On Ricci solitons in $N(k)$ - quasi Einstein manifolds

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Abstract: He object of the present paper is to study $N(k)$-quasi Einstein manifolds satisfying certain curvature conditions. Further we study gradient Ricci solitons on $N(k)$-quasi Einstein manifolds.

Keywords: Quasi Einstein manifold, projective curvature tensor, pseudosymmetry, Ricci solitons.

1 Introduction

Einstein manifolds play a vital part in Riemannian geometry as well as in the general theory of relativity (*GTR*). Also, Einstein manifolds build a natural subclass of various class of Riemannian or semi-Riemannian manifolds by a curvature condition enforced on their Ricci tensor ([1];p.432-433). A non-flat n- dimensional Riemannian manifold (M,g) $(n > 2)$ is said to be an Einstein manifold if the condition

$$S = \frac{r}{n}g \tag{1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M,g) respectively. A Ricci soliton is a generalization of an Einstein metirc. In a Riemannian manifold (M,g), g is called a Ricci soliton if

$$L_{vg} + 2S(X,Y) + 2\lambda g = 0, \tag{2}$$

where L denotes Lie derivative with respect to a complete vector field V, S is the Ricci tensor and λ is a constant.

A Ricci soliton is *expanding*, *shrinking* or *steady* as λ is positive, negative or zero respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (2) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{3}$$

Theoretical physicist have also been looking into the equation of Ricci soliton in relation with string theory. Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hyper surfaces of semi-Euclidean spaces. Quasi Einstein manifolds have some importance in *GTR*. For instance, the Robertson-Walker spacetimes are quasi Einstein manifolds. Further, quasi-Einstein manifolds can be taken as a model of the perfect fluid spacetime in *GTR* [8]. If the generator ξ of quasi Einstein manifold belongs to some k-nullity distribution $N(k)$, then the quasi Einstein manifold is called an $N(k)$- quasi Einstein manifold [20]. In [20], it was shown that an n- dimensional conformally flat quasi Einstein manifold is a $N(\frac{a+b}{n-1})$- quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is a $N(\frac{a+b}{n-1})$- quasi Einstein manifold. Further, De, Sengupta and Saha in [5] studied conformally flat and semisymmetric quasi Einstein manifolds and some physical examples of $N(k)$- quasi

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Einstein Manifolds was cited by Özgür in [16]. Then, Taleshian and Hosseinzadeh ([13,18]), Yildiz, De and Centinkaya [22] and singh et al. studied $N(k)$- quasi Einstein manifolds satisfying certain curvatre conditions extensively. Also $N(k)$-mixed quasi Einstein manifold was studied in [15] by Nagaraja.

2 Preliminaries

An almost contact structure on a n-dimensional smooth manifold M is given by a triple (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor, ξ a global vector field and η a one-form, such that

$$\phi^2 = -Id + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 0. \tag{4}$$

A Riemannian metric g on an almost contact manifold M is said to be compatible with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{5}$$

for any vector fields X, Y on M. A non-flat n-dimensional Riemannian manifold (M, g) $(n > 2)$, is said to be a quasi Einstein [3] if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \quad X, Y \in TM, \tag{6}$$

for some smooth functions a and $b \neq 0$, where η is a non-zero 1 form such that

$$g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1, \tag{7}$$

for the associated vector field ξ. Then η is called associated 1- from and the unit vector field ξ is called the generator of the manifold. Throughout this paper $M_{a,b}^n$ denotes n- dimensional $N(k)$-quasi Einstein manifold. From (6) and (7) it follows that

$$QX = aX + b\eta(X)\xi, \quad S(X, \xi) = (a+b)\eta(X). \tag{8}$$

and

$$r = na + b, \tag{9}$$

where r is the scalar curvature of M. Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$- of a Riemannian manifold M [19] is defined by

$$N(k) : p \rightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y], \tag{10}$$

k being some smooth function. In a quasi Einstein manifold M, if the generator $\hat{I}\frac{3}{4}$ belongs to some k-nullity distribution $N(k)$, then M is said to be a $N(k)$-quasi Einstein manifold [17]. In an n-dimensional $N(k)$-quasi Einstein manifold [17] it follows that $k = \frac{a+b}{n-1}$. Now, it is immediate to note that in an n-dimensional $N(k)$-quasi-Einstein manifold [17]

$$R(X,Y)\xi = \frac{a+b}{n-1}\{\eta(Y)X - \eta(X)Y\}. \tag{11}$$

The projective curvature tensor is defined by [16]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}. \tag{12}$$

From (13), it follows that

$$P(X,Y)\xi = 0. \tag{13}$$

Definition 1. $M_{a,b}^n(\xi)$ *is called regular if* $k \neq 0$, *equivalently* $a + b \neq 0$.

Theorem 1. [4] *Fix a* $M_{a,b}^n(\xi)$ *not necessarily regular. Then* (g, ξ, λ) *is a Ricci soliton on* $M_{a,b}^n(\xi)$ *if and only if* $k = -\lambda$ *is a constant and* $L_\xi g = 2b(g - \eta \times \eta)$.

From theorem 1 and Eqn.(2), we have

$$S(X,Y) = -(\lambda + b)g(X,Y) + b\eta(X)\eta(Y). \tag{14}$$

3 N(k)-quasi Einstein manifold

We consider an n-dimensional $N(k)$-quasi Einstein manifold M satisfying the condition

$$S((X,\xi).P)(Y,Z)U = 0. \tag{15}$$

By definition, we have $S((X,\xi).P)(Y,Z)U = ((X \wedge_s \xi).P)(Y,Z)U$, where the endomorphism $(X \wedge_s Y)Z$ is defined by $(X \wedge_s Y)Z = S(Y,Z)X - S(X,Z)Y$.

Therefore (15) takes the form,

$$\begin{aligned}
&S(\xi, P(Y,Z)U)X - S(X, P(Y,Z)U)\xi - S(\xi,Y)P(X,Z)U + S(X,Y)P(\xi,Z)U - S(\xi,Z)P(Y,X)U \\
&- S(X,Z)P(Y,\xi)U - S(\xi,U)P(\xi,Z)U - S(\xi,Z)P(Y,X)U + S(X,Z)P(Y,\xi)U - S(\xi,U)P(Y,Z)X \\
&+ S(X,U)P(Y,Z)\xi = 0.
\end{aligned} \tag{16}$$

Contracting this with ξ, we get

$$\begin{aligned}
&S(\xi, P(Y,Z)U)\eta(X) - S(X, P(Y,Z)U) - S(\xi,Y)\eta(P(X,Z)U) + S(X,Y)\eta(P(\xi,Z)U) - S(\xi,Z)\eta(P(Y,X)U) \\
&- S(X,Z)\eta(P(Y,\xi)U) - S(\xi,U)\eta(P(\xi,Z)U) - S(\xi,Z)\eta(P(Y,X)U) + S(X,Z)\eta(P(Y,\xi)U) \\
&- S(\xi,U)\eta(P(Y,Z)X) + S(X,U)\eta(P(Y,Z)\xi) = 0.
\end{aligned} \tag{17}$$

Using (12) in (17), we have

$$\begin{aligned}
&\frac{ab}{n-1}\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U) + g(X,Y)\eta(Z)\eta(U) - g(X,Z)\eta(Y)\eta(U)\} \\
&+ \frac{b^2}{n-1}\{g(X,Y)\eta(Z)\eta(U) - g(X,Z)\eta(Y)\eta(U)\} + \frac{a^2}{n-1}\{g(Z,U)g(X,Y) \\
&- g(Y,U)g(X,Z)\} - ag(X, R(Y,Z)U) = 0.
\end{aligned} \tag{18}$$

Taking $Z = \xi$ in (18), we obtain

$$\frac{(a+b)b}{n-1}\{g(X,Y) - \eta(X)\eta(Y)\}\eta(W) = 0. \tag{19}$$

Since, in a quasi Einstein manifold $b \neq 0$, from (19) it follows that $a + b = 0$. Thus we state that.

Theorem 2. *An n- dimensional N(k)- quasi Einstein manifold M satisfies* $S(X,\xi) \cdot P = 0$ *if and only if the sum of the associated scalars is zero.*

Definition 2. *A Riemannian manifold is said to be projectively pseudosymmetric [21] if at every point of the manifold the following relation*

$$(R(X,Y) \cdot P)(U,V)W = L_P((X \wedge Y) \cdot P)(U,V)W, \tag{20}$$

holds for any vector fields $X,Y,U,V,W \in TM$, *where* $L_P(\neq \frac{a+b}{n-1})$ *is a function on M.*

The endomorphism $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y.$$

Putting $Y = W = \xi$ in (20), we have

$$(R(X,\xi) \cdot P)(U,V)\xi = L_P((X \wedge \xi) \cdot P)(U,V)\xi. \tag{21}$$

Now

$$L_P((X \wedge \xi) \cdot P)(U,V)\xi = L_P\{(X \wedge \xi)P(U,V)\xi - P((X \wedge \xi)U,V)\xi - P(U,(X \wedge \xi)V)\xi - P(U,V)(X \wedge \xi)\xi\} \tag{22}$$

In view of (11) and (12), (21) takes the form

$$\frac{a+b}{n-1}P(U,V)X = L_P P(U,V)X. \tag{23}$$

The above equation yields

$$\left(L_P - \frac{a+b}{n-1}\right)P(U,V)X = 0. \tag{24}$$

By definition 2 $L_P \neq \frac{a+b}{n-1}$, hence we get

$$P(U,V)X = 0,$$

for any vector fields U,V and X. Conversely, $P = 0$, then the (21) holds trivially. Thus we can state the following.

Theorem 3. *An n- dimensional $N(k)$- quasi Einstein manifold is projectively pseudosymmetric if and only if it is projectively flat.*

4 Ricci solitons and Gradient Ricci solitons on $N(k)$-quasi Einstein manifold

Let (M,g) be an n- dimensional $N(k)$- quasi Einstein manifold and g be a gradient Ricci soliton. Then (3) can be written as

$$\nabla_Y Df = QY + \lambda Y, \tag{25}$$

for all vector fields X in M, where D denotes gradient operator of g. From (25) it follows that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X, \; X,Y \in TM. \tag{26}$$

By substituting $X = \xi$ in (26) and taking inner product with ξ, we get

$$g(R(\xi,Y)Df,\xi) = g\left(\frac{a+b}{n-1}(Df - (\xi f)\xi),Y\right), \; Y \in TM. \tag{27}$$

Then we have

$$g\left(\left[(\nabla_\xi Q)Y - (\nabla_Y Q)\xi\right],\xi\right) = 0. \tag{28}$$

From (27) and (28), we get

$$\frac{a+b}{n-1}(Df - (\xi f)\xi) = 0,$$

That is, either $a+b = 0$ or

$$Df = (\xi f)\xi. \tag{29}$$

If $a+b \neq 0$ and using (29) in (25), we get

$$S(X,Y) + \lambda g(X,Y) = Y(\xi f)\eta(X). \tag{30}$$

Symmetrizing with respect to X and Y, we arrive at

$$2S(X,Y) + 2\lambda g(X,Y) = Y(\xi f)\eta(X) + X(\xi f)\eta(Y). \tag{31}$$

Substituting ξ for Y in (31), we get

$$X(\xi f) = 2((a+b) + \lambda)\eta(X). \tag{32}$$

From (31) and (32), we have

$$S(X,Y) = 2[(a+b) + \lambda]\eta(X)\eta(Y) - \lambda g(X,Y). \tag{33}$$

Using (33) in (25), we get

$$\nabla_Y Df = 2[(a+b) + \lambda]\eta(X)\xi. \tag{34}$$

Using (34) we compute $R(X,Y)Df$ and obtain

$$g(R(X,Y)(\xi f)\xi, \xi) = 2[(a+b) + \lambda]d\eta(X,Y). \tag{35}$$

Then we get

$$\lambda = -(a+b). \tag{36}$$

Therefore from (32) we have

$$X(\xi f) = 0, X \in TM.$$

i.e. $\xi f = c$, where c is a constant. Thus (29) gives

$$df = c\eta.$$

Taking exterior derivative on both sides of the above equation, we get that

$$cd\eta = 0.$$

Hence $c = 0$. Thus f is a constant. Consequently, the equation (25) reduces to $S(X,Y) = -\lambda g(X,Y)$, i.e., g is Einstein. Hence we state the following

Theorem 4. *A regular $N(k)$-quasi Einstein manifold $M_{a,b}^n$ with generator ξ does not admit a gradient Ricci soliton.*

Consider an n - dimensional $N(k)$- quasi Einstein manifold M satisfying the condition

$$P(\xi, X) \cdot S = 0. \tag{37}$$

This implies

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = 0. \tag{38}$$

Which in view of (7) gives

$$0 = \frac{b}{n-1}\{g(X,Y)S(\xi,Z) - \eta(X)\eta(Y)S(\xi,Z) + g(X,Z)S(Y,\xi) - \eta(X)\eta(Z)S(Y,\xi)\}. \tag{39}$$

Since $b \neq 0$, in view of (2), it follows that

$$0 = \lambda\{-g(X,Y)\eta(Z)g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)\}. \tag{40}$$

Taking $Z = \xi$ in (40), we get

$$0 = \lambda\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{41}$$

Therefore $\lambda = 0$. Hence we can state the following

Theorem 5. *A Ricci soliton in an n-dimensional $N(k)$- quasi Einstein manifold which satisfies the condition $P(\xi,X)\cdot S = 0$ is steady.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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