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On the Geometric Singularities of Surfaces of Pseudo-Euclidean Space

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ABSTRACT

In this article we explore the space of constant curvature. We consider the principal bundle over pseudoconformal plane. The elements of differential geometry are found for a surface of pseudo-Euclidean space. The elements of the matrix of the metric tensor, as well as the coefficients of the Riemannian connection, are calculated.

Keywords: *principal bundle; space of constant curvature; pseudo-Euclidean space; Riemannian connection.*

AMS Subject Classification (2010): *53B20; 53B30; 53C21.*

1. Introduction

In 1895, the "Screw Count" [2] of A.P. Kotelnikov was appeared, in which to the geometry of the groups of motions of the Euclidean space $\mathbb{E}^3$ and to the geometry of the manifold of lines in this space the geometry of three-dimensional and two-dimensional spheres was applied in dual Euclidean spaces.

Ideas of A.P. Kotelnikov D.N. Zeiliger [16] developed, systematically studied the line differential geometry of the Euclidean space by means of the transfer method. In 1925 P.A. Shirokov introduced for the first time an important class of A-spaces, subsequently known as the Keler [11], he studied the geometry of symmetric spaces of this type [12]. He also considered the application of screw calculus to the differential geometry, specifying various surface-related lines using dual vectors [13].

A.P. Shirokov [9] constructed a theory of biplanar spaces - multidimensional generalizations of A.P. Norden biaxial spaces. The study of spaces with structures, defined by algebras of general form (associative and unital), was followed. In subsequent years in the works of A.P. Shirokov and his students various aspects of the theory of spaces over algebras developed, its numerous applications to the linear geometry, geometry of non-Euclidean spaces, to the theory of tangent bundles [15], [10].

The structures defined by algebras naturally arise on foliated manifolds of various types.

Such structures represent special interest in connection with the fact that they find numerous applications in mathematics, mechanics and theoretical physics.

So, A.P. Shirokov showed, that tangent bundles of an arbitrary order carry on themselves a natural structure, determined by the algebras of plural numbers [8].

V.V. Vishnevsky constructed a theory of semitangent bundles and showed that they carry a nilpotent affinor structure of the most general kind [14].

It should be noted the works of B.N. Shapukov [7] on the theory of vector and, in particular, tensor bundles.

From all that has been said, it follows that questions of development of theoretical and practical positions on the study of various bundles with structures of algebraic type are relevant and of scientific interest.

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2. The group of invertible elements of the algebra of antiquaternions

We introduce the four-dimensional algebra $\mathbb{A}$ of antiquaternions [3] with the basis 1, f, e, i, defined by the multiplication table

	1	f	e	i
1	1	f	e	i
f	f	1	i	e
e	e	$-i$	1	$-f$
i	i	$-e$	f	-1

We write the antiquaternion in the form $\mathbf{a} = a^0 + a^1 f + a^2 e + a^3 i$. In the algebra of antiquaternions we define the conjugate element $\bar{\mathbf{a}} = a^0 - a^1 f - a^2 e - a^3 i$, for which the condition $\overline{\mathbf{ab}} = \bar{\mathbf{b}}\bar{\mathbf{a}}$ holds. The multiplication $\mathbf{a}\bar{\mathbf{a}}$= $(a^0)^2 - (a^1)^2 - (a^2)^2 + (a^3)^2$ is a real number. We define the scalar product of two antiquaternions: $\mathbf{ab} = \frac{1}{2}(\mathbf{a}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{a}})$. Then under these conditions in $\mathbb{A}$ the four-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ appears.

The antiquaternion module $|\mathbf{a}|$ is the square root $\sqrt{\mathbf{a}\bar{\mathbf{a}}}$. For $|\mathbf{a}| \neq 0$ the inverse element is determined, which is denoted by $\mathbf{a}^{-1}$ and is equal to $\frac{\bar{\mathbf{a}}}{|\mathbf{a}|^2}$.

The set of invertible elements of the algebra $\mathbb{A}$ of antiquaternions

$$\tilde{\mathbb{A}} = \{\mathbf{a} \,|\, |\mathbf{a}|^2 = \mathbf{a}\bar{\mathbf{a}} \neq \mathbf{0}\}$$

forms a Lie group.

Any two-dimensional subspace of the algebra of antiquaternions, containing a unit, is a subalgebra, which is isomorphic to the two-dimensional algebra of complex, double or dual numbers [1].

In this paper we consider the third case - the two-dimensional subalgebra $\mathbb{R}(\epsilon)$ of dual numbers with basis $\{1, \epsilon\}$, where $\epsilon = f + i$. The set of invertible elements of the subalgebra $\mathbb{R}(\epsilon)$

$$\tilde{\mathbb{R}}(\epsilon) = \{\mu = x + y\epsilon \,|\, x \neq 0\}, \quad x, y \in \mathbb{R}$$

is a Lie subgroup of the group $\tilde{\mathbb{A}}$, namely a two-dimensional plane without a double line.

Antiquaternions can be written in different ways [4].

We write the antiquaternion as

$$\mathbf{a} = a^0 + a^2 + a^1\epsilon + (a^3 - a^1 + a^2\epsilon)i = c_1 + c_2 i, \quad c_1, c_2 \in \mathbb{R}(\epsilon).$$

We supplement the table of multiplication of the antiquaternion algebra by elements ϵ and $\hat{\epsilon}$, where $\hat{\epsilon} = f - i$, $\hat{\epsilon}^2 = 0$. In this case, the multiplication table is extended and the following multiplications hold:

$$\epsilon f = -\epsilon i = f\hat{\epsilon} = i\hat{\epsilon} = \frac{1}{2}\epsilon\hat{\epsilon} = 1 - e\,, \quad f\epsilon = -i\epsilon = \hat{\epsilon} f = \hat{\epsilon} i = \frac{1}{2}\hat{\epsilon}\epsilon = 1 + e\,,$$

$$\epsilon e = -e\epsilon = i + f, \quad \hat{\epsilon} e = -e\hat{\epsilon} = i - f.$$

According to these equalities we introduce the necessary for us multiplications.

Let $c = x + y\epsilon \in \mathbb{R}(\epsilon)$ and $\hat{c} = x + y\hat{\epsilon} \in \mathbb{R}(\hat{\epsilon})$, then

$$i\bar{c} = \hat{c}i\,, \quad i\hat{c} = \bar{c}i\,.$$

Thus,

$$\bar{\mathbf{a}} = \bar{c}_1 - \hat{c}_2 i, \quad \mathbf{a}\bar{\mathbf{a}} = (c_1\bar{c}_1 + c_2\bar{c}_2) + (c_2\hat{c}_1 - c_1\hat{c}_2)i, \tag{2.1}$$

where $\hat{c}_1 = a^0 + a^2 + a^1\hat{\epsilon}$, $\hat{c}_2 = a^3 - a^1 + a^2\hat{\epsilon}$.

We introduce the factorset of right cosets $\tilde{\mathbb{A}}/\tilde{\mathbb{R}}(\epsilon)$. Antiquaternions $\mathbf{a}, \mathbf{b} \in \tilde{\mathbb{A}}$ belong to the same right coset of $\tilde{\mathbb{R}}(\epsilon)$ if and only if when $\mathbf{ab}^{-1} \in \tilde{\mathbb{R}}(\epsilon)$ holds. $\mathbf{ab}^{-1} = \frac{\mathbf{a}\bar{\mathbf{b}}}{|\mathbf{b}|^2}$ and

$$\mathbf{a}\bar{\mathbf{b}} = (c_1 + c_2 i)(\bar{d}_1 - \hat{d}_2 i) = (c_1\bar{d}_1 + c_2\bar{d}_2) + (c_2\hat{d}_1 - c_1\hat{d}_2)i. \tag{2.2}$$

If the following conditions

$$c_2\hat{d}_1 - c_1\hat{d}_2 = 0, \;\; c_1\bar{d}_1 + c_2\bar{d}_2 \neq 0$$

are satisfied, then antiquaternion is a nonzero dual number.

The first condition means that $c_1 : c_2 = \hat{d}_1 : \hat{d}_2$ and, consequently, the second condition is written down $\hat{d}_1\bar{d}_1 + \hat{d}_2\bar{d}_2 \neq 0$ and by analogy $\hat{c}_1\bar{c}_1 + \hat{c}_2\bar{c}_2 \neq 0$. Then, firstly, the canonical projection $\pi : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}/\tilde{\mathbb{R}}(\epsilon)$ takes the form

$$\pi(\mathbf{a}) = (c_1 : c_2). \tag{2.3}$$

Secondly, the subset N

$$N = \{[c_1 : c_2] \in P(\epsilon) \,|\, \hat{c}_1\bar{c}_1 + \hat{c}_2\bar{c}_2 \neq 0\}$$

of the dual projective line $P(\epsilon)$ is a factorset of right cosets.

It is covered by two cards:

$$V_1 = \{[c_1 : c_2] \,|\, |c_2| \neq 0\} \quad \text{with the coordinate } c = \frac{c_1}{c_2} \tag{2.4}$$

and

$$V_2 = \{[c_1 : c_2] \,|\, |c_1| \neq 0\} \quad \text{with the coordinate } c' = \frac{c_2}{c_1}\,. \tag{2.5}$$

Consequently, the subset N is a pseudoconformal plane [6] without a pair of parallel straight lines $|c|^2 = 1$.

Consider the following relations:

$$\mathbf{a} = c_1 + c_2 i = c_2(c + i) = c_2\left(\frac{c_1}{c_2} + i\right) \quad \text{under } |c_2| \neq 0,$$

$$\mathbf{a} = c_1 + c_2 i = c_1(1 + c' i) = c_1\left(1 + \frac{c_2}{c_1} i\right) \quad \text{under } |c_1| \neq 0.$$

From these formulas we get the following:

$$\psi_1 : \pi^{-1}(V_1) \longrightarrow V_1 \times \tilde{\mathbb{R}}(\epsilon), \quad \psi_1(c_1 + c_2 i) = \left(\frac{c_1}{c_2},\ c_2\right),$$

$$\psi_2 : \pi^{-1}(V_2) \longrightarrow V_2 \times \tilde{\mathbb{R}}(\epsilon), \quad \psi_2(c_1 + c_2 i) = \left(\frac{c_2}{c_1},\ c_1\right).$$

We have obtained the mappings of trivialization. The inverse to them will be written as follows:

$$\psi_1^{-1}(c,\ \mu) = \mu(c + i),$$

$$\psi_2^{-1}(c',\ \mu) = \mu(1 + c' i).$$

Thus, we obtain the principal bundle $E = (\tilde{\mathbb{A}}, \pi, N)$ of right cosets $\tilde{\mathbb{R}}(\epsilon)\mathbf{a}$ with the structural group $\tilde{\mathbb{R}}(\epsilon)$, which acts on the left $\tilde{\mathbb{R}}(\epsilon)\mathbf{a} \to \tilde{\mathbb{R}}(\epsilon)(\tilde{\mathbb{R}}(\epsilon)\mathbf{a}) = \tilde{\mathbb{R}}(\epsilon)\mathbf{a}$.

Then the gluing function takes the form:

$$\varphi_{12}(c,\ \mu) = \psi_2 \circ \psi_1^{-1} = \left(\frac{1}{c},\ \mu c\right).$$

Thus, we have the locally trivial fibration.

So, the following theorem holds

Theorem 2.1. *The bundle $(\tilde{\mathbb{A}}, \pi, N)$, defined by the equality (2.3), is a principal locally trivial bundle over a pseudoconformal plane without a pair of parallel lines with a typical fiber that is diffeomorphic to the two-dimensional plane without a double line and a structural group $\tilde{\mathbb{R}}(\epsilon)$.*

Let $g = p_0 + \epsilon p_1 \in N \subset P(\epsilon)$ over the neighborhood V_1. The preimages of any point of the base of the mapping π are two-dimensional planes M_2, given by the equation $c_1 - gc_2 = 0$. In real coordinates we obtain a system of two equations:

$$\begin{cases} a^0 + \frac{1}{1+p_1}((p_0^2 + p_1 + 1)a^2 - p_0 a^3) & = & 0\,, \\ a^1 - \frac{1}{1+p_1}(p_0 a^2 + p_1 a^3) & = & 0\,. \end{cases} \tag{2.6}$$

Let us study the sections of the planes M_2 (2.6) by an isotropic cone with equation $(a^0)^2 - (a^1)^2 - (a^2)^2 + (a^3)^2 = 0$. We have $F(p_0a^2 - a^3)^2 = 0$, where $F = p_0^2 + 2p_1 + 1$.
Under $F \neq 0$ $(p_0a^2 - a^3)^2 = 0$ holds. The section is one double isotropic line. Consequently, a semi-Euclidean two-dimensional plane that is tangent to the isotropic cone of the space $\mathbb{E}_2^4$ is the fiber $\pi^{-1}(g)$ for every point $g \in V_1$.
Under $F = 0$ the equality $F(p_0a^2 - a^3)^2 = 0$ holds for any a^2 and a^3. Thus, the planes (2.6) are generators of the isotropic cone.

Let $g' = p_0' + \epsilon p_1' \in V_2$. Then we have semi-Euclidean two-dimensional planes $M_2' : c_2 - g'c_1 = 0$, which are given by the following equations

$$\begin{cases} a^2 + \frac{1}{p_1'-1}(p_1'a^0 + p_0'a^1) & = & 0\,, \\ a^3 + \frac{1}{p_1'-1}(p_0'a^0 + (p_0'^2 - p_1' + 1)a^1) & = & 0. \end{cases} \tag{2.7}$$

The properties of the planes (2.7) are analogous to the properties of the two-dimensional planes (2.6).

3. The principal bundle of the sphere of real radius of the pseudo-Euclidean space

Consider a sphere of real radius $S_2^3(1)$ with the equation

$$(a^0)^2 - (a^1)^2 - (a^2)^2 + (a^3)^2 = 1. \tag{3.1}$$

in the pseudo-Euclidean space $\mathbb{E}_2^4$.

Geometrically, the group of antiquaternions of the unit module, satisfying the condition $\mathbf{a}\bar{\mathbf{a}} = 1$, can be represented as the sphere $S_2^3(1) \subset \mathbb{E}_2^4$.

Let $\mathbf{d} = b_1 + b_2 i$, $\mathbf{a} = c_1 + c_2 i$, where b_1, b_2, c_1, c_2 - dual numbers. Taking into account the condition (2.1), the multiplication $\mathbf{da}$ is

$$\mathbf{a}' = \mathbf{da} = (b_1 + b_2 i)(c_1 + c_2 i) = (b_1c_1 - b_2\hat{\hat{c}}_2) + (b_1c_2 + b_2\hat{\hat{c}}_1)i\,.$$

Under $\mathbf{d} \in S_2^3(1)$ we obtain, that the transformations $\mathbf{a}' = \mathbf{da}$ and $\mathbf{a}' = \mathbf{ad}$ in view of the fact that $|\mathbf{a}'|^2 = |\mathbf{d}|^2|\mathbf{a}|^2$ are rotations in $\mathbb{E}_2^4$ if the rotation parameter is independent of the vector a. Such transformations are called paratactic turns [6]. We call them, respectively, left and right and find the matrices of these transformations.

Let

$$b_1 = m^0 + m^2 + m^1\epsilon,\; b_2 = m^3 - m^1 + m^2\epsilon,$$
$$c_1 = a^0 + a^2 + a^1\epsilon,\; c_2 = a^3 - a^1 + a^2\epsilon\,.$$

Then the corresponding fourth-order matrix with real elements of the transformation $\mathbf{a}' = \mathbf{da}$ takes the form

$$D^{\mathbb{R}} = \begin{pmatrix} m^0 & m^1 - 2m^2 & m^2 & 2m^2 - m^3 \\ m^1 & m^0 - m^2 & m^3 & 0 \\ m^2 & 2m^2 - m^3 & m^0 & m^1 - 2m^2 \\ m^3 & -2m^2 & m^1 & m^0 + m^2 \end{pmatrix}, \quad \det D^{\mathbb{R}} = 1. \tag{3.2}$$

$D^{\mathbb{R}}$ - is a special pseudoorthogonal matrix.

Consider transformations of the form $\mathbf{a}' = \mathbf{ad}$ with $\mathbf{d} \in S_2^3(1)$.

$$\mathbf{a}' = \mathbf{ad} = (c_1 + c_2 i)(b_1 + b_2 i) = (b_1c_1 - \hat{\hat{b}}_2c_2) + (b_2c_1 + \hat{\hat{b}}_1c_2)i\,.$$

Thereby,

$$c_1' = b_1c_1 - \hat{\hat{b}}_2c_2,\; c_2' = b_2c_1 + \hat{\hat{b}}_1c_2 \tag{3.3}$$

and the corresponding matrix

$$D = \begin{pmatrix} b_1 & -\hat{\hat{b}}_2 \\ b_2 & \hat{\hat{b}}_1 \end{pmatrix}.$$

We restrict the bundle $E = (\tilde{\mathbb{A}}, \pi, N)$ and consider the bundle $(S_2^3(1), \pi, N)$, where $S_2^3(1) \subset \tilde{\mathbb{A}}$ is the sphere of real radius.

Consider the sections of the planes M_2 of the sphere $S_2^3(1)$.

If $F \neq 0$, then $(p_0 a^2 - a^3)^2 = \frac{(1+p_1)^2}{F}$. Here two cases take place:
1) if $F = p_0^2 + 2p_1 + 1 < 0$, then in the section we obtain imaginary curves;
2) if $F = p_0^2 + 2p_1 + 1 > 0$, then in the section we obtain semi-Euclidean circles, which are represented by a pair of the parallel lines $(p_0 a^2 - a^3)^2 = d > 0$ under the projection onto the plane (a^2, a^3).

Therefore, it makes sense to consider only the bundle in the area $p_0^2 + 2p_1 + 1 > 0$. The restriction of the subgroup $\tilde{\mathbb{R}}(\epsilon)$ of dual numbers to the sphere $S_2^3(1)$ is the subgroup s of dual numbers of the unit module.

Theorem 3.1. *The bundle $(S_2^3(1), \pi, N)$ is the principal bundle of the group $S_2^3(1)$ onto right cosets by the Lie subgroup s of the antiquaternions $\mathbf{d} = (b_1, 0) : b_1 \bar{b}_1 = 1$.*

Proof. Consider two arbitrary antiquaternions $\mathbf{a} = c_1 + c_2 i$ and $\mathbf{a}' = c_1' + c_2' i$, where $\mathbf{a}$ and $\mathbf{a}' \in S_2^3(1)$. The inverse element $\mathbf{a}^{-1}$ for the element $\mathbf{a}$ takes the form $\mathbf{a}^{-1} = \bar{\mathbf{a}} = \bar{c}_1 - \hat{c}_2 i$ and under the condition (2.2) we get

$$\mathbf{a}'\mathbf{a}^{-1} = \mathbf{a}'\bar{\mathbf{a}} = (c_1'\bar{c}_1 + c_2'\bar{c}_2) + (c_2'\hat{c}_1 - c_1'\hat{c}_2)i.$$

Then, when $c_2 = c_2' = 0$, we obtain that s is a subgroup of the group $S_2^3(1)$, $\mathbf{a}'\mathbf{a}^{-1} = c_1'\bar{c}_1$. By the Cartan theorem, it follows that the closed submanifold s of dimension 1 of $S_2^3(1)$ is a Lie subgroup. Moreover, s is a large circle, which is represented by a pair of parallel lines. If the condition $\mathbf{a}'\mathbf{a}^{-1} \in s$ is fulfilled, namely, if $\mathbf{a}'\mathbf{a}^{-1} = \pm 1 + \epsilon\varphi$ for some φ, then the antiquaternions $\mathbf{a}$ and $\mathbf{a}'$ belong to the same right coset by s. Hence, $\mathbf{a}' = (\pm 1 + \epsilon\varphi)\mathbf{a}$ is the action of the structure group on $S_2^3(1)$ and

$$c_1' = (\pm 1 + \epsilon\varphi)c_1, \quad c_2' = (\pm 1 + \epsilon\varphi)c_2 \quad . \quad \square \tag{3.4}$$

Taking into account the matrix form $D^{\mathbb{R}}$ (3.2) the corresponding pseudoorthogonal matrix of the action (3.4) is written

$$D = \begin{pmatrix} \pm 1 & \varphi & 0 & -\varphi \\ \varphi & \pm 1 & \varphi & 0 \\ 0 & -\varphi & \pm 1 & \varphi \\ \varphi & 0 & \varphi & \pm 1 \end{pmatrix}. \tag{3.5}$$

4. Pseudo-Riemannian metric of the surface S_2^3

We choose coordinates adapted to our fibration on the surface $S_2^3(1)$. The coordinates of the point $c = \frac{c_1}{c_2} = p_0 + \epsilon p_1 \in N \subset P(\epsilon)$ and the rotation parameter, originating from some fixed point $\mathbf{h}$ of the orbit $(c_1(\varphi), c_2(\varphi))$ are the adapted coordinates. Then, under the action (3.4), we get $c_1(\varphi) = (1+\epsilon\varphi)c_1$, $c_2(\varphi) = (1+\epsilon\varphi)c_2$, and the point $\mathbf{a} = (1+\epsilon\varphi)\mathbf{h}$ has the following dual coordinates

$$c_1 = (1+\epsilon\varphi)(h^0 + h^2 + h^1\epsilon), \quad c_2 = (1+\epsilon\varphi)(h^3 - h^1 + h^2\epsilon).$$

Consider a hyperplane with the equation $h^2 = 0$. For this we have $c_1 = (1+\epsilon\varphi)(h^0 + h^1\epsilon)$, $c_2 = (1+\epsilon\varphi)(h^3 - h^1)$. We define the initial point c of the orbit in given hyperplane. Therefore,

$$c = \frac{c_1}{c_2} = \frac{h^0 + h^1\epsilon}{h^3 - h^1} = p_0 + \epsilon p_1.$$

Thus, the real coordinates of the starting point are written

$$h^0 = \frac{1}{\sqrt{F}}p_0, \; h^1 = \frac{1}{\sqrt{F}}p_1, \; h^2 = 0, \; h^3 = \frac{1}{\sqrt{F}}(1+p_1),$$

where $F = p_0^2 + 2p_1 + 1 > 0$. The parametric equation of the sphere $S_2^3(1)$ takes the form

$$\begin{cases} a^0 &= \frac{1}{\sqrt{F}}(p_0 - \varphi), \\ a^1 &= \frac{1}{\sqrt{F}}(p_1 + p_0\varphi), \\ a^2 &= \frac{1}{\sqrt{F}}\varphi, \\ a^3 &= \frac{1}{\sqrt{F}}(1 + p_1 + p_0\varphi). \end{cases} \tag{4.1}$$

From the parametric equation (4.1) we pass to the equation in vector form. To do this, consider two pseudoorthogonal vectors

$$\mathbf{d} = \frac{1}{\sqrt{F}}(p_0, p_1, 0, 1+p_1), \; \tilde{\mathbf{d}} = \frac{1}{\sqrt{F}}(-1, p_0, 1, p_0).$$

Then

$$\mathbf{a} = \mathbf{d} + \tilde{\mathbf{d}}\varphi \tag{4.2}$$

is the vector equation of the sphere $S_2^3(1)$. In this case, for the coordinates (p_0, p_1, φ) introduced on $S_2^3(1)$, adapted to the bundle, we can define a pseudo-Riemannian metric of the sphere.

From the vector equation of $S_2^3(1)$ we obtain

$$\begin{cases} \mathbf{a}_{p_0} &= \mathbf{d}_{p_0} + \tilde{\mathbf{d}}_{p_0}\varphi, \\ \mathbf{a}_{p_1} &= \mathbf{d}_{p_1} + \tilde{\mathbf{d}}_{p_1}\varphi, \\ \mathbf{a}_{\varphi} &= \tilde{\mathbf{d}}. \end{cases}$$

Let us calculate the matrix elements of the metric tensor $g_{XY} = (\mathbf{a}_X, \mathbf{a}_Y)$, $(X, Y = 1, 2, 3)$ and $\mathbf{a}_1 = \mathbf{a}_{p_0}$, $\mathbf{a}_2 = \mathbf{a}_{p_1}$, $\mathbf{a}_3 = \mathbf{a}_{\varphi}$.
Thus,

$$(g_{XY}) = \begin{pmatrix} \frac{2p_1+1}{F^2} & \frac{-p_0}{F^2} & \frac{-1}{F} \\ \frac{-p_0}{F^2} & \frac{-1}{F^2} & 0 \\ \frac{-1}{F} & 0 & 0 \end{pmatrix}. \tag{4.3}$$

The transformations $\varphi \to \varphi'$ are motions of the three-dimensional sphere $S_2^3(1)$. This corresponds to the matrix (g_{XY}), which does not depend on φ. We obtained the indefinite metric (4.3).

5. The space of constant curvature

We can not determine horizontal distribution, orthogonal to the fibers, since the subalgebra of dual numbers is tangent to an isotropic cone.

Let us find the coefficients of the Riemannian connection for the metric (4.3). These take the form

$$\Gamma^1_{11} = -\frac{2p_0}{F}, \ \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^3_{23} = \Gamma^3_{32} = -\frac{1}{F},$$

$$\Gamma^2_{22} = -\frac{2}{F}, \ \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = -\frac{p_0}{F}, \ \Gamma^2_{13} = \Gamma^2_{31} = 1.$$

All remaining coefficients of connection are zero. Thus, we get the connection in the bundle $(S_2^3(1), \pi, N)$.

Consequently, the pseudo-Riemannian space $(S_2^3(1), g_{XY})$ is the space of constant curvature with $K = -1$.

Acknowledgments

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