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Research Article

Study of 2m-th order parabolic equation in non-symmetric conical domains

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Abstract

This article is devoted to the study of a N-space dimensional linear high-order parabolic equation, subject to Cauchy-Dirichlet boundary conditions. The problem is set in a non-symmetric conical domain. The analysis is performed in the framework of weighted anisotropic Sobolev spaces by using the domain decomposition method.

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1. Introduction

This work is devoted to the study of the following high-order parabolic problem

$$\begin{cases} \partial_t u + Au = f \in L^2_\omega(Q), \\ \partial_\nu^k u\big|_{\partial Q \setminus \Gamma_T} = 0, \ k = 0, ..., m-1, \end{cases} \tag{1.1}$$

where $A = (-1)^m(\partial_{x_1}^{2m} + \partial_{x_2}^{2m} + ... + \partial_{x_N}^{2m})$, m belongs to the set of all nonzero natural numbers $\mathbb{N}^*$, ∂Q is the boundary of Q, Γ_T is the part of the boundary of Q where $t = T$, and ∂_ν stands for the normal derivative. $L^2_\omega(Q)$ is the space of square-integrable functions on Q with the measure $\omega dt dx_1 dx_2 ... dx_N$, where the weight ω is a real-valued function defined on $[0, T]$, differentiable on $]0, T]$. Here Q is the non-symmetric conical domain

$$Q = \left\{ (t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : \begin{array}{l} 0 \le \frac{x_1^2}{h^2(t)\varphi^2(t)} + \frac{x_2^2}{\varphi^2(t)} + ... + \frac{x_N^2}{\varphi^2(t)} < 1, \\ 0 < t < T \end{array} \right\},$$

where $T > 0$, φ and h are two Lipschitz continuous real-valued functions on $[0, T]$ satisfying

$$\varphi(0) = 0, \ \text{and} \ \varphi(t) > 0, \ \forall t \in]0, T], \tag{1.2}$$

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$$0 < \delta \leq h(t) \leq \beta,\ \forall t \in [0, T], \tag{1.3}$$

where δ and β are positive constants.

The difficulty related to this kind of problems comes from the fact that the domain Q considered here is nonstandard since it shrinks at $t = 0$ ($\varphi(0) = 0$), which prevents the domain Q to be transformed into a regular domain without the appearance of some degenerate terms in the parabolic equation, see for example Sadallah [24]. On the other hand, we cannot recast such problems in semi groups setting. Indeed, since the initial condition is defined on a set measure zero, then the semi group generating the solution cannot be defined.

It is well known that there are two main approaches for the study of boundary value problems in such non-smooth domains. We can work directly in the non-regular domains and we obtain singular solutions (see, for example [13, 16, 17, 25]), or we impose conditions on the non-regular domains to obtain regular solutions (see, for example [10, 11, 21, 24]). It is the second approach that we follow in this work. So, let us consider the anisotropic weighted Sobolev space

$$H_{0,\omega}^{1,2m}(Q) := \left\{ u \in H_{\omega}^{1,2m}(Q) : \partial_{\nu}^{k} u\big|_{\partial Q \setminus \Gamma_T} = 0,\ k = 0, ..., m-1 \right\},$$

with

$$H_{\omega}^{1,2m}(Q) = \{u : \partial_t u, \partial^{\alpha} u \in L_{\omega}^{2}(Q), |\alpha| \leq 2m\}$$

where

$$\alpha = (i_1, i_2, ..., i_N) \in \mathbb{N}^2, |\alpha| = i_1 + i_2 + ... + i_N, \partial^{\alpha} u = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u.$$

The space $H_{\omega}^{1,2m}(Q)$ is equipped with the natural norm, that is

$$\|u\|_{H_{\omega}^{1,2m}(Q)} = \left(\|\partial_t u\|_{L_{\omega}^{2}(Q)}^{2} + \sum_{|\alpha| \leq 2m} \|\partial^{\alpha} u\|_{L_{\omega}^{2}(Q)}^{2} \right)^{1/2}.$$

In this paper we prove that Problem (1.1) admits a unique solution u in $H_{\omega}^{1,2m}(Q)$, under the following additional conditions on the functions φ and ω

$$\varphi'(t)\, \varphi^{m}(t) \quad \to \quad 0 \qquad \text{as } t \to 0, \quad m \in \mathbb{N}^{*}, \tag{1.4}$$

$$\forall t \in [0, T] : \omega(t) > 0, \tag{1.5}$$

$$\omega \text{ is a decreasing function on }]0, T]. \tag{1.6}$$

Our main result is the following.

Theorem 1.1. *Let us assume that the functions ω, φ and h verify assumptions (1.5), (1.6), (1.2) and (1.3). Then, Problem (1.1) admits a unique solution $u \in H_{\omega}^{1,2m}(Q)$ in one of these two cases:*

(1) *the functions $(h\varphi)$ and φ are increasing in a neighborhood of* 0,
(2) *the function φ verifies condition (1.4).*

The case $m = 1$ corresponding to a second-order parabolic equation is studied in [9] and [12] both in bi-dimensional and multidimensional cases. In Sadallah [26] and Kheloufi et al. [14], the second-order parabolic problem has been studied in the case of a symmetric conical domain; i.e., in the case where $h = 1$, both in bi-dimensional and multidimensional cases.

Whereas second-order parabolic equations in non-smooth domains are well studied, the literature concerning higher-order parabolic problems in non-cylindrical domains does not seem to be very rich. The solvability of the first boundary-value problem for higher-order parabolic equations in non-cylindrical domains in Sobolev spaces was considered in Mikhailov [22] for the one-dimensional case, and in [23] for the multidimensional case. The author considered a class of "backward" paraboloid for which the parabolic boundary lies below the characteristic plane $t = 0$. In the case of Hölder spaces functional framework, in

Baderko [1] and [2], we can find solvability results of boundary value problems for a 2m-th order parabolic equation for non-cylindrical domains (of the same kind but which can not include our domain) with a non-smooth (in t) lateral boundary. In [7] the authors obtained well posedness results for the solution of a boundary value-problem for the parabolic equation

$$Lu = \partial_t u + (-1)^m (\sum_{j=1}^{N} \partial_{x_j}^{2m} + \partial_{x_{N+1}}^{2m})u = f$$

in a noncylindrical domain with respect to one spatial variable. More precisely, the spatial domain considered is

$$D = \left\{(x, x_{N+1}) : x \in \mathbb{R}^N, \alpha_1(x) < x_{N+1} < \alpha_2(x)\right\}$$

with $\alpha_k \in C^1(\mathbb{R}^N)$, $k \in \{1, 2\}$. Further references on the analysis of higher-order parabolic problems in non-cylindrical domains are: Cherepova [4], Labbas and Sadallah [18], Galaktionov [6] and Cherfaoui et al. [5].

The plan of this paper is as follows. First, we prove a uniqueness result for Problem (1.1). Then, to prove the existence of the solution of Problem (1.1), we divide the study into two steps:

a) We prove a uniqueness and existence result with estimates, for a Problem (3.1) similar to (1.1) where Q is replaced by the truncated domain

$$Q_n = \left\{(t, x_1, x_2, ..., x_N) \in Q : \frac{1}{n} < t < T\right\}, \; n \in \mathbb{N}^*,$$

(Theorem 3.1 and Proposition 3.10).

b) We build a solution u of Problem (1.1) when T is small enough, by considering $\widetilde{u_n}$ the 0-extension to Q of the solution of (3.1) (u_n exists by Theorem 3.1), and showing (in virtue of Proposition 3.10) that $\widetilde{u_{n_k}} \rightharpoonup u$, weakly in $L^2_\omega(Q)$, for a suitable increasing sequence of integers $(n_k)_{k\geq 1}$. The obtained existence local result can be extended to a global in time one by considering

$$D_1 = \{(t, x_1, x_2, ..., x_N) \in Q : 0 < t < T_1\}$$

and

$$D_2 = \{(t, x_1, x_2, ..., x_N) \in Q : T_1 < t < T\}$$

with T_1 small enough and applying the previous case.

2. Uniqueness of solutions and technical lemmas

Proposition 2.1. *Under the assumptions (1.5) and (1.6) on the weight function ω, Problem (1.1) admits at most one solution.*

Proof. Let us consider $u \in H^{1,2m}_{0,\omega}(Q)$ a solution of Problem (1.1) with a null right-hand side term. So, the calculations show that the inner product $\langle \partial_t u + Au, u\rangle$ in $L^2_\omega(Q)$ gives

$$\begin{aligned} 0 \;=\; & \textstyle\int_{\Gamma_T} |u|^2 \omega(T)\, dx_1 dx_2 ... dx_N - \int_Q \frac{1}{2} |u|^2 \omega'(t)\, dt\; dx_1 dx_2 ... dx_N \\ & + \textstyle\int_Q \left(\left|\partial^m_{x_1} u\right|^2 + \left|\partial^m_{x_2} u\right|^2 + ... + \left|\partial^m_{x_N} u\right|^2 \right) \omega(t) dt\; dx_1 dx_2 ... dx_N. \end{aligned}$$

Thanks to the conditions (1.5) and (1.6), this implies that $\left|\partial^m_{x_1} u\right|^2 + \left|\partial^m_{x_2} u\right|^2 + ... + \left|\partial^m_{x_N} u\right|^2 = 0$ and consequently $\partial^{2m}_{x_1} u = \partial^{2m}_{x_2} u = ... = \partial^{2m}_{x_N} u = 0$. Then, the equation of Problem (1.1) gives $\partial_t u = 0$. Thus, $u = \sum_{k=0}^{m-1} a_k x^k$, $a_k \in \mathbb{R}$, $k = 0, 1, ..., m-1$. The boundary conditions imply that $u = 0$ in Q. This proves the uniqueness of the solution of Problem (1.1). □

Remark 2.2. In the sequel, we will be interested only in the question of the existence of the solution of Problem (1.1).

The following result is well known (see, for example, [20]).

Lemma 2.3. *Let $B(0,1)$ be the unit ball of $\mathbb{R}^N$. Then, the operator*

$$A : H^{2m}(B(0,1)) \cap H_0^m(B(0,1)) \longrightarrow L^2(B(0,1)), \; v \mapsto Av = (-1)^m \sum_{j=1}^{N} \partial_{x_j}^{2m} v$$

is an isomorphism. Moreover, there exists a constant $C > 0$ such that

$$\|v\|_{H^{2m}(B(0,1))} \leq C \|Av\|_{L^2(B(0,1))}, \; \forall v \in H^{2m}(B(0,1)) \cap H_0^m(B(0,1)).$$

In the above lemma, H^{2m} and H_0^m are the usual Sobolev spaces defined, for instance, in Lions-Magenes [20]. In Section 3, we will need the following result.

Lemma 2.4. *For a fixed $t \in \left]0,T\right[$, there exists a constant $C > 0$ such that for each $u \in H^{2m}(\Omega_t)$, we have*

$$\left\|\partial_{x_j}^l u\right\|_{L^2(\Omega_t)}^2 \leq C\varphi^{2(2m-l)}(t) \|Au\|_{L^2(\Omega_t)}^2, \; l = 0,1,...,2m-1, \; j = 1,2,...,N.$$

Here, Ω_t is the section of Q defined (for a fixed $t \in]0,T[$) by

$$\Omega_t = \left\{(x_1,x_2,...,x_N) \in \mathbb{R}^N : \; 0 \leq \frac{x_1^2}{h^2(t)\varphi^2(t)} + \frac{x_2^2}{\varphi^2(t)} + ... + \frac{x_N^2}{\varphi^2(t)} < 1\right\}.$$

Proof. It is a direct consequence of Lemma 2.3. Indeed, let $t \in \left]0,T\right[$ and define the following change of variables

$$\begin{array}{lll} B(0,1) & \longrightarrow & \Omega_t, \\ (x_1,x_2,...,x_N) & \longmapsto & (h(t)\varphi(t)x_1, \varphi(t)x_2, ..., \varphi(t)x_N) = (x_1', x_2', ..., x_N'). \end{array}$$

Set $v(x_1,x_2,...,x_N) = u(x_1',x_2',...,x_N')$, then if $v \in H^{2m}(B(0,1))$, u belongs to $H^{2m}(\Omega_t)$.

(a) We have

$$\begin{aligned} \left\|\partial_{x_1}^l v\right\|_{L^2(B(0,1))}^2 &= \int_{B(0,1)} \left(\partial_{x_1}^l v\right)^2 (x_1,x_2,...,x_N)\, dx_1 dx_2 ... dx_N \\ &= \int_{\Omega_t} (h(t)\varphi(t))^{2l} \left(\partial_{x_1'}^l u\right)^2 (x_1',x_2',...,x_N') \tfrac{1}{h(t)\varphi^N(t)} dx_1' dx_2' ... dx_N' \\ &= h^{2l-1}(t)\varphi^{2l-N}(t) \int_{\Omega_t} \left(\partial_{x_1'}^l u\right)^2 (x_1',x_2',...,x_N')\, dx_1' dx_2' ... dx_N' \\ &= h^{2l-1}(t)\varphi^{2l-N}(t) \left\|\partial_{x_1'}^l u\right\|_{L^2(\Omega_t)}^2 \end{aligned}$$

where $l \in \{0,1,...,2m-1\}$. On the other hand, we have

$$\begin{aligned} &\|Av\|_{L^2(B(0,1))}^2 \\ &\quad = \int_{B(0,1)} \left[(-1)^m \left(\partial_{x_1}^{2m} v + \partial_{x_2}^{2m} v + ... + \partial_{x_N}^{2m} v\right)(x_1,x_2,...,x_N)\right]^2 \, dx_1 dx_2 ... dx_N \\ &\quad = \int_{\Omega_t} \left[(h(t)\varphi(t))^{2m} \partial_{x_1'}^{2m} u + \textstyle\sum_{k=2}^N \varphi^{2m}(t) \partial_{x_k'}^{2m} u\right]^2 (x_1',x_2',...,x_N') \\ &\qquad \times \tfrac{1}{(h\varphi^N)(t)} dx_1' dx_2' ... dx_N' \\ &\quad = \tfrac{\varphi^{4m}(t)}{h(t)\varphi^N(t)} \int_{\Omega_t} \left(h^{2m}(t)\partial_{x_1'}^{2m} u + \textstyle\sum_{k=2}^N \partial_{x_k'}^{2m} u\right)^2 (x_1',x_2',...,x_N')\, dx_1' dx_2' ... dx_N' \\ &\quad \leq \tfrac{\varphi^{4m-N}(t)}{\delta} \left(\max(\beta^{2m},1)\right)^2 \int_{\Omega_t} \left(\partial_{x_1'}^{2m} u + \textstyle\sum_{k=2}^N \partial_{x_k'}^{2m} u\right)^2 dx_1' dx_2' ... dx_N' \\ &\quad \leq \tfrac{\varphi^{4m-N}(t)}{\delta} K \int_{\Omega_t} \left(\partial_{x_1'}^{2m} u + \textstyle\sum_{k=2}^N \partial_{x_k'}^{2m} u\right)^2 dx_1' dx_2' ... dx_N' \\ &\quad \leq \tfrac{\varphi^{4m-N}(t)}{\delta} K \|Au\|_{L^2(\Omega_t)}^2. \end{aligned}$$

where $K = \left(\max(\beta^{2m},1)\right)^2$ and δ and β are the constants which appear in (1.3). Using the Lemma 2.3 and the condition (1.3), we obtain the desired inequality.

(b) For $j = 2, ..., N$, we have

$$\begin{aligned}
\left\|\partial_{x_j}^l v\right\|_{L^2(B(0,1))}^2 &= \int_{B(0,1)} \left(\partial_{x_j}^l v\right)^2 (x_1, x_2, ..., x_N)\, dx_1 dx_2 ... dx_N \\
&= \int_{\Omega_t} \varphi^{2l}(t) \left(\partial_{x_j'}^l u\right)^2 (x_1', x_2', ..., x_N') \tfrac{1}{h(t)\varphi^N(t)} dx_1' dx_2' ... dx_N' \\
&= \tfrac{\varphi^{2l-N}(t)}{h(t)} \int_{\Omega_t} \left(\partial_{x_j'}^l u\right)^2 (x_1', x_2', ..., x_N')\, dx_1' dx_2' ... dx_N' \\
&= \tfrac{\varphi^{2l-N}(t)}{h(t)} \left\|\partial_{x_j'}^l u\right\|_{L^2(\Omega_t)}^2
\end{aligned}$$

where $l \in \{0, 1, ..., 2m-1\}$. On the other hand, we have

$$\|Av\|_{L^2(B(0,1))}^2 \le \frac{\varphi^{4m-N}(t)}{\delta} K \|Au\|_{L^2(\Omega_t)}^2 .$$

Using the inequality

$$\left\|\partial_{x_j}^l v\right\|_{L^2(B(0,1))}^2 \le C \|Av\|_{L^2(B(0,1))}^2$$

of Lemma 2.3 and condition (1.3), we obtain the desired inequality

$$\left\|\partial_{x_j'}^l u\right\|_{L^2(\Omega_t)}^2 \le C\varphi^{2(2m-l)}(t) \|Au\|_{L^2(\Omega_t)}^2 .$$

□

Remark 2.5. In Lemma 2.4 we can replace $\|.\|_{L^2}$ by $\|.\|_{L^2_\omega}$.

3. Existence result for problem (1.1)

We divide the proof of Theorem 1.1 into two steps.

3.1. Step 1: Existence result in truncated domains Q_n

In this subsection, we replace Q by Q_n, $n \in \mathbb{N}^*$ and $\frac{1}{n} < T$:

$$Q_n = \left\{(t, x_1, x_2, ..., x_N) \in Q : \frac{1}{n} < t < T\right\}.$$

Theorem 3.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the problem*

$$\begin{cases} \partial_t u_n + A u_n = f_n \in L^2_\omega(Q_n), \\ u_n|_{t=\frac{1}{n}} = 0, \\ \partial_\nu^k u_n\Big|_{\partial Q_n \setminus (\Gamma_T \cup \{t=\frac{1}{n}\})} = 0, \ k = 0, 1, ..., m-1, \end{cases} \tag{3.1}$$

where $f_n = f|_{Q_n}$ admits a unique solution $u_n \in H^{1,2m}_\omega(Q_n)$.

Proof of Theorem 3.1. The change of variables

$$(t, x_1, x_2, ..., x_N) \mapsto (t, y_1, y_2, ..., y_N) = \left(t, \tfrac{x_1}{h(t)\varphi(t)}, \tfrac{x_2}{\varphi(t)}, ..., \tfrac{x_N}{\varphi(t)}\right)$$

transforms Q_n into the cylinder $P_n = \left]\frac{1}{n}, T\right[\times B(0,1)$, where $B(0,1)$ is the unit ball of $\mathbb{R}^N$. Putting $u_n(t, x_1, x_2, ..., x_N) = v_n(t, y_1, y_2, ..., y_N)$ and $f_n(t, x_1, x_2, ..., x_N) = g_n(t, y_1, y_2, ..., y_N)$, then problem (3.1) is transformed, in P_n into the following variable-coefficient parabolic problem

$$\begin{cases} \partial_t v_n + \frac{(-1)^m}{\varphi^{2m}(t)}[\frac{1}{h^{2m}(t)}\partial_{y_1}^{2m} v_n + \sum_{j=2}^N \partial_{y_j}^{2m} v_n] - \frac{(h\varphi')(t)y_1}{(h\varphi)(t)}\partial_{y_1} v_n - \frac{\varphi'(t)}{\varphi(t)}\sum_{j=2}^N y_j \partial_{y_j} v_n = g_n, \\ v_n|_{t=\frac{1}{n}} = 0, \\ \partial_\nu^k v_n\Big|_{\partial P_n \smallsetminus (\Sigma_T \cup \{t=\frac{1}{n}\})} = 0,\ k = 0, 1, ..., m-1, \end{cases}$$

where Σ_T is the part of the boundary of P_n where $t = T$. The above change of variables conserves the spaces L_ω^2 and $H_\omega^{1,2m}$ because $\frac{(-1)^m}{h^{2m}\varphi^{2m}}$, $\frac{(-1)^m}{\varphi^{2m}}$, $\frac{(h\varphi)'}{h\varphi}$ and $\frac{\varphi'}{\varphi}$ are bounded functions when $t \in]\frac{1}{n}, T[$. In other words

$$f_n \in L_\omega^2(Q_n) \Longleftrightarrow g_n \in L_\omega^2(P_n),\ u_n \in H_\omega^{1,2m}(Q_n) \Longleftrightarrow v_n \in H_\omega^{1,2m}(P_n).$$

Proposition 3.2. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the following operator is compact*

$$-\frac{(h\varphi')(t)\,y_1}{(h\varphi)(t)}\partial_{y_1} - \frac{\varphi'(t)}{\varphi(t)}\sum_{j=2}^N y_j \partial_{y_j} : H_{0,\omega}^{1,2m}(P_n) \longrightarrow L_\omega^2(P_n).$$

Proof. P_n has the "horn property" of Besov (see [3]). So, for $j = 1, 2, ..., N$

$$\partial_{y_j} : H_{0,\omega}^{1,2m}(P_n) \longrightarrow H_\omega^{1-\frac{1}{2m},2m-1}(P_n),\ v \longmapsto \partial_{y_j} v,$$

is continuous. Since P_n is bounded, the canonical injection is compact from $H_\omega^{1-\frac{1}{2m},2m-1}(P_n)$ into $L_\omega^2(P_n)$ (see for instance [3]), where

$$H_\omega^{1-\frac{1}{2m},2m-1}(P_n) = L^2\left(\frac{1}{n}, T; H^{2m-1}(B(0,1))\right) \cap H^{1-\frac{1}{2m}}\left(\frac{1}{n}, T; L^2(B(0,1))\right).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces, see for instance [20]. Consider the composition

$$\partial_{y_j} : H_{0,\omega}^{1,2m}(P_n) \to H_\omega^{1-\frac{1}{2m},2m-1}(P_n) \to L_\omega^2(P_n),\ v \mapsto \partial_{y_j} v \mapsto \partial_{y_j} v,$$

then ∂_{y_j} is a compact operator from $H_{0,\omega}^{1,2m}(P_n)$ into $L_\omega^2(P_n)$. Since $-\frac{(h\varphi')(t)}{h(t)\varphi(t)}$ and $-\frac{\varphi'(t)}{\varphi(t)}$ are a bounded functions for $\frac{1}{n} < t < T$, the operators $-\frac{(h\varphi')(t)y_1}{(h\varphi)(t)}\partial_{y_1}$, $-\frac{\varphi'(t)y_k}{\varphi(t)}\partial_{y_k}$, $k = 2, 3, ..., N$ are also compact from $H_{0,\omega}^{1,2m}(P_n)$ into $L_\omega^2(P_n)$. Consequently,

$$-\frac{(h\varphi')(t)\,y_1}{(h\varphi)(t)}\partial_{y_1} - \frac{\varphi'(t)}{\varphi(t)}\sum_{j=2}^N y_j \partial_{y_j}$$

is compact from $H_{0,\omega}^{1,2m}(P_n)$ into $L_\omega^2(P_n)$. □

So, thanks to Proposition 3.2, to complete the proof of Theorem 3.1, it is sufficient to show that the operator

$$\partial_t + \frac{(-1)^m}{h^{2m}(t)\,\varphi^{2m}(t)}\partial_{y_1}^{2m} + \frac{(-1)^m}{\varphi^{2m}(t)}\sum_{j=2}^N \partial_{y_j}^{2m}$$

is an isomorphism from $H_{0,\omega}^{1,2m}(P_n)$ into $L_\omega^2(P_n)$.

Lemma 3.3. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the operator*

$$\partial_t + \frac{(-1)^m}{h^{2m}(t)\,\varphi^{2m}(t)}\partial_{y_1}^{2m} + \frac{(-1)^m}{\varphi^{2m}(t)}\sum_{j=2}^N \partial_{y_j}^{2m}$$

is an isomorphism from $H_{0,\omega}^{1,2m}(P_n)$ into $L_\omega^2(P_n)$.

Proof. Since the coefficient $\dfrac{(-1)^m}{h^{2m}(t)\,\varphi^{2m}(t)}, \dfrac{(-1)^m}{\varphi^{2m}(t)}$ are bounded in $\overline{P_n}$, the optimal regularity is given by Ladyzhenskaya, Solonnikov and Ural'tseva [19]. □

We shall need the following result in order to justify the calculus of this section.

Lemma 3.4. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the space*

$$\left\{ v_n \in H^{2m}(P_n) : v_n|_{t=\frac{1}{n}} = \partial_\nu^k v_n\Big|_{\partial P_n \smallsetminus (\Sigma_T \cup \{t=\frac{1}{n}\})} = 0,\ k = 0, 1, ..., m-1 \right\}$$

is dense in the space

$$\left\{ v_n \in H^{1,2m}(P_n) : v_n|_{t=\frac{1}{n}} = \partial_\nu^k v_n\Big|_{\partial P_n \smallsetminus (\Sigma_T \cup \{t=\frac{1}{n}\})} = 0,\ k = 0, 1, ..., m-1 \right\}.$$

Here, H^{2m} stands for the usual Sobolev space defined, for instance, in Lions-Magenes [20].

The proof of the above lemma may be found in [20].

Remark 3.5. In Lemma 3.4, we can replace P_n by Q_n with the help of the change of variables defined above.

3.2. Step 2: Existence result in the conical domain Q

3.2.1. An "energy" type estimate. Now, we return to the conical domain Q and we suppose that the functions h and φ satisfy conditions (1.2), (1.3) and (1.4).
For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we denote $f_n = f|_{Q_n}$ and $u_n \in H_\omega^{1,2m}(Q_n)$ the solution of Problem (1.1) in Q_n. Such a solution exists by Theorem 3.1. We look for a constant $K_1 > 0$ independent of n satisfying the estimate

$$\|u_n\|_{H_\omega^{1,2m}(Q_n)} \leq K_1 \|f_n\|_{L_\omega^2(Q_n)} \leq K_1 \|f\|_{L_\omega^2(Q)}. \tag{3.2}$$

Let us denote the inner product in $L_\omega^2(Q_n)$ by $\langle .,. \rangle$, then we have

$$\begin{aligned} \|f_n\|^2_{L_\omega^2(Q_n)} &= \langle \partial_t u_n + A u_n, \partial_t u_n + A u_n \rangle \\ &= \|\partial_t u_n\|^2_{L_\omega^2(Q_n)} + \|A u_n\|^2_{L_\omega^2(Q_n)} + 2\langle \partial_t u_n, A u_n \rangle. \end{aligned}$$

We need the following result which is a consequence of Lemma 2.4 and Grisvard-Looss [8, Theorem 2.2].

Lemma 3.6. *There exists a constant $C > 0$ independent of n such that*

$$\sum_{|\alpha|=2m} \|\partial^\alpha u_n\|^2_{L_\omega^2(Q_n)} \leq C \|A u_n\|^2_{L_\omega^2(Q_n)}.$$

In the sequel, we will estimate the inner product $\langle \partial_t u_n, A u_n \rangle$ making use of the boundary conditions

$$u_n|_{t=\frac{1}{n}} = \partial_\nu^k u_n\Big|_{\partial Q_n \smallsetminus (\Gamma_T \cup \{t=\frac{1}{n}\})} = 0,\ k = 0, ..., m-1,$$

which are equivalent to

$$u_n|_{t=\frac{1}{n}} = \partial_{x_j}^k u_n\Big|_{\partial Q_n \smallsetminus (\Gamma_T \cup \{t=\frac{1}{n}\})} = 0,\ k = 0, ..., m-1; j = 1, 2, ..., N$$

This equivalence can be proved, for instance, by induction.

Lemma 3.7. *One has*

$$2\langle \partial_t u_n, Au_n\rangle = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_{\frac{1}{n}}^{T}\left((h\varphi)'(t)\cos^2\theta_1 + (h\varphi')(t)\sin^2\theta_1\right)$$
$$\times\left(\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\varphi(t)\omega(t)\,dtd\theta_1\ldots d\theta_{N-2}d\theta_{N-1}$$
$$+\int_{\Gamma_T}\left(\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right)(T,x_1,x_2,...,x_N).\omega(T)\,dx_1dx_2...dx_N$$
$$-\int_{Q_n}\left(\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\omega'(t)\,dtdx_1dx_2...dx_N.$$

Proof. We have

$$\partial_t u_n.Au_n = \sum_{j=1}^{N}\left[\sum_{k=0}^{m-1}\partial_{x_j}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m} + \frac{1}{2}\partial_t\left(\partial_{x_j}^m u_n\right)^2\right].$$

Then

$$\begin{aligned}2\langle \partial_t u_n, Au_n\rangle &= 2\textstyle\int_{Q_n}\partial_t u_n.Au_n.\omega(t)\,dtdx_1dx_2...dx_N\\
&= 2\textstyle\int_{Q_n}\sum_{j=1}^{N}\sum_{k=0}^{m-1}\partial_{x_j}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}\\
&\quad\times\omega(t)\,dtdx_1dx_2...dx_N\\
&\quad+\textstyle\int_{Q_n}\partial_t\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2.\omega(t)\,dtdx_1dx_2...dx_N\\
&= 2\textstyle\int_{\partial Q_n}\sum_{j=1}^{N}\sum_{k=0}^{m-1}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}\nu_{x_j}.\omega(t)\,d\sigma\\
&\quad+\textstyle\int_{\partial Q_n}\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\nu_t.\omega(t)\,d\sigma\\
&\quad-\textstyle\int_{Q_n}\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2.\omega'(t)\,dtdx_1dx_2...dx_N\end{aligned}$$

with $\nu_t,\nu_{x_1},...,\nu_{x_N}$ are the components of the unit outward normal vector at ∂Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Q_n where $t=\dfrac{1}{n}$, we have $u_n=0,\ \nu_t=-1$ and $\nu_{x_j}=0,\ j=1,2,...,N$. Consequently the corresponding boundary integral vanishes. On the part of the boundary where $t=T$, we have $\nu_{x_j}=0,\ j=1,2,...,N$ and $\nu_t=1$. Accordingly, the corresponding boundary integral

$$\int_{\Gamma_T}\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2(T,x_1,x_2,...,x_N).\omega(T)\,dx_1dx_2...dx_N$$

is nonnegative. On the part Γ_1 of ∂Q_n defined by

$$\Gamma_1=\left\{(t,x_1,x_2,...,x_N):\frac{x_1^2}{h^2(t)\varphi^2(t)}+\frac{x_2^2}{\varphi^2(t)}+...+\frac{x_N^2}{\varphi^2(t)}=1\right\},$$

we have, for $k=1,2,...,N-1$,

$$\nu_{x_k}=\frac{h(t)\sin\theta_1...\sin\theta_{k-1}\cos\theta_k}{\sqrt{\left(\varphi'(t)h(t)\sin^2\theta_1+(h\varphi)'(t)\cos^2\theta_1\right)^2+\left(h(t)\sin\theta_1\right)^2+\cos^2\theta_1}},$$

$$\nu_{x_N}=\frac{h(t)\sin\theta_1...\sin\theta_{N-2}\sin\theta_{N-1}}{\sqrt{\left(\varphi'(t)h(t)\sin^2\theta_1+(h\varphi)'(t)\cos^2\theta_1\right)^2+\left(h(t)\sin\theta_1\right)^2+\cos^2\theta_1}},$$

$$\nu_t = \frac{-\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)}{\sqrt{\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)^2 + \left(h(t)\sin\theta_1\right)^2 + \cos^2\theta_1}}$$

and

$$\partial_{x_j}^k u_n(t, h(t)\,\varphi(t)\cos\theta_1, \varphi(t)\sin\theta_1\cos\theta_2, ..., \varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\cos\theta_{N-1}, \varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\sin\theta_{N-1}) = 0,\ k = 0, ..., m-1;\ j = 1, 2, ..., N.$$

Let us denote

$$I = 2\int_{\Gamma_1}\sum_{j=1}^{N}\sum_{k=0}^{m-1}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}\,\nu_{x_j}.\omega(t)\,d\sigma.$$

We have

$$\begin{aligned} I &= 2\int_{\Gamma_1}\sum_{j=1}^{N}\left(\partial_t u_n.\partial_{x_j}^{2m-1}u_n\right)(-1)^m\,\nu_{x_j}.\omega(t)\,d\sigma \\ &\quad +2\int_{\Gamma_1}\sum_{j=1}^{N}\sum_{k=1}^{m-2}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}\,\nu_{x_j}.\omega(t)\,d\sigma \\ &\quad -2\int_{\Gamma_1}\sum_{j=1}^{N}\left(\partial_{x_j}^{m-1}\partial_t u_n.\partial_{x_j}^{m}u_n\right)\nu_{x_j}.\omega(t)\,d\sigma \\ &= I_0 + I_1 + I_{m-1}. \end{aligned}$$

a) Estimation of $I_0 = 2\int_{\Gamma_1}\sum_{j=1}^{N}\left(\partial_t u_n.\partial_{x_j}^{2m-1}u_n\right)(-1)^m\,\nu_{x_j}.\omega(t)\,d\sigma$:
We have

$$u_n(t, h(t)\,\varphi(t)\cos\theta_1, \varphi(t)\sin\theta_1\cos\theta_2, ..., \varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\cos\theta_{N-1}, \varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\sin\theta_{N-1}) = 0.$$

Differentiating with respect to t, we obtain

$$\begin{aligned} \partial_t u_n &= -\varphi'(t)\left[\sum_{k=2}^{N-1}\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + \sin\theta_1 ... \sin\theta_{N-1}.\partial_{x_N}u_n\right] \\ &\quad -(h\varphi)'(t)\cos\theta_1.\partial_{x_1}u_n = 0. \end{aligned}$$

So, the boundary integral I_0 vanishes.

b) Estimation of $I_1 = 2\int_{\Gamma_1}\sum_{j=1}^{N}\sum_{k=1}^{m-2}\left(\partial_{x_j}^k\partial_t u_n.\partial_{x_j}^{2m-k-1}u_n\right)(-1)^{k+m}\,\nu_{x_j}.\omega(t)\,d\sigma$:
We have

$$\partial_{x_j}^k u_n(t, h(t)\,\varphi(t)\cos\theta_1, \varphi(t)\sin\theta_1\cos\theta_2, ..., \varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\cos\theta_{N-1}, \varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\sin\theta_{N-1}) = 0, \quad k = 1, ..., m-2;\ j = 1, 2, ..., N.$$

Differentiating with respect to t, we obtain

$$\begin{aligned} \partial_t\partial_{x_j}^k u_n &= -\varphi'(t)\sum_{l=2}^{N-1}\sin\theta_1 \ldots \sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^k u_n \\ &\quad -\varphi'(t)\sin\theta_1 \ldots \sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^k u_n \\ &\quad -(h\varphi)'(t)\cos\theta_1.\partial_{x_1}\partial_{x_j}^k u_n, \quad k = 1, ..., m-2;\ j = 1, 2, \ldots, N. \end{aligned}$$

The Dirichlet boundary conditions on Γ_1 lead to

$$\begin{aligned} \partial_t\partial_{x_j}^k u_n = &-\varphi'(t)\sum_{l=2, l\neq j}^{N-1}\sin\theta_1 \ldots \sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^k u_n \\ &-\varphi'(t)\sin\theta_1 \ldots \sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^k u_n \\ &-(h\varphi)'(t)\cos\theta_1.\partial_{x_1}\partial_{x_j}^k u_n, \quad k = 1, ..., m-2;\ j = 2, \ldots, N-1, \end{aligned}$$

$$\begin{aligned} \partial_t\partial_{x_1}^k u_n = &-\varphi'(t)\sum_{l=2}^{N-1}\sin\theta_1 \ldots \sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_1}^k u_n \\ &-\varphi'(t)\sin\theta_1 \ldots \sin\theta_{N-1}.\partial_{x_N}\partial_{x_1}^k u_n, \end{aligned}$$

and

$$\begin{aligned} \partial_t\partial_{x_N}^k u_n = &-\varphi'(t)\sum_{l=2}^{N-1}\sin\theta_1 \ldots \sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_N}^k u_n \\ &-(h\varphi)'(t)\cos\theta_1.\partial_{x_1}\partial_{x_N}^k u_n, \qquad k = 1, ..., m-2. \end{aligned}$$

Now, differentiating the formula

$$\partial_{x_j}^k u_n(t, h(t)\varphi(t)\cos\theta_1, \varphi(t)\sin\theta_1\cos\theta_2, ..., \varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\cos\theta_{N-1},$$
$$\varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\sin\theta_{N-1}) = 0, \quad k = 1, ..., m-2;\ j = 1, 2, ..., N,$$

with respect to $\theta_1, \ldots, \theta_{N-2}$ and θ_{N-1}, we obtain for $p = 2, \ldots, N-2$,

$$\begin{aligned}\sin\theta_p.\partial_{x_p}\partial_{x_j}^k u_n = &\cos\theta_p \textstyle\sum_{l=p+1}^{N-1} \sin\theta_{p+1}\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^k u_n\\ &+\cos\theta_p\sin\theta_{p+1}\ldots\sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^k u_n,\end{aligned}$$

$$\begin{aligned}h(t).\sin\theta_1.\partial_{x_1}\partial_{x_j}^k u_n =&\cos\theta_1\sum_{l=2}^{N-1}\sin\theta_2\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^k u_n\\ &+\cos\theta_1\sin\theta_2\ldots\sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^k u_n,\end{aligned}$$

and

$$\sin\theta_{N-1}\partial_{x_{N-1}}\partial_{x_j}^k u_n = \cos\theta_{N-1}\partial_{x_N}\partial_{x_j}^k u_n$$

where $k = 1, \ldots, m-2;\ j = 1, \ldots, N$. The Dirichlet boundary conditions on Γ_1 lead to

$$\partial_{x_i}\partial_{x_j}^k u_n = 0,\ k = 1, ..., m-2;\ i = 1, \ldots, N,\ j = 1, \ldots, N$$

and consequently

$$\partial_t\partial_{x_j}^k u_n = 0, k = 1, \ldots, m-2; j = 1, \ldots, N.$$

So, the boundary integral I_1 vanishes.

c) Estimation of $I_{m-1} = -2\int_{\Gamma_1}\sum_{j=1}^N \left(\partial_{x_j}^{m-1}\partial_t u_n.\partial_{x_j}^m u_n\right)\nu_{x_j}.\omega(t)\,d\sigma$:

We have

$$\partial_{x_j}^{m-1} u_n(t, h(t)\varphi(t)\cos\theta_1, \varphi(t)\sin\theta_1\cos\theta_2, ..., \varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\cos\theta_{N-1},$$
$$\varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\sin\theta_{N-1}) = 0,\ j = 1, 2, ..., N.$$

Differentiating with respect to t, we obtain

$$\begin{aligned}\partial_t\partial_{x_j}^{m-1} u_n =& -\varphi'(t)\sum_{l=2}^{N-1}\sin\theta_1\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^{m-1} u_n\\ &-\varphi'(t)\sin\theta_1\ldots\sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^{m-1} u_n\\ &-(h\varphi)'(t)\cos\theta_1.\partial_{x_1}\partial_{x_j}^{m-1} u_n,\ j = 1, 2, \ldots, N.\end{aligned}$$

The Dirichlet boundary conditions on Γ_1 lead to

$$\begin{aligned}\partial_t\partial_{x_j}^{m-1} u_n =& -\varphi'(t)\textstyle\sum_{l=2, l\neq j}^{N-1}\sin\theta_1\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^{m-1} u_n\\ &-\varphi'(t)\sin\theta_1\ldots\sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^{m-1} u_n\\ &-(h\varphi)'(t)\cos\theta_1.\partial_{x_1}\partial_{x_j}^{m-1} u_n,\ j = 2, \ldots, N-1,\end{aligned}$$

$$\begin{aligned}\partial_t\partial_{x_1}^{m-1} u_n =& -\varphi'(t)\textstyle\sum_{l=2}^{N-1}\sin\theta_1\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_1}^{m-1} u_n\\ &-\varphi'(t)\sin\theta_1\ldots\sin\theta_{N-1}.\partial_{x_N}\partial_{x_1}^{m-1} u_n,\end{aligned}$$

and

$$\begin{aligned}\partial_t\partial_{x_N}^{m-1} u_n =& -\varphi'(t)\textstyle\sum_{l=2}^{N-1}\sin\theta_1\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_N}^{m-1} u_n\\ &-(h\varphi)'(t)\cos\theta_1.\partial_{x_1}\partial_{x_N}^{m-1} u_n.\end{aligned}$$

Now, differentiating the formula

$$\partial_{x_j}^{m-1} u_n(t, h(t)\varphi(t)\cos\theta_1, \varphi(t)\sin\theta_1\cos\theta_2, ..., \varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\cos\theta_{N-1},$$
$$\varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\sin\theta_{N-1}) = 0, \qquad j = 1, 2, ..., N.$$

with respect to $\theta_1, \ldots, \theta_{N-2}$ and θ_{N-1}, we obtain for $p = 2, \ldots, N-2$,

$$\sin\theta_p.\partial_{x_p}\partial_{x_j}^{m-1}u_n = \cos\theta_p\left[\sum_{l=p+1}^{N-1}\sin\theta_{p+1}\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^{m-1}u_n\right] + \cos\theta_p\left[\sin\theta_{p+1}\ldots\sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^{m-1}u_n\right], \quad j = 1, \ldots, N$$

$$\begin{aligned} h(t).\sin\theta_1.\partial_{x_1}\partial_{x_j}^{m-1}u_n &= \cos\theta_1\left[\textstyle\sum_{l=2}^{N-1}\sin\theta_2\ldots\sin\theta_{l-1}\cos\theta_l.\partial_{x_l}\partial_{x_j}^{m-1}u_n\right] \\ & \cos\theta_1\left[\sin\theta_2\ldots\sin\theta_{N-1}.\partial_{x_N}\partial_{x_j}^{m-1}u_n\right], \end{aligned}$$

and

$$\sin\theta_{N-1}\partial_{x_{N-1}}\partial_{x_j}^{m-1}u_n = \cos\theta_{N-1}\partial_{x_N}\partial_{x_j}^{m-1}u_n, \ j = 1, \ldots, N.$$

Taking into account these relationships we deduce

$$\begin{aligned} I_{m-1} &= 2\textstyle\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_{\frac{1}{n}}^{T}\left((h\varphi)'(t)\cos^2\theta_1 + (h\varphi')(t)\sin^2\theta_1\right) \\ &\times\left(\textstyle\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\varphi(t)\omega(t)\,dtd\theta_1\ldots d\theta_{N-2}d\theta_{N-1} \end{aligned}$$

Finally

$$\begin{aligned} 2\langle\partial_t u_n, Au_n\rangle &= \textstyle\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_{\frac{1}{n}}^{T}\left((h\varphi)'(t)\cos^2\theta_1 + (h\varphi')(t)\sin^2\theta_1\right) \\ &\times\left(\textstyle\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\varphi(t)\omega(t)\,dtd\theta_1\ldots d\theta_{N-2}d\theta_{N-1} \\ &+\textstyle\int_{\Gamma_T}\left(\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right)(T, x_1, x_2, ..., x_N).\omega(T)\,dx_1dx_2...dx_N \\ &-\textstyle\int_{Q_n}\left(\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\omega'(t)\,dtdx_1dx_2...dx_N. \end{aligned} \tag{3.3}$$

□

Remark 3.8. Observe that the integrals

$$\int_{\Gamma_T}\left(\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right)(T, x_1, x_2).\omega(T)\,dx_1dx_2...dx_N$$

and

$$-\int_{Q_n}\left(\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\omega'(t)\,dtdx_1dx_2...dx_N,$$

which appear in the last formula are nonnegative thanks to the assumptions (1.5) and (1.6) on the weight function ω. This is a good sign for our estimate because we can deduce immediately

$$\begin{aligned} \|f_n\|^2_{L^2_\omega(Q_n)} &\geq \|\partial_t u_n\|^2_{L^2_\omega(Q_n)} + \|Au_n\|^2_{L^2_\omega(Q_n)} \\ &+\textstyle\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_{\frac{1}{n}}^{T}\left((h\varphi)'(t)\cos^2\theta_1 + (h\varphi')(t)\sin^2\theta_1\right) \\ &\times\left(\textstyle\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\varphi(t)\omega(t)\,dtd\theta_1\ldots d\theta_{N-2}d\theta_{N-1}. \end{aligned}$$

So, if φ and $h\varphi$ are increasing functions in the interval $(\frac{1}{n}, T)$, then

$$\begin{aligned} &\textstyle\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_{\frac{1}{n}}^{T}\left((h\varphi)'(t)\cos^2\theta_1 + (h\varphi')(t)\sin^2\theta_1\right) \\ &\times\left(\textstyle\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\right).\varphi(t)\omega(t)\,dtd\theta_1\ldots d\theta_{N-2}d\theta_{N-1} \geq 0. \end{aligned}$$

Consequently,

$$\|f_n\|^2_{L^2_\omega(Q_n)} \geq \|\partial_t u_n\|^2_{L^2_\omega(Q_n)} + \|Au_n\|^2_{L^2_\omega(Q_n)}. \tag{3.4}$$

But, thanks to Lemma 2.4 and since φ is bounded in $(0,T)$, there exists a constant $C'>0$ such that

$$\left\|\partial_{x_j}^{l}u_n\right\|_{L_\omega^2(Q_n)}^2 \leq C'\left\|Au_n\right\|_{L_\omega^2(Q_n)}^2, \quad l=0,1,...,2m-1; j=1,2,...,N.$$

Taking into account Lemma 3.6 and estimate (3.4), this proves the desired estimate (3.2).

So, it remains to establish the estimate (3.2) under the hypothesis (1.4). For this purpose, we need the following lemma.

Lemma 3.9. *One has*

$$\begin{aligned} 2\langle\partial_t u_n, Au_n\rangle = \; & 2\int_{Q_n}\left(\frac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}^m u_n+\frac{\varphi'}{\varphi}\sum_{j=2}^N x_j\partial_{x_j}^m u_n\right)Au_n \\ & \times\omega(t)\,dtdx_1dx_2...dx_N \\ & -(N-2)\int_{Q_n}\left(\frac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}^m u_n\right)^2+\frac{\varphi'}{\varphi}\sum_{j=2}^N\left(\partial_{x_j}^m u_n\right)^2\right) \\ & \times\omega(t)\,dtdx_1dx_2\ldots dx_N \\ & +\int_{\Gamma_T}\sum_{j=1}^N\left(\partial_{x_j}^m u_n\right)^2(T,x_1,x_2).\omega(T)\,dx_1dx_2\ldots dx_N. \end{aligned}$$

Proof. This result can be obtained by following step by step the proof of [9, Lemma 3.8]. □

Proposition 3.10. *Let us assume that the functions ω, φ and h verify assumptions (1.5), (1.6), (1.2) and (1.3). Then for T small enough, there exists a constant K_1 independent of n such that*

$$\|u_n\|_{H_\omega^{1,2m}(Q_n)} \leq K_1\|f_n\|_{L_\omega^2(Q_n)} \leq K_1\|f\|_{L_\omega^2(Q)}$$

if one of the following conditions is satisfied

1. *the functions $h\varphi$ and φ are increasing in a neighborhood of* 0,
2. *the function φ verifies condition (1.4).*

Proof. The case when $h\varphi$ and φ are increasing functions in a neighborhood of 0 has been treated in Remark 3.8. Then assume that φ verifies the condition (1.4).

Remark 3.11. Let $\epsilon>0$ be a real which we will choose small enough. The hypothesis (1.4) implies the existence of a real number $T>0$ small enough such that

$$\forall t\in(0,T), \left|\varphi'(t)\varphi^m(t)\right| \leq \epsilon. \tag{3.5}$$

Now, we continue the proof of Proposition 3.10. We have

$$\begin{aligned} & \left|\int_{Q_n}\left(\frac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}^m u_n+\frac{\varphi'}{\varphi}\sum_{j=2}^N x_j\partial_{x_j}^m u_n\right)Au_n.\omega(t)\,dtdx_1dx_2\right| \\ & \quad\leq\|Au_n\|_{L_\omega^2(Q_n)}\left\|\frac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}^m u_n\right\|_{L_\omega^2(Q_n)}+\|Au_n\|_{L_\omega^2(Q_n)}\left\|\frac{\varphi'}{\varphi}\sum_{j=2}^N x_j\partial_{x_j}^m u_n\right\|_{L_\omega^2(Q_n)}, \end{aligned}$$

but Lemma 2.4 yields for $j=2,\ldots,N$

$$\begin{aligned} \left\|\frac{\varphi'}{\varphi}x_j\partial_{x_j}^m u_n\right\|_{L_\omega^2(Q_n)}^2 & = \int_{\frac{1}{n}}^T\varphi'^2(t)\int_{\Omega_t}\left(\frac{x_j}{\varphi(t)}\right)^2\left(\partial_{x_j}^m u_n\right)^2.\omega(t)\,dtdx_1dx_2...dx_N \\ & \leq \int_{\frac{1}{n}}^T\varphi'^2(t)\int_{\Omega_t}\left(\partial_{x_j}^m u_n\right)^2.\omega(t)\,dtdx_1dx_2...dx_N \\ & \leq C^2\int_{\frac{1}{n}}^T\left(\varphi^m(t)\varphi'(t)\right)^2\int_{\Omega_t}\left(Au_n\right)^2.\omega(t)\,dtdx_1dx_2...dx_N \\ & \leq C^2\epsilon^2\|Au_n\|_{L_\omega^2(Q_n)}^2, \end{aligned}$$

since $(\varphi^m(t)\varphi'(t))\leq\epsilon$ thanks to the condition (3.5). Similarly, we have

$$\left\|\frac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}^m u_n\right\|_{L_\omega^2(Q_n)}^2 \leq C^2\epsilon^2\|Au_n\|_{L_\omega^2(Q_n)}^2$$

Then

$$\left|\int_{Q_n}\left(\frac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}^m u_n+\frac{\varphi'}{\varphi}\sum_{j=2}^{N}x_j\partial_{x_j}^m u_n\right)Au_n.\omega\left(t\right)dtdx_1dx_2...dx_N\right|\leq NC\epsilon\left\|Au_n\right\|^2_{L^2_\omega(Q_n)}.$$

Therefore, Lemma 3.9 shows that

$$\begin{aligned}|2\langle\partial_t u_n,Au_n\rangle|\geq&-2\left|\int_{Q_n}\left(\frac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}^m u_n+\frac{\varphi'}{\varphi}\sum_{j=2}^{N}x_j\partial_{x_j}^m u_n\right)Au_n.\omega\left(t\right)dtdx_1dx_2...dx_N\right|\\&-\left|(N-2)\int_{Q_n}\left(\frac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}^m u_n\right)^2+\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\frac{\varphi'}{\varphi}\right).\omega\left(t\right)dtdx_1dx_2...dx_N\right|\\&+\int_{\Gamma_T}\sum_{j=1}^{N}\left(\partial_{x_j}^m u_n\right)^2\left(T,x_1,x_2,...,x_N\right).\omega\left(T\right)dx_1dx_2...dx_N\\&\geq-N^2C\epsilon\left\|Au_n\right\|^2_{L^2_\omega(Q_n)}.\end{aligned}$$

Hence

$$\begin{aligned}\left\|f_n\right\|^2_{L^2_\omega(Q_n)}&=\left\|\partial_t u_n\right\|^2_{L^2_\omega(Q_n)}+\left\|Au_n\right\|^2_{L^2_\omega(Q_n)}+2\langle\partial_t u_n,Au_n\rangle\\&\geq\left\|\partial_t u_n\right\|^2_{L^2_\omega(Q_n)}+\left(1-N^2C\epsilon\right)\left\|Au_n\right\|^2_{L^2_\omega(Q_n)}.\end{aligned}$$

Then, it is sufficient to choose ϵ such that $1-N^2C\epsilon>0$ to get a constant $K_0>0$ independent of n such that

$$\left\|f_n\right\|_{L^2_\omega(Q_n)}\geq K_0\left\|u_n\right\|_{H^{1,2m}_\omega(Q_n)},$$

and since

$$\left\|f_n\right\|_{L^2_\omega(Q_n)}\leq\left\|f\right\|_{L^2_\omega(Q)},$$

there exists a constant $K_1>0$, independent of n satisfying

$$\left\|u_n\right\|_{H^{1,2m}_\omega(Q_n)}\leq K_1\left\|f_n\right\|_{L^2_\omega(Q_n)}\leq K_1\left\|f\right\|_{L^2_\omega(Q)}.$$

This completes the proof of Proposition 3.10. □

3.2.2. Passage to the limit. Choose a sequence $(Q_n)_{n\in\mathbb{N}^*}$ of the domains defined above (see subsection 3.1), such that $Q_n\subseteq Q$. Then, we have $Q_n\to Q$, as $n\to\infty$. Consider the solution $u_n\in H^{1,2m}_\omega\left(Q_n\right)$ of the Cauchy-Dirichlet problem

$$\begin{cases}\partial_t u_n+Au_n=f_n\in L^2_\omega\left(Q_n\right),\\ u_n|_{t=\frac{1}{n}}=0,\\ \partial_\nu^k u_n\Big|_{\partial Q_n\smallsetminus(\Gamma_T\cup\{t=\frac{1}{n}\})}=0,\ k=0,1,...,m-1,\end{cases}$$

where $f_n=f|_{Q_n}$. Such a solution u_n exists by Theorem 3.1. Let $\widetilde{u_n}$ the 0-extension of u_n to Q. In virtue of Proposition 3.10 for T small enough, we know that there exists a constant C such that

$$\left\|\widetilde{u_n}\right\|_{L^2_\omega(Q)}+\left\|\widetilde{\partial_t u_n}\right\|_{L^2_\omega(Q)}+\sum_{1\leq|\alpha|\leq 2m}\left\|\widetilde{\partial^\alpha u_n}\right\|_{L^2_\omega(Q)}\leq C\left\|f\right\|_{L^2_\omega(Q)}.$$

This means that $\widetilde{u_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial^\alpha u_n}$ for $1\leq|\alpha|\leq 2m$ are bounded functions in $L^2_\omega\left(Q\right)$. So, for a suitable increasing sequence of integers n_k, $k=1,2,...$, there exist functions

$$u,\ v\text{ and }v_\alpha\ 1\leq|\alpha|\leq 2m$$

in $L^2_\omega\left(Q\right)$ such that

$$\widetilde{u_{n_k}}\rightharpoonup u,\ \widetilde{\partial_t u_{n_k}}\rightharpoonup v,\ \widetilde{\partial^\alpha u_{n_k}}\rightharpoonup v_\alpha$$

weakly in $L^2_\omega(Q)$ as $k \to \infty$, $1 \le |\alpha| \le 2m$. Clearly,

$$v = \partial_t u, v_\alpha = \partial^\alpha u, 1 \le |\alpha| \le 2m$$

in the sense of distributions in Q and so in $L^2_\omega(Q)$. So, $u \in H^{1,2m}_\omega(Q)$ and

$$\partial_t u + Au = f \text{ in } Q.$$

On the other hand, the solution u satisfies the boundary conditions

$$\partial^k_\nu u\Big|_{\partial Q \smallsetminus \Gamma_T} = 0,\ k = 0, 1, ..., m-1,$$

since

$$\forall n \in \mathbb{N}^*, \qquad u|_{Q_n} = u_n.$$

This ends the proof of Theorem 1.1 in the case of T small enough.

Remark 3.12. The obtained local in time result can be extended easily to a global in time one by considering

$$D_1 = \{(t, x_1, x_2, ..., x_N) \in Q : 0 < t < T_1\}$$

and

$$D_2 = \{(t, x_1, x_2, ..., x_N) \in Q : T_1 < t < T\}$$

with T_1 small enough and applying the previous case. For more details, see [15].

Remark 3.13. Note that this work may be extended at least in the following directions:

1. The function f on the right-hand side of the equation of Problem (1.1), may be taken in $L^p_\omega(Q)$, $p \in]1, \infty[$. The domain decomposition method used here does not seem to be appropriate for the space $L^p_\omega(Q)$ when $p \neq 2$.

2. The conical domain Q may be replaced by a twisted-conical domain, i.e., the function φ may also depend on an angle $\theta \in (0, 2\pi)$

$$Q = \left\{ (t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : \begin{array}{l} 0 \le \frac{x_1^2}{h^2(t)\varphi^2(t,\theta)} + \frac{x_2^2}{\varphi^2(t,\theta)} + ... + \frac{x_N^2}{\varphi^2(t,\theta)} < 1, \\ 0 < t < T \end{array} \right\},$$

where φ is a Lipschitz continuous real-valued function on $[0, T] \times [0, 2\pi]$ satisfying the conditions $\varphi(0, \theta) = 0$ for $0 \le \theta \le 2\pi$ and $\varphi(t, 0) = \varphi(t, 2\pi)$.

These questions will be developed in forthcoming works.

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