g(JAe_2,e_2)g(JAe_1,e_1) \\ &\quad - g^2(JAe_1,e_2) + \frac{\epsilon}{2} + \left(h_{12}^{n+1}\right)^2 + \sum_{i\neq j, i,j\geq 2}\left(h_{ij}^{n+1}\right)^2 - \left(h_{12}^{n+1}\right)^2 - g(\phi Se_2,e_2) \\ &\quad - g(\phi Se_1,e_1) + \eta(e_2)^2 + \eta(e_1)^2 \\ &\geq 3g^2(\phi e_1,e_2) + g(Ae_2,e_2)g(Ae_1,e_1) - g^2(Ae_1,e_2) + g(JAe_2,e_2)g(JAe_1,e_1) \\ &\quad - g^2(JAe_1,e_2) - g(\phi Se_2,e_2) - g(\phi Se_1,e_1) + \eta(e_2)^2 + \eta(e_1)^2 + \tilde{\tau} \\ &\quad - \frac{3(n-1)-2}{2} - \frac{g^2(AN,N)}{2} - \frac{n^2(n-2)}{2(n-1)}||\tilde{\mathcal{H}}||^2 + (n-1)g(\phi Se_i,e_i) \end{aligned}$$

Thus, we derive

$$\begin{aligned} \tilde{\tau}(p) - \tilde{\mathcal{K}}(\pi) &\leq (n-2)\Big\{\frac{n^2||\tilde{\mathcal{H}}||^2}{2(n-1)} + \frac{3(n-1)-2}{2(n-2)}\Big\} - 3g^2(\phi e_1,e_2) - g(Ae_2,e_2)g(Ae_1,e_1) \\ &\quad + g^2(Ae_1,e_2) - g(JAe_2,e_2)g(JAe_1,e_1) + g^2(JAe_1,e_2) + g(\phi Se_2,e_2) \\ &\quad + g(\phi Se_1,e_1) - \eta(e_2)^2 - \eta(e_1)^2 + \frac{g^2(AN,N)}{2} - (n-1)g((\phi Se_i,e_i) \end{aligned}$$

or, equivalently

$$\tilde{\tau}(p) - \tilde{\mathcal{K}}(\pi) \leq (n-2)\left\{\frac{n^2||\tilde{\mathcal{H}}||^2}{2(n-1)} + \frac{3(n-1)-2}{2(n-2)}\right\} + g^2(Ae_1, e_2) + g^2(JAe_1, e_2)$$
$$+ \frac{g^2(AN, N)}{2} + g(\phi Se_1, e_1) + g(\phi Se_2, e_2) + (n-1)g(\phi Se_i, e_i) \tag{6.8}$$

Now, finally we get equality in (6.1) at $p \in \mathcal{M}$ if and only if we have the equality case of lemma.

$$h_{ij}^{n+1} = 0 \text{ for all } i \neq j,$$
$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = h_{44}^{n+1} = \ldots = h_{n\,n}^{n+1}.$$

Thus, we may have the choice for $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$. Hence, the matrix of the shape operator has the form (6.2). □

Corollary 6.1. *Let $\mathcal{M}$ be a real hypersurface of Q^m admitting SSMC $\tilde{\nabla}$. Then, for a $\mathcal{U}$-principal normal vector field, we have*

$$\tilde{\tau}(p) - \mathcal{K}(\pi) \leq (n-2)\left\{\frac{n^2||\tilde{\mathcal{H}}||^2}{2(n-1)} + \frac{3(n-1)-1}{2(n-2)}\right\} + g^2(Ae_1, e_2) + g^2(JAe_1, e_2)$$
$$+ g(\phi Se_1, e_1) + g(\phi Se_2, e_2) + (n-1)g(\phi Se_i, e_i),$$

where $\pi \subset T_p\mathcal{M}$ is a 2-plane section spanned by tangent vectors e_1 and e_2.

Corollary 6.2. *Let $\mathcal{M}$ be a real hypersurface of Q^m admitting SSMC $\tilde{\nabla}$. Then, for a $\mathcal{U}$-isotropic normal vector field, we have*

$$\tilde{\tau}(p) - \mathcal{K}(\pi) \leq (n-2)\left\{\frac{n^2||\tilde{\mathcal{H}}||^2}{2(n-1)} + \frac{3(n-1)-2}{2(n-2)}\right\} + g^2(Ae_1, e_2) + g^2(JAe_1, e_2)$$
$$+ g(\phi Se_1, e_1) + g(\phi Se_2, e_2) + (n-1)g(\phi Se_i, e_i),$$

where $\pi \subset T_p\mathcal{M}$ is a 2-plane section spanned by tangent vectors e_1 and e_2.

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