Extremities Involving B. Y. Chen’s Invariants for Real Hypersurfaces in Complex Quadric

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ABSTRACT

The article is concerned with the study of real hypersurfaces of the complex quadric $Q^m$. We establish B. Y. Chen’s inequalities for real hypersurfaces of the complex quadric $Q^m$ and by considering the equality case, we obtain some consequences. Also, we establish an inequality in terms of the warping function and the scalar curvature for a warped product real hypersurface of $Q^m$ and some obstructions have been given. Moreover, we investigate the expression of the curvature tensor of a real hypersurface in the complex quadric $Q^m$ admitting semi-symmetric metric connection. Using this curvature, we derive inequalities involving Chen $\delta$-invariant admitting a semi-symmetric metric connection. Furthermore, the equality case is considered.

Keywords: real hypersurface; complex quadric; scalar curvature; Chen $\delta$-invariant; semi-symmetric metric connection.

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1. Introduction

In 1968, S. S. Chern raised a question involving minimal isometric immersion into Euclidean space [12]. Then, Chen found some obstructions to Chern’s problem and proposed inequalities for submanifolds in Riemannian space form concerning the sectional curvature, the scalar curvature and the squared mean curvature [9]. Moreover, he proposed inequality concerning $\delta(n_1, n_2, ..., n_k)$ and the squared mean curvature for the submanifolds in real space form [10].

Afterwards, many papers have been appeared in submanifolds of space forms in the version of real and complex like, generalised complex space forms [11], $(k, \mu)$-contact space forms [1] and Sasakian space forms [13]. Further, the geometry of the complex quadric has been studied by H. Reckziegel [16] in 1995 and Y. J. Suh, obtained some analyzing results on real hypersurfaces in the complex quadric by considering some geometric conditions like parallel Ricci tensor [17], Reeb parallel shape operator [18]. Also, the classifications of real hypersurfaces of the complex quadric with isometric Reeb flow were obtained by Berndt and Suh [5] and many more work have been studied by different authors considering the same ambient space ([2]-[4],[19]).

However, Hayden [14] originated the idea of a semi-symmetric metric connection on a Riemannian manifold. Yano [20] deliberated this connection and found some properties of a Riemannian manifold with the same connection. Also, A. Mihai and C. Özgür studied the Chen extremities for submanifolds of the real space forms with same connection [15].

Here, we first establish Chen’s extremities for real hypersurfaces of the complex quadric $Q^m$ and considering the equality case, we obtain some consequences. Also, we establish an inequality in terms of the warping function and the scalar curvature for warped product real hypersurface of $Q^m$ and some obstructions have been given. Then, we study real hypersurface of $Q^m$ admitting semi-symmetric metric connection and find the curvature tensor of a real hypersurface in $Q^m$ with the semi-symmetric metric connection. Additionally, using this curvature we develop Chen’s inequality for a real hypersurfaces of the complex quadric $Q^m$ admitting semi symmetric metric connection.

As long as, by virtue of simpleness, throughout a paper we denote semi-symmetric metric connection, Levi-Civita connection and Warped product by SSMC, LC connection and WP, respectively.
2. The complex quadric $Q^m$

For more details of the geometry of complex quadric we refer to ([5],[16],[17]). The complex hypersurface of $\mathbb{CP}^{m+1}$ is known as the complex quadric $Q^m$ defined by the equation $z_1^2 + \ldots + z_{m+1}^2 = 0$, where $z_1, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{CP}^{m+1}$ equipped with the induced Riemannian metric $g$. Then, naturally the canonical Kähler structure $(J,g)$ on $Q^m$ is induced by Kähler structure on $\mathbb{CP}^{m+1}$ [18]. The 1-dimensional quadric $Q^1$ is congruent to the round 2-sphere $S^2$. The 2-dimensional quadric $Q^2$ is congruent to the Riemannian product $S^2 \times S^2$. For this, we will assume $m \geq 3$ throughout the paper.

Apart from $J$ there is one more geometric structure on $Q^m$, known as the complex conjugation $A$ on the tangent spaces of $Q^m$ which is a parallel rank-two vector bundle $U$ containing $S^1$-bundle of real structures. For $x \in Q^m$, let $A_x$ be the shape operator of $Q^m$ in $\mathbb{CP}^{m+1}$. Then we have $A_x W = W$ for $W \in T_xQ^m$, that is, $A$ is an involution or $A$ is a complex conjugation restricted to $T_x Q^m$. Now, $T_x Q^m$ is decomposed as [18]:

$$ T_x Q^m = V(A_x) \oplus JV(A_x), $$

such that $V(A_x)$ and $JV(A_x)$, respectively denote the $(+1)$-eigenspace and $(-1)$-eigenspace of the involution $A_x^2 = I$ on $T_x Q^m$, $x \in Q^m$.

Now, a tangent vector $W \neq 0$ in $T_x Q^m$ is known as the singular if it is tangent to more than one maximal flat in $Q^m$. Classification of singular tangent vectors for $Q^m$ are given as [19]:

1. If there exists $A \in U$ such that $W$ is an eigenvector corresponding to an eigenvalue (+1), then the singular tangent vector $W$ is known as $U$-principal.
2. If there exists $A \in U$ and orthonormal vectors $U,V \in \mathcal{V}(A)$ such that $W/\|W\| = (U + JV)/\sqrt{2}$, then the singular tangent vector $W$ is known as $U$-isotropic.

Let $\mathcal{M}^n$ be a real hypersurface of $Q^m$ with a connection $\nabla$ induced from the LC connection $\nabla$ in $Q^m$. Then, the transform $JU$ of the Kähler structure $J$ on $Q^m$ is defined by $JU = \phi U + \eta(U)N$ where $\phi U$ is the tangential component of $JU$ and $N \in T^*_U \mathcal{M}$, for $U \in T_p \mathcal{M}$. Here, $\mathcal{M}$ associates an induced almost contact metric structure $(\phi, \xi, \eta, g)$ satisfying the following relations [6]:

$$ \xi = -JN, \eta(\xi) = 1, \eta(U) = g(\xi, U), \phi^2 U + U = \eta(U)\xi, \phi \xi = 0, $$

$$ \eta(\phi U) = 0, g(\phi U, \phi V) + \eta(U)\eta(V) = g(U, V), g(\phi U, V) = -g(U, \phi V). $$

Moreover, the real hypersurface $\mathcal{M}$ of $Q^m$ satisfy

$$ \nabla_U \xi = \phi SU, $$

where $S$ is the shaper operator of $\mathcal{M}$.

On the other hand, the Gauss and the Weingarten formulas for $M$ follows

$$ \nabla_U V = \nabla_U V + h(U, V) \quad \text{and} \quad \nabla_U N = -SU, $$

respectively, for $U, V \in T_p \mathcal{M}$ and $N \in T^*_p \mathcal{M}$. The second fundamental form $h$ and the shape operator $S$ of $\mathcal{M}$ are related by

$$ g(h(U, V), N) = g(SN U, V) = g(SU, V). $$

Now, we take $A \in U_x$ such that $N = \cos(t)Z_1 + \sin(t)JZ_2$, where $Z_1, Z_2$ are orthonormal vectors in $\mathcal{V}(A)$ and $0 \leq t \leq \frac{\pi}{2}$ (see Proposition 3 [16]) which is a function on $\mathcal{M}$. Since $\xi = -JN$, we have

$$ N = \cos(t)Z_1 + \sin(t)JZ_2, $$

$$ AN = \cos(t)Z_1 - \sin(t)JZ_2, $$

$$ \xi = \sin(t)Z_2 - \cos(t)JZ_1, $$

$$ A\xi = \sin(t)Z_2 + \cos(t)JZ_1, $$

from which it follows that $g(\xi, AN) = 0$. 

35
3. B. Y. Chen inequality for a real hypersurface of $Q^m$

Here, we obtain the general inequality associated with the Chen $\delta$-invariant for a real hypersurfaces $\mathcal{M}$ of the complex quadric $Q^m$.

Now, from the Gauss equation, the Riemannian curvature tensor $R$ of connection $\nabla$ in terms of $J$ and $A \in \mathcal{U}$ is defined as [18]:

$$R(U, V)W = g(V, W)U - g(U, W)V + g(\phi V, W)\phi U - g(\phi U, W)\phi V - 2g(\phi U, V)\phi W + g(AV, W)AU - g(AU, W)AV + g(JAV, W)JAU - g(JAU, W)JAV + g(SV, W)SU - g(SU, W)SV,$$

where $U, V, W \in T_p\mathcal{M}$.

Then, we can see

$$g(R(U, V)W + R(V, W)U + R(W, U)V, W') = 0, \quad \text{for } U, V, W, W' \in T_p\mathcal{M}$$

that is, the first Bianchi Identity holds for $\mathcal{M}$ of LC connection $\nabla$.

Next, the curvature tensor $R$ of the Hopf hypersurface $\mathcal{M}$ (i.e. $\alpha = g(S\xi, \xi)$), where $\alpha$ is a smooth function on $\mathcal{M}$ satisfies

$$R(U, \xi)V = \eta(V)[U + \alpha SU] - [g(U, V) + \alpha g(SU, V)]\xi + g(A\xi, V)AU - g(AU, V)A\xi - g(AN, V)JAU + g(JAU, V)AN,$$

$$R(U, V)\xi = \eta(V)[U + \alpha SU] - \eta(U)[V + \alpha SV] + g(AV, \xi)AU - g(AU, \xi)AV - g(AN, V)JAU + g(AU, N)JAV.$$

Moreover, for a real hypersurface $\mathcal{M}$ and $U, V, W, W' \in T_p\mathcal{M}$, the relation (3.1) produce

$$g(R(U, V)W, W') = g(V, W)g(U, W') - g(U, W)g(V, W') + g(\phi V, W)g(\phi U, W') - g(\phi U, W)g(\phi V, W') - g(\phi U, V)g(\phi W, W') + g(AV, W)g(\phi W, W') - g(AW, V)g(\phi V, W') + g(JAV, W)g(\phi W, W') - g(JAW, W)g(\phi V, W') + g(SV, W)g(\phi SU, W') - g(SU, W)g(\phi SV, W').$$

By taking $U = W' = e_i$ in (3.3), one can have [17]

$$\text{Ric}(V, W) = n\eta(V, W) - 3\eta(V)\eta(W) - g(AN, V)g(AV, W) + g(AW, N)g(AV, N) + g(AW, \xi)g(AV, \xi) + \text{tr}(S)g(SV, W) - g(S^2V, W),$$

where the Ricci tensor of $\mathcal{M}$ with connection $\nabla$ is symbolized by $\text{Ric}$ which satisfy

$$\text{Ric}(U, \xi) = (2n - 4 + \alpha h - \alpha^2)\eta(X) - 2g(AN, N)g(AU, \xi).$$

Consider an orthonormal basis $\{e_i\}^n_1$ and $\{e_{n+1}\} = N$ of $T_p\mathcal{M}$ and $T_p^p\mathcal{M}$ respectively, where $n + 1 = 2m$. Conveniently, let $h_{ij}^{n+1} = g(h(e_i, e_j), e_{n+1}) = g(h(e_i, e_j), N)$ for $i, j \in \{1, \ldots, n\}$. Now, one defines the squared mean curvature $||\mathcal{H}||^2$ of $\mathcal{M}$ in $Q^m$ and the squared norm $||h||^2$ of $h$ are given by:

$$||\mathcal{H}||^2 = \frac{1}{n^2}(\sum_{i,j=1}^{n} h_{ij}^{n+1})^2, \quad ||h||^2 = \sum_{i,j=1}^{n} (h_{ij}^{n+1})^2,$$

respectively.

Now, the scalar curvature $\tau$ has the expression

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where $K(e_i)$ denotes the sectional curvature of $\mathcal{M}$ involved with a plane section $\pi \subset T_p\mathcal{M}$ and is spanned by tangent vectors $\{e_i, e_j\}$ and $\sum_{1 \leq i < j \leq n} K(e_i \wedge e_j) = \sum_{1 \leq i < j \leq n} g(R(e_i, e_j)e_i, e_j)$.

Revoke that the Chen first invariant ([9],[10]) is defined by

$$\delta_m(p) = \tau(p) - \inf\{K(e_i) | \pi \subset T_p\mathcal{M}, \dim \pi = 2\},$$

where $\tau(p)$ is the scalar curvature at $p$.

We give one algebraic result which we will use to proof our result.

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Lemma 3.1. [9] Let \( a_1, a_2, ..., a_k, b \) be \((k + 1)(k + 2)\) real numbers satisfying
\[
(\sum_{i=1}^{k} a_i)^2 = (k - 1)(\sum_{i=1}^{k} a_i^2 + b)
\]
Then \( 2a_1a_2 \geq b \), with equality holding if and only if \( a_1 + a_2 = a_3 = ... = a_k \).

Theorem 3.1. For a real hypersurface \( M \) of \( Q^n \) with 2-plane section \( \pi \subset T_pM \) spanned by tangent vectors \( e_1 \) and \( e_2 \), we have
\[
\tau(p) - K(\pi) \leq \frac{n^2}{2 \left( 1 + \frac{n-2}{n-1} ||\mathcal{H}||^2 \right)} + g^2(AN, N) + g^2(JAe_1, e_2).
\]
Moreover, equality holds in (3.5) at \( p \in M \) if and only if there exist an orthonormal basis \( \{e_i\}_{n+1}^n \) of \( T_pM \) and orthonormal normal frame \( \{e_{n+1} = N\} \) of \( T_p^1M \), such that the matrix of the shape operator \( S \) takes the following form
\[
S = \begin{pmatrix}
  p' & 0 & 0 \\
  0 & q' & 0 \\
  0 & 0 & M
\end{pmatrix},
\]
where \( M \) is the diagonal matrix of order \( n - 2 \) with diagonal entry \( r = p' + q' \).

Proof. From (3.4), we deduce that
\[
2\tau = n^2 + g^2(AN, N) - 1 + n^2||\mathcal{H}||^2 - ||h||^2
\]
where we have used
\[
||h||^2 = g(h(e_i, e_j), h(e_i, e_j)) = g(g(Se_i, e_j)N, g(Se_i, e_j)N) = \text{tr}(S^2).
\]
Let us denote
\[
\epsilon = 2\tau - n^2 - g^2(AN, N) + 1 - \frac{n^2(n-2)}{n-1}||\mathcal{H}||^2.
\]
We obtain
\[
\epsilon = n^2||\mathcal{H}||^2 - ||h||^2 - \frac{n^2(n-2)}{n-1}||\mathcal{H}||^2
\]
which provide
\[
n^2||\mathcal{H}||^2 = (n-1)\{\epsilon + ||h||^2\}.
\]
or, equivalently
\[
(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\epsilon + \sum_{i=1}^{n} (h_{ij}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2\}.
\]
Using lemma (3.1) together with equation (3.10), we obtain
\[
2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i \neq j} (h_{ij}^{n+1})^2 + \epsilon.
\]
Also, the Gauss equation implies that
\[
K(\pi) = g(R(e_1, e_2)e_2, e_1)
= 1 + 3g^2(\phi e_1, e_2) + g(Ae_2, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2)
+ g(JAe_2, e_2)g(JAe_1, e_1) - g^2(JAe_1, e_2) - (h_{11}^{n+1})^2 + h_{22}^{n+1}h_{11}^{n+1}.
\]
Incorporating (3.11) in (3.12) yields
\[ K(\pi) \geq 1 + 3g^2(\varphi e_1, e_2) + g(Ae_2, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2) + g(J Ae_2, e_2)g(J Ae_1, e_1) \\
- g^2(J Ae_1, e_2) + \frac{1}{2} \left\{ \sum_{i \neq j} (h_{ij}^{n+1})^2 + \epsilon \right\} - (h_{12}^{n+1})^2 \]
\[ = 1 + 3g^2(\varphi e_1, e_2) + g(Ae_2, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2) + g(J Ae_2, e_2)g(J Ae_1, e_1) \\
- g^2(J Ae_1, e_2) + \tau - n_2 \frac{\|H\|^2}{2} + \frac{1}{2} - n_2^{(n-2) \frac{\|H\|^2}{2} + 1} + \frac{1}{2} \sum_{i \neq j, i, j \geq 2} (h_{ij}^n)^2 \]
\[ \geq 1 + 3g^2(\varphi e_1, e_2) + g(Ae_2, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2) + g(J Ae_2, e_2)g(J Ae_1, e_1) \\
- g^2(J Ae_1, e_2) + \tau - n_2 \frac{\|H\|^2}{2} + \frac{1}{2} - n_2^{(n-2) \frac{\|H\|^2}{2} + 1} \]

Thus, finally we have
\[ \tau(p) - K(\pi) \leq \frac{n^2}{2} \left\{ 1 + \left( \frac{n-2}{n-1} \right) \|H\|^2 \right\} + g^2(Ae_1, e_2) + g^2(J Ae_1, e_2) + \frac{g^2(AN, N)}{2} \]

or
\[ \tau(p) - K(\pi) \leq \frac{n^2}{2} \left\{ 1 + \left( \frac{n-2}{n-1} \right) \|H\|^2 \right\} + g^2(Ae_1, e_2) + g^2(J Ae_1, e_2) + \frac{1}{1} \]

(3.13)

Now, finally we get the equality in (13) at \( p \in M \) if and only if we have the equality case of lemma i.e.,
\[ h_{ij}^{n+1} = 0 \quad \text{for all} \quad i \neq j, \]
\[ h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = h_{44}^{n+1} = ... = h_{nn}^{n+1}. \]

Thus, we may have the choice for \( \{e_1, e_2\} \) such that \( h_{12}^{n+1} = 0 \). Hence, the matrix of the shape operator has the form (3.6).

**Corollary 3.1.** Let \( M \) be a real hypersurface of \( Q^n \) with 2-plane section \( \pi \subset T_p M \) spanned by tangent vectors \( e_1 \) and \( e_2 \) such that the normal vector field is \( U \)-principal. Then, we have
\[ \tau(p) - K(\pi) \leq \frac{n^2}{2} \left\{ 1 + \left( \frac{n-2}{n-1} \right) \|H\|^2 \right\} + g^2(Ae_1, e_2) + g^2(J Ae_1, e_2) + \frac{1}{1} \]

(3.14)

Moreover, equality holds in (3.14) at \( p \in M \) if and only if there exist an orthonormal basis \( \{e_i\}^n \) of \( T_p M \) and orthonormal normal frame \( \{e_{n+1} = N\} \) of \( T_p^\perp M \), such that the matrix of the shape operator \( S \) takes the following form
\[ S = \begin{pmatrix} p' & 0 & 0 \\
0 & q' & 0 \\
0 & 0 & M \end{pmatrix}, \]
(3.15)

where \( M \) is the diagonal matrix of order \( n - 2 \) with diagonal entry \( r = p' + q' \).

**Corollary 3.2.** Let \( M \) be a real hypersurface of \( Q^n \) with 2-plane section \( \pi \subset T_p M \) spanned by tangent vectors \( e_1 \) and \( e_2 \) such that the normal vector field is \( U \)-isotropic. Then, we have
\[ \tau(p) - K(\pi) \leq \frac{n^2}{2} \left\{ 1 + \left( \frac{n-2}{n-1} \right) \|H\|^2 \right\} + g^2(Ae_1, e_2) + g^2(J Ae_1, e_2) \]

(3.16)

Moreover, equality holds in (3.16) at \( p \in M \) if and only if there exist an orthonormal basis \( \{e_i\}^n \) of \( T_p M \) and orthonormal normal frame \( \{e_{n+1} = N\} \) of \( T_p^\perp M \), such that the matrix of the shape operator \( S \) takes the following form
\[ S = \begin{pmatrix} p' & 0 & 0 \\
0 & q' & 0 \\
0 & 0 & M \end{pmatrix}, \]
(3.17)

where \( M \) is the diagonal matrix of order \( n - 2 \) with diagonal entry \( r = p' + q' \).
4. WP real hypersurface of $Q^m$

In this section, we develop inequalities involving the warping function of a WP real hypersurface $M$ of $Q^m$.

Next, we consider two Riemannian manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of dimensions $n_1$ and $n_2$ equipped with Riemannian metrics $\varsigma_1$ and $\varsigma_2$ respectively. Let $\varsigma$ be a positive function on $\mathcal{M}_1$. The WP manifold $\mathcal{M}_1 \otimes \varsigma \mathcal{M}_2$ is defined to be the product manifold $\mathcal{M}_1 \otimes \mathcal{M}_2$ with the warped metric $g = \varsigma_1 + \varsigma_2^2$ [7].

Consider an isometric immersion $\Psi : M = \mathcal{M}_1 \otimes \varsigma \mathcal{M}_2 \rightarrow Q^m$ of a WP manifold $\mathcal{M}_1 \otimes \varsigma \mathcal{M}_2$ into a Riemannian manifold $Q^m$. Let $h$ be the second fundamental form of $\Psi$ and the mean curvature vectors denoted by $\mathcal{H}_i = \frac{1}{n_i} \mathcal{H}(h_i)$ where $\mathcal{H}(h_i)$ is the trace of $h$ restricted to $\mathcal{M}_i (i = 1, 2)$.

**Theorem 4.1.** Let $\Psi : M^n = M_1 \otimes \varsigma \mathcal{M}_2 \rightarrow Q^m$ be an isometric immersion of a WP real hypersurface into $Q^m$ with $\varsigma \in T_p M_1$. Then

$$n_2 \frac{\Delta \varsigma}{\varsigma} \leq -\frac{n^2}{2} + \frac{1}{2} g^2 (\mathcal{H}N, N) - 2n + 2n_1n_2 + \frac{11}{2} + \frac{n^2}{4} ||\mathcal{H}||^2 + 2 \sum_{i=1}^{n} g(A^2 e_i, e_i),$$

where $n_i = \text{dim} \mathcal{M}_i$ for $i = 1, 2$, $\Delta$ is the Laplacian operator of $\mathcal{M}_1$ and $\{ e_i \}_{1}^{n}$ is an orthonormal basis of $T_p M$.

**Proof.** Let us consider an isometric immersion $\Psi : M = \mathcal{M}_1 \otimes \varsigma \mathcal{M}_2 \rightarrow N(s)$ of a WP real hypersurface $\mathcal{M}_1 \otimes \varsigma \mathcal{M}_2$ into $Q^m$ whose structure vector field $\varsigma \in T_p M_1$. Then, one can easily have [8]

$$\mathcal{K}(X \wedge Z) = \frac{1}{\varsigma} (\nabla_X \varsigma) - \varsigma X^2 \varsigma.)$$

Now we choose an orthonormal basis $\{ e_i \}_{1}^{n}$ of $T_p M$ such that $e_1, ..., e_{n_1}$ are tangent to $M_1$ and $e_{n_1+1}, ..., e_n$ are tangent to $M_2$. Then, with the virtue of above defined relation, we obtain

$$\frac{\Delta \varsigma}{\varsigma} = \sum_{1 \leq i \leq n_1} \sum_{1 \leq j \leq n} \mathcal{K}(e_i \wedge e_j). \quad (4.1)$$

By definition of scalar curvature $\tau$ and (4.1) yields

$$n_2 \frac{\Delta \varsigma}{\varsigma} = \tau - \sum_{1 \leq i \leq n_1} \mathcal{K}(e_i \wedge e_j) - \sum_{n_1+1 \leq j \leq n} \mathcal{K}(e_i \wedge e_j) \quad (4.2)$$

From (3.7), we have

$$n^2 ||\mathcal{H}||^2 = 2(\delta + ||h||^2) \quad (4.3)$$

where

$$\delta = 2 \tau - n^2 - g^2 (\mathcal{H}N, N) + 1 - \frac{n^2}{2} ||\mathcal{H}||^2. \quad (4.4)$$

Moreover, in local coordinates (4.3) has the following expression

$$\left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = 2 \left( \delta + \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 \right)$$

or, equivalently

$$\left( h_{11}^{n+1} + \sum_{i=2}^{n_1} h_{ii}^{n+1} + \sum_{i=n_1+1}^{n} h_{ii}^{n+1} \right)^2 = 2 \left\{ \delta + \left( h_{11}^{n+1} \right)^2 + \sum_{i=2}^{n_1} (h_{ii}^{n+1})^2 + \sum_{i=n_1+1}^{n} (h_{ii}^{n+1})^2 \right\}$$

$$+ \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2$$

$$= 2 \left\{ \delta + \left( h_{11}^{n+1} \right)^2 + \left( \sum_{i=2}^{n_1} h_{ii}^{n+1} \right)^2 \right\} - \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n_1+1} h_{kk}^{n_1+1}$$

$$+ \sum_{i=1}^{n} h_{ii}^{n+1} - \sum_{n_1+1 \leq j \neq k \leq n} h_{jj} h_{kk} + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 \right\}$$
Using lemma (3.1), we have
\[ \sum_{1 \leq j \neq k \leq n_1} h_{ij}^{n+1} h_{jk}^{n+1} + \sum_{n_1 + 1 \leq j \neq k \leq n} h_{ij}^{n+1} h_{jk}^{n+1} \geq \frac{\delta}{2} + \sum_{1 \leq i < j \leq n} (h_{ij}^{n+1})^2 \] (4.5)

Furthermore, equality holds if and only if
\[ \sum_{i = 1}^{n_1} h_{ii}^{n+1} = \sum_{i = n_1 + 1}^{n} h_{ii}^{n+1} \] (4.6)

We also know that
\[ \tau = \sum_{1 \leq i < j \leq n} \mathcal{K}(e_i \wedge e_j) \]
\[ = \sum_{1 \leq i < j \leq n_1} \mathcal{K}(e_i \wedge e_j) + \sum_{n_1 + 1 \leq i < j \leq n} \mathcal{K}(e_i \wedge e_j) + \sum_{j = n_1 + 1}^{n} \sum_{i = 1}^{n_1} \mathcal{K}(e_i \wedge e_j) \]

So, from (4.1), we derive
\[ n_2 \frac{\Delta \zeta}{\zeta} = \sum_{j = n_2 + 1}^{n} \sum_{i = 1}^{n_1} \mathcal{K}(e_i \wedge e_j) \]
\[ = \tau - \sum_{1 \leq i < j \leq n_1} \mathcal{K}(e_i \wedge e_j) - \sum_{n_1 + 1 \leq i < j \leq n} \mathcal{K}(e_i \wedge e_j) \]
\[ = \tau - n_1(n_1 - 1) - 3(n_1 - 1) - n_2(n_2 - 1) - 3(n_2 - 1) + 2 \sum_{i = 1}^{n} g(A^2 e_i, e_i) \]
\[ - \left\{ \sum_{1 \leq i < j \leq n_1} g(A e_i, e_i) g(A e_j, e_j) + \sum_{n_1 + 1 \leq i < j \leq n} g(A e_i, e_i) g(A e_j, e_j) \right\} \]
\[ - \left\{ \sum_{1 \leq i < j \leq n_1} g(J A e_i, e_i) g(J A e_j, e_j) + \sum_{n_1 + 1 \leq i < j \leq n} g(J A e_i, e_i) g(J A e_j, e_j) \right\} \]
\[ - \left\{ \sum_{1 \leq i < j \leq n_1} h_{ii}^{n+1} h_{jj}^{n+1} + \sum_{n_1 + 1 \leq i < j \leq n} h_{ii}^{n+1} h_{jj}^{n+1} \right\} + \left\{ \sum_{1 \leq i < j \leq n_1} (h_{ij}^{n+1})^2 \right\} \]
\[ + \sum_{n_1 + 1 \leq i < j \leq n} (h_{ij}^{n+1})^2 \}

Using (4.5), we have
\[ n_2 \frac{\Delta \zeta}{\zeta} \leq \tau - n_1(n_1 - 1) - 3(n_1 - 1) - n_2(n_2 - 1) - 3(n_2 - 1) + 2 \sum_{i = 1}^{n} g(A^2 e_i, e_i) \]
\[ - \left\{ \sum_{1 \leq i < j \leq n_1} g(A e_i, e_i) g(A e_j, e_j) + \sum_{n_1 + 1 \leq i < j \leq n} g(A e_i, e_i) g(A e_j, e_j) \right\} \]
\[ - \left\{ \sum_{1 \leq i < j \leq n_1} g(J A e_i, e_i) g(J A e_j, e_j) + \sum_{n_1 + 1 \leq i < j \leq n} g(J A e_i, e_i) g(J A e_j, e_j) \right\} \]
\[ - \left\{ \frac{1}{2} \delta + \sum_{1 \leq i < j \leq n_1} (h_{ij}^{n+1})^2 \right\} + \left\{ \sum_{1 \leq i < j \leq n_1} (h_{ij}^{n+1})^2 + \sum_{n_1 + 1 \leq i < j \leq n} (h_{ij}^{n+1})^2 \right\} \]

which gives
\[ n_2 \frac{\Delta \zeta}{\zeta} \leq \tau - n_1(n_1 - 1) - 3(n_1 - 1) - n_2(n_2 - 1) - 3(n_2 - 1) + 2 \sum_{i = 1}^{n} g(A^2 e_i, e_i) - \frac{1}{2} \delta. \]
Incorporating (4.4) with the above relation, we derive
\[ n_2 \frac{\Delta \zeta}{\zeta} \leq -\frac{n^2}{2} - 2n + 2n_1n_2 + \frac{11}{2} + \frac{n^2}{4}||H||^2 + 2\sum_{i=1}^{n} g(A^2e_i, e_i) - \frac{1}{2} \delta \]
from which we conclude our result.

**Corollary 4.1.** Let \( \Psi : M^n = M_1 \otimes_\zeta M_2 \to Q^m \) be an isometric immersion of a WP real hypersurface into \( Q^m \) with \( \xi \in T_pM_1 \). Then, for a \( \eta \)-principal normal vector field, we have the inequality
\[ n_2 \frac{\Delta \zeta}{\zeta} \leq -\frac{n^2}{2} - 2n + 2n_1n_2 + 6 + \frac{n^2}{4}||H||^2 + 2\sum_{i=1}^{n} g(A^2e_i, e_i) \]
where \( n_i = \text{dim}M_i, \) for \( i = 1, 2, \Delta \) is the Laplacian operator of \( M_1 \) and \( \{e_i\}_n \) is an orthonormal basis of \( T_pM \).

**Corollary 4.2.** Let \( \Psi : M^n = M_1 \otimes_\zeta M_2 \to Q^m \) be an isometric immersion of a WP real hypersurface into \( Q^m \) with \( \xi \in T_pM_1 \). Then, for a \( \eta \)-isotropic normal vector field, we have the inequality
\[ n_2 \frac{\Delta \zeta}{\zeta} \leq -\frac{n^2}{2} - 2n + 2n_1n_2 + \frac{11}{2} + \frac{n^2}{4}||H||^2 + 2\sum_{i=1}^{n} g(A^2e_i, e_i) \]
where \( n_i = \text{dim}M_i, \) for \( i = 1, 2, \Delta \) is the Laplacian operator of \( M_1 \) and \( \{e_i\}_n \) is an orthonormal basis of \( T_pM \).

### 5. Curvature tensor of real hypersurface \( M \) in \( Q^m \) admitting SSMC

In this section, we study SSMC and then we obtain the curvature tensor of a real hypersurface \( M \) in \( Q^m \) with respect to SSMC and then we find the intrinsic scalar curvature with respect to SSMC.

Consider a Riemannian manifold \( (M^n, g) \) with linear connection \( \overline{\nabla} \). Then, \( \overline{\nabla} \) is called *semi-symmetric connection* [20] if its torsion tensor \( \overline{T} \), defined by
\[ \overline{T}(U, V) = \overline{\nabla}_U V - \overline{\nabla}_V U - [U, V], \] (5.1)
satisfy
\[ \overline{T}(U, V) = \eta(V)U - \eta(U)V, \] (5.2)
for \( U, V \in T_pM \) and a 1-form \( \eta \). In addition, a semi-symmetric linear connection is said to be SSMC \( \overline{\nabla} \) if it holds
\[ \overline{\nabla} g = 0, \] (5.3)
for all \( U, V \in T_pM \), otherwise it is said to be a *semi-symmetric non-metric connection*.

A SSMC \( \overline{\nabla} \) in terms of the LC connection \( \nabla \) on \( M \) is defined by
\[ \overline{\nabla}_U V = \nabla_U V + \eta(V)U - g(U, V)\zeta, \] (5.4)
for \( U, V \in T_pM \).

Now, let us consider the complex quadric \( Q^m \) admitting SSMC \( \overline{\nabla} \) and the LC connection \( \nabla \). Next, let \( M \) be a real hypersurface of \( Q^m \) with the induced SSMC \( \overline{\nabla} \) and the induced LC connection \( \nabla \). Let \( \overline{\Gamma} \) and \( \Gamma \) be the curvature tensors of \( Q^m \) with respect to the connections \( \overline{\nabla} \) and \( \nabla \) respectively. Put \( \tilde{h} \) as the curvature tensor field of \( \overline{\nabla} \) and \( h \) as the curvature tensor field of \( \nabla \) on \( M \). Then the Gauss formulae with respect to \( \overline{\nabla} \) and \( \nabla \) has the expression
\[ \overline{\nabla}_U V = \overline{\nabla}_U V + \tilde{h}(U, V), \quad \nabla_U V = \nabla_U V + h(U, V) \]
respectively, where \( \tilde{h} \) is the \((0,2)\)-tensor of \( M \) in \( Q^m \) and from these two relations, one can easily get \( \tilde{h}(U, V) = h(U, V) \).
Furthermore, using (5.4) for $U, V \in T_p\mathcal{M}$, we have

$$
\begin{align*}
(\tilde{\nabla}_U \eta)(V) &= (\nabla_U \eta)(V) + g(\phi U, \phi V) = g(\phi SU, V) + g(\phi U, \phi V), \\
(\tilde{\nabla}_U \phi)(V) &= (\nabla_U \phi)(V) - g(U, \phi V)\xi - \eta(V)\phi U \\
&= \eta(V)SU - \eta(V)\phi U - g(SU, V)\xi + g(\phi U, V)\xi,
\end{align*}
$$

and the covariant derivative of torsion tensor of $\tilde{\nabla}$ with respect to SSMC follows

$$(\tilde{\nabla}_U \tilde{T})(V, W) = g(\phi SU, V)W - g(\phi SU, W)V + g(U, V)W - g(U, W)V - \eta(U)[\eta(V)W - \eta(W)V],$$

for $U, V, W \in T_p\mathcal{M}$.

Now, we know the curvature tensor $\tilde{R}$ can be calculated by

$$
\tilde{R}(U, V)W = \tilde{\nabla}_U \tilde{\nabla}_V W - \tilde{\nabla}_V \tilde{\nabla}_U W - \tilde{\nabla}_{[U, V]} W.
$$

Thus, using the relation (5.4), we obtain the relation between curvature tensor vector $\tilde{R}$ and $R$ of $\mathcal{M}$ in $Q^m$ admitting SSMC $\tilde{\nabla}$ and LC connection $\nabla$ given by

$$
\tilde{R}(U, V)W = R(U, V)W + g(\phi SU, W)V - g(\phi SV, W)U + \eta(W)[\eta(V)U - \eta(U)V] - g(U, W)[\phi SU + U - \eta(U)\xi] + g(U, W)[\phi SV + V - \eta(V)\xi] \tag{5.5}
$$

Then from (5.5), one can easily obtain

$$
\begin{align*}
\tilde{R}(U, \xi)W &= R(U, \xi)W + g(\phi SU, W)\xi - \eta(W)\phi SU, \\
\tilde{R}(U, V)\xi &= R(U, V)\xi - \eta(V)\phi SU + \eta(U)\phi SV.
\end{align*}
$$

Also for $U, V, W, W' \in T_p\mathcal{M}$, we have

$$
\begin{align*}
g(\tilde{R}(U, V)W, W') &= -g(\tilde{R}(U, V)W, W'), \\
g(\tilde{R}(U, V)W', W) &= -g(\tilde{R}(U, V)W, W').
\end{align*}
$$

Now, if we assume that $\mathcal{M}$ satisfies $\phi S + S\phi = 0$, then we derive

$$
\begin{align*}
g(\tilde{R}(W, W')U, V) &= g(\tilde{R}(U, V)W, W'), \\
g(\tilde{R}(U, V)W + \tilde{R}(V, W)U + \tilde{R}(W, U)V, W') &= 0.
\end{align*}
$$

Thus, we are able to state the following results

**Theorem 5.1.** Let $\mathcal{M}$ be a real hypersurface $\mathcal{M}$ in $Q^m$ admitting SSMC. Then for $U, V, W, W' \in T_p\mathcal{M}$, we have

(a) The curvature tensor of $\mathcal{M}$ with SSMC is given by (5.5)
(b) $g(\tilde{R}(V, U)W, W') + g(\tilde{R}(U, V)W, W') = 0$
(c) $g(\tilde{R}(U, V)W', W) + g(\tilde{R}(U, V)W, W') = 0$.

**Proposition 5.1.** In a real hypersurface $\mathcal{M}$ of $Q^m$ admitting SSMC together with $\phi S + S\phi = 0$, we have

(a) $g(\tilde{R}(U, V)W, W') - g(\tilde{R}(W, W')U, V) = 0$ for $U, V, W, W' \in T_p\mathcal{M}$
(b) $\mathcal{M}$ holds first Bianchi identity with respect to SSMC.

**Proof.** By using the assumption, the result follows immediately. \(\Box\)

Relation (5.5) can be rewritten as

$$
\begin{align*}
g(\tilde{R}(U, V)W, W') &= g(\tilde{R}(U, V)W, W') + \eta(U)[\eta(W)g(SV, W') - g(SV, W')\eta(W')] - \eta(V)[\eta(W)g(SU, W') - g(SU, W')\eta(W')] - g(\phi SU, V)g(\phi W, W') \\
&\quad + g(\phi SV, U)g(\phi W, W').
\end{align*}
$$

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Now, on contracting $U$ and $W$ in above defined relation, we derive
\[ \tilde{Ric}(V, W) = Ric(V, W) - (n - 2)g(\phi SV, W) + (n - 2)\eta(V)\eta(W) \]
\[ -g(V, W)\left[ \sum_{i=1}^{n} g(\phi Se_{i}, e_{i}) + (n - 2) \right], \]
(5.6)
where $\tilde{Ric}(V, W)$ and $Ric(V, W)$ are the Ricci tensors of the connection $\tilde{\nabla}$ and $\nabla$ respectively.

Again, by applying contraction on $V$ and $W$, the scalar curvature $\tilde{\tau}$ with SSMC has the following expression
\[ 2\tilde{\tau} = 3(n - 1) - 2 + g^2(AN, N) + n^2||\tilde{H}||^2 - ||h||^2 - 2(n - 1)g(\phi Se_{i}, e_{i}). \]
(5.7)
Thus, we have

**Lemma 5.1.** In a real hypersurface $M$ of $Q^m$ admitting SSMC such that $\phi S = S\phi$, we have
(a) $\tilde{Ric}(V, W) = Ric(V, W) - (n - 2)[g(\phi SV, W) + g(\phi V, \phi W)]$
(b) $QV = QV - (n - 2)[V - \eta(V)\xi - \phi SV]$
(c) $Ric(V, \xi)$ coincides with $Ric(V, \xi)$
for all $V, W \in T_pM$.

**Proof.** Let us assume that $\phi S = S\phi$. Then, we have
\[ g(\phi Se_{i}, e_{i}) = g(S\phi e_{i}, e_{i}) \]
\[ = -g(\phi Se_{i}, e_{i}) \]
This results $g(\phi Se_{i}, e_{i}) = 0$, which together with (5.6) follows (a) and hence (b). By using the assumption and inserting $W = \xi$ in (5.6), we get (c).

Also, we know that the Ricci operator $Q$ of SSMC is defined by
\[ \tilde{Ric}(V, W) = g(QV, W), \forall \ V, W \in T_pM. \]

From this incorporating (5.6) together with the assumption, we have
\[ QV = QV - (n - 2)[V - \eta(V)\xi - \phi SV]. \]

\[ \square \]

6. Chen’s inequality for a real hypersurface $M$ of $Q^m$ with SSMC

Here, we obtain inequality for the mean curvature, the scalar and the sectional curvature associated with the induced SSMC for a real hypersurfaces $M$ of $Q^m$.

Here, we have the squared mean curvature $||\tilde{H}||^2$ of $M$ in $Q^M$ and the squared norm $||h||^2$ of $h$ as
\[ ||\tilde{H}||^2 = \frac{1}{n^2}\left( \sum_{i,j=1}^{n} h_{ij}^{n+1} \right)^2 \] and
\[ ||h||^2 = \sum_{i,j=1}^{n} (h_{ij}^{n+1})^2 \]
respectively, where $h_{ij}^{n+1} = g(h(e_{i}, e_{j}), N)$ and the mean curvature vector field $\tilde{H}$ of $\tilde{\nabla}$ and $H$ of $\nabla$ are invariant.

Now, the scalar curvature $\tilde{\tau}$ for an orthonormal basis $\{e_{i}\}_{n}$ reads
\[ \tilde{\tau} = \sum_{1 \leq i < j \leq n} K(e_{i} \land e_{j}). \]

**Theorem 6.1.** Let $M$ be a real hypersurface of $Q^m$ admitting SSMC $\tilde{\nabla}$. Then, for 2-plane section $\pi \subset T_pM$ spanned by tangent vectors $e_1$ and $e_2$, we have
\[ \tilde{\tau}(x) - K(\pi) \leq (n - 2)\left\{ \frac{n^2||\tilde{H}||^2}{2(n - 1)} + \frac{3(n - 1) - 2}{2(n - 2)} \right\} + g^2(Ae_1, e_2) + \frac{g^2(AN, N)}{2} \]
\[ + g(\phi Se_1, e_1) + g(\phi Se_2, e_2) + g^2(JAe_1, e_2) + (n - 1)g(\phi Se_i, e_i). \]
(6.1)
Moreover, equality holds in (6.1) at \( p \in M \) if and only if there exist an orthonormal basis \( \{e_i\}_i^n \) of \( T_p M \) and orthonormal normal frame \( \{e_{n+1} = N\} \) of \( T_p^\perp M \), such that the matrix of the shape operator \( S \) takes the following form

\[
S = \begin{pmatrix}
p' & 0 & 0 \\
0 & q' & 0 \\
0 & 0 & M \\
\end{pmatrix},
\]

where \( M \) is the diagonal matrix of order \( n - 2 \) with diagonal entry \( r = p' + q' \).

**Proof.** First of all we put

\[
e = 2\tilde{\tau} - 3(n - 1) + 2 - g^2(AN, N) + 2(n - 1)g(\phi Se_i, e_i) - \frac{n^2(n - 2)}{n - 1}||\tilde{H}||^2
\]

Thus, we have

\[
n^2||\tilde{H}||^2 = (n - 1)\{e + ||h||^2\}.
\]

Moreover, we can write

\[
\left(\sum_{i=1}^{n} h_i^{n+1}\right)^2 = (n - 1)\left\{e + \sum_{i=1}^{n} (h_i^{n+1})^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2\right\}
\]

Using lemma (3.1), we obtain

\[
2h_1^{n+1}h_2^{n+1} \geq \sum_{i\neq j} (h_{ij}^{n+1})^2 + e
\]

The Gauss equation yields

\[
\bar{K}(\pi) = g(\tilde{H}(e_1, e_2)e_1, e_1)
\]

\[
= 3g^2(\phi e_1, e_2) + g(Ae_1, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2) + g(J Ae_1, e_1) - (h_{12}^{n+1})^2 + h_1^{n+1}h_2^{n+1} - g(\phi Se_2, e_2) - g(\phi Se_1, e_1) + \eta(e_2)^2 + \eta(e_1)^2 - g^2(J Ae_1, e_2)
\]

Inserting (6.6) into (6.7) yields

\[
\bar{K}(\pi) \geq 3g^2(\phi e_1, e_2) + g(Ae_1, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2) + g(J Ae_1, e_1) - (h_{12}^{n+1})^2 + h_1^{n+1}h_2^{n+1} - g(\phi Se_2, e_2) - g(\phi Se_1, e_1) + \eta(e_2)^2 + \eta(e_1)^2
\]

\[
\geq 3g^2(\phi e_1, e_2) + g(Ae_2, e_2)g(Ae_1, e_1) - g^2(Ae_1, e_2) + g(J Ae_2, e_2)g(J Ae_1, e_1) - g^2(J Ae_1, e_2) - g(\phi Se_2, e_2) - g(\phi Se_1, e_1) + \eta(e_2)^2 + \eta(e_1)^2 + \tilde{\tau} + \frac{3(n - 1) - 2}{2} - \frac{n^2(n - 2)}{2(n - 1)}||\tilde{H}||^2 + (n - 1)g(\phi Se_e, e_1)
\]

Thus, we derive

\[
\tilde{\tau}(p) - \bar{K}(\pi) \leq (n - 2)\left\{\frac{n^2||\tilde{H}||^2}{2(n - 1)} + \frac{3(n - 1) - 2}{2(n - 2)}\right\} - 3g^2(\phi e_1, e_2) - g(Ae_2, e_2)g(Ae_1, e_1)
\]

\[
+ g^2(Ae_1, e_2) - g(J Ae_2, e_2)g(J Ae_1, e_1) + g^2(J Ae_1, e_2) + g(\phi Se_2, e_2)
\]

\[
+ g(\phi Se_1, e_1) - \eta(e_2)^2 - \eta(e_1)^2 + \frac{g^2(AN, N)}{2} - (n - 1)g(\phi Se_1, e_1)
\]
or, equivalently
\[
\tau(p) - \tilde{K}(\pi) \leq (n - 2) \left( \frac{n^2|\tilde{H}|^2}{2(n-1)} + \frac{3(n-1) - 2}{2(n-2)}} \right) + g^2(Ae_1, e_2) + g^2(J Ae_1, e_2) + g\left(\phi Se_1, e_1\right) + g\left(\phi Se_2, e_2\right) + (n-1)g(\phi Se_i, e_i)\]
\[+ g^2(AN, N) + \frac{1}{2}\]
\[= 0 \text{ for all } i \neq j,
\]
\[h_{i1}^{n+1} = h_{i2}^{n+1} = ... = h_{n}^{n+1}.
\]

Thus, we may have the choice for \{e_1, e_2\} such that \(h_{i2}^{n+1} = 0\). Hence, the matrix of the shape operator has the form (6.2).

**Corollary 6.1.** Let \(\mathcal{M}\) be a real hypersurface of \(Q^n\) admitting SSMC \(\tilde{\nabla}\). Then, for a \(U\)-principal normal vector field, we have
\[
\tau(p) - K(\pi) \leq (n - 2) \left( \frac{n^2|\tilde{H}|^2}{2(n-1)} + \frac{3(n-1) - 2}{2(n-2)}} \right) + g^2(Ae_1, e_2) + g^2(J Ae_1, e_2)
\]
\[+ g(\phi Se_1, e_1) + g(\phi Se_2, e_2) + (n-1)g(\phi Se_i, e_i),\]

where \(\pi \subset T_p\mathcal{M}\) is a 2-plane section spanned by tangent vectors \(e_1\) and \(e_2\).

**Corollary 6.2.** Let \(\mathcal{M}\) be a real hypersurface of \(Q^n\) admitting SSMC \(\tilde{\nabla}\). Then, for a \(U\)-isotropic normal vector field, we have
\[
\tau(p) - K(\pi) \leq (n - 2) \left( \frac{n^2|\tilde{H}|^2}{2(n-1)} + \frac{3(n-1) - 2}{2(n-2)}} \right) + g^2(Ae_1, e_2) + g^2(J Ae_1, e_2)
\]
\[+ g(\phi Se_1, e_1) + g(\phi Se_2, e_2) + (n-1)g(\phi Se_i, e_i),\]

where \(\pi \subset T_p\mathcal{M}\) is a 2-plane section spanned by tangent vectors \(e_1\) and \(e_2\).

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Extremities involving B. Y. Chen’s invariants


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