Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 254 – 272
DOI : 10.15672/hujms.455030

Research Article

An extension of z-ideals and z°-ideals

Ali R. Aliabad[1], Mehdi Badie*[2], Sajad Nazari[1]

[1] *Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran*
[2] *Department of Basic Sciences, Jundi-Shapur University of Technology, Dezful, Iran*

Abstract

Let R be a commutative with unity, $Y \subseteq \mathrm{Spec}(R)$, and $h_Y(S) = \{P \in Y : S \subseteq P\}$, for every $S \subseteq R$. An ideal I is said to be an $\mathcal{H}_Y$-ideal whenever it follows from $h_Y(a) \subseteq h_Y(b)$ and $a \in I$ that $b \in I$. A strong $\mathcal{H}_Y$-ideal is defined in the same way by replacing an arbitrary finite set F instead of the element a. In this paper these two classes of ideals (which are based on the spectrum of the ring R and are a generalization of the well-known concepts semiprime ideal, z-ideal, z°-ideal (d-ideal), sz-ideal and sz°-ideal (ξ-ideal)) are studied. We show that the most important results about these concepts, "Zariski topology", "annihilator", and etc. can be extended in such a way that the corresponding consequences seems to be trivial and useless. This comprehensive look helps to recognize the resemblances and differences of known concepts better.

Mathematics Subject Classification (2010). 13A15, 54C40

Keywords. z-ideal, z°-ideal, strong z-ideal, strong z°-ideal, prime ideal, semiprime ideal, Zariski topology, Hilbert ideal, rings of continuous functions.

1. Introduction

The concept of z-ideal, was first studied in the rings of continuous functions as an ideal I of $C(X)$ that $Z(f) \subseteq Z(g)$ and $f \in I$ implies that $g \in I$, see [12]. Then this concept was studied more generally for the commutative rings, in [18], as an ideal I of R that whenever two elements of R are contained in the same family of maximal ideals and I contains one of them, then it follows that I contains the other one. If we use $(Z(f))^\circ \subseteq (Z(g))^\circ$ instead of the above inclusion relation and the minimal prime ideals instead of the maximal ideals in the above definitions, then we obtain the concept of z°-ideal (d-ideal) in $C(X)$ and the commutative rings, which are introduced and carefully studied in [9, 10, 15]. The concepts of z-ideal and z°-ideal can be generalized to the concepts of sz-ideal and sz°-ideal (ξ-ideal), respectively, based on the finite subsets of the ideals instead of the single points in the ideal, and are studied in [3, 7, 18].

In this paper, we define and carefully study the $\mathcal{H}_Y$-ideals and the strong $\mathcal{H}_Y$-ideals which are a generalization of all of the above concepts. It is not difficult to see that a large amount of the results of the above mentioned papers and generally the papers in the literature about these topics, are special cases of the results of this paper.

*Corresponding Author.
Email addresses: aliabady_r@scu.ac.ir (A.R. Aliabad), badie@jsu.ac.ir (M. Badie), sajadnazari1983@gmail.com (S. Nazari)
Received: 24.08.2018; Accepted: 30.10.2018

In the next section we recall some pertinent definitions. In Section 3, we define, characterize and give examples of $\mathcal{H}_Y$-ideals, strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals and study relations among them. We give new characterizations of z°-ideals and sz°-ideals. It is shown that the minimal prime ideals over a (strong) $\mathcal{H}_Y$-ideal are again (strong) $\mathcal{H}_Y$-ideals and so every (strong) $\mathcal{H}_Y$-ideal is the intersection of minimal prime (strong) $\mathcal{H}_Y$-ideals containing it. In $C(X)$ the concepts of $\mathcal{H}_Y$-ideals and strong $\mathcal{H}_Y$-ideals coincide and the conditions under which these two classes of ideals coincide in an arbitrary ring are also considered in this section. The family of all $h_Y(F)$'s, where F is an arbitrary finite subset of R, is closed under the finite intersection and union, hence it forms a distributive lattice. The study of (minimal prime, prime and maximal) filters of this distributive lattice and their correspondence with the (minimal prime, prime and maximal) strong $\mathcal{H}_Y$-ideals of R is the subject of Section 4. Section 5 is devoted to propositions which generate a rich source of examples of $\mathcal{H}_Y$-ideals, strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals. For example if I is a (strong) $\mathcal{H}_Y$-ideal, then $(J : I)$ and I_A are (strong) $\mathcal{H}_Y$-ideals, where A is a multiplicatively closed subset of R disjoint from I. Moreover we give characterizations of Von Neumann regular rings, according to the (strong) $\mathcal{H}_Y$-ideals. In Section 6 we answer the natural questions that arise about the product, contraction, extension and quotients of (strong) $\mathcal{H}_Y$-ideals and Y-Hilbert ideals. In the last section we characterize certain (strong) $\mathcal{H}_Y$-ideals over or contained in an arbitrary ideal. For example for every ideal I, the smallest (strong) $\mathcal{H}_Y$-ideal containing I exists and is shown by $I_{\mathcal{H}}$ ($I_{S\mathcal{H}}$). We give a precise characterization of these ideals and their properties.

2. Preliminaries

In this article, any ring R is commutative with unity. A semiprime ideal is an ideal which is an intersection of prime ideals. The set of all ideals of R is denoted by $\mathcal{I}(R)$. For each ideal $I \in \mathcal{I}(R)$ and each element a of R, we denote the ideal $\{x \in R : ax \in I\}$ by $(I : a)$. When $I = \langle 0\rangle$ we write $\mathrm{Ann}(a)$ instead of $(\langle 0\rangle : a)$ and call this the annihilator of a. A prime ideal P containing an ideal I is said to be a minimal prime over I, if there is no any prime ideal strictly contained in P that contains I. $\mathrm{Spec}(R)$, $\mathrm{Min}(R)$, $\mathrm{Max}(R)$, $\mathrm{Rad}(R)$ and $\mathrm{Jac}(R)$ denote the set of all prime ideals, all minimal prime ideals, all maximal ideals of R and their intersections, respectively. By $\mathrm{Min}(I)$ we mean the set of minimal prime ideals over I. In fact $\mathrm{Min}(\langle 0\rangle) = \mathrm{Min}(R)$. A ring R is said to be reduced if $\mathrm{Rad}(R) = \langle 0\rangle$. If $\mathrm{Jac}(R) = \langle 0\rangle$, then we call R semiprimitive. The socle of a ring R is the sum of all minimal ideals of R.

A prime ideal P is called a Bourbaki associated prime divisor of an ideal I if $(I : x) = P$, for some $x \in R$. We denote the set of all Bourbaki associated prime divisors of an ideal I by $\mathcal{B}(I)$. We use $\mathcal{B}(R)$ instead of $\mathcal{B}(\langle 0\rangle)$. A representation $I = \bigcap_{P\in\mathcal{P}} P$ of I as an intersection of prime ideals is called irredundant if no $P \in \mathcal{P}$ may be omitted. Let I be a semiprime ideal, $P_\circ \in \mathrm{Min}(I)$ is called irredundant with respect to I, if $I \neq \bigcap_{P_\circ\neq P\in\mathrm{Min}(I)} P$. If I is equal to the intersection of all irredundant with respect to I, then we call I a fixed-place ideal, exactly, by [1, Theorem 2.1], we have $I = \bigcap \mathcal{B}(I)$.

In this paper, all $Y \subseteq \mathrm{Spec}(R)$ is considered by Zariski topology; i.e., by assuming as a base for the closed sets of Y, the sets $h_Y(a)$ where $h_Y(a) = \{P \in Y : a \in P\}$. Hence, closed sets of Y are of the form $h_Y(I) = \bigcap_{a\in I} h_Y(a) = \{P \in Y : I \subseteq P\}$, for some ideal I in R. Also, we set $h^c_Y(I) = Y \setminus h_Y(I)$. For any subset S of Y, we denote the kernel of S by $k(S) = \bigcap_{P\in S} P$ and we have $\overline{S} = cl_Y S = h_Y k(S)$. When $Y = \mathrm{Spec}(R)$, we omit the index Y and when $Y = \mathrm{Max}(R)$ ($Y = \mathrm{Min}(R)$) we write M (m) instead of Y in the index. By these notations, for every $S \subseteq R$, we can use the notations $kh_m(S)$ and $kh_M(S)$ instead of P_S and M_S (which is usually used in the context of $C(X)$), respectively. We use the following well-known lemma frequently, one may see [17, Lemma 4.1] or [6, Proposition 2.9] for the proof.

Lemma 2.1. *Let R be a ring, $Y \subseteq Spec(R)$ and $k(Y) = I$. Then $(I : S) = kh_Y^c(S)$, for every $S \subseteq R$. In particular, if $k(Y) = \langle 0 \rangle$, then $Ann(S) = kh_Y^c(S)$.*

Throughout the paper $C(X)$ (resp., $C^*(X)$) is the ring of all (resp., bounded) real valued continuous functions on a Tychonoff space X. Suppose that $f \in C(X)$, we denote $f^{-1}\{0\}$ by $Z(f)$ and $X \setminus Z(f)$ by $Coz(f)$. Every subset of X of the form $Z(f)$ (resp., $Coz(f)$), for some $f \in C(X)$ is called zero set (resp., cozero set). A space X is called pseudocompact, if $C(X) = C^*(X)$. $\overline{Coz(f)}$ is called the support of f. The family of all functions in $C(X)$ with compact (resp., pseudocompact) support is denoted by $C_K(X)$ (resp., $C_\psi(X)$).

Recall that if L is a lattice, then $\emptyset \neq F \subseteq L$ is a filter if F is closed under the finite meet and whenever $a \in F$ and $b \geq a$, then it follows that $b \in F$. A filter F is called prime if for every $a, b \in L$, $a \vee b \in F$ implies that $a \in F$ or $b \in F$.

The reader is referred to [8, 11–13, 21, 22] for undefined terms and notations.

3. $\mathcal{H}_Y$-ideals, $\mathcal{H}_Y$-filters, strong $\mathcal{H}_Y$-ideals, Y-Hilbert ideals and their characterizations

First, for a set A, we designed by $\mathbf{F}(A)$ the set of all finite subsets of A. Recall that a ring of sets is a collection of subsets of some set A which is closed under the finite unions and intersections. A ring of sets is obviously a distributive lattice. Now, for a ring R, we denote the collection $\{h_Y(F) : F \in \mathbf{F}(R)\} = \{h_Y(I) : I \text{ is a finitely generated ideal of } R\}$ by $\mathcal{H}_Y$. Since for arbitrary ideals I and J of R, $h_Y(I) \cap h_Y(J) = h_Y(I+J)$, $h_Y(I) \cup h_Y(J) = h_Y(IJ)$, $h_Y(\langle 0 \rangle) = Y$ and $h_Y(R) = h_Y(\langle 1 \rangle) = \emptyset$, also since the sum and the product of two finitely generated ideals are finitely generated, $\mathcal{H}_Y$ is a ring of sets and so it is a bounded distributive lattice. We call a filter of the distributive lattice $\mathcal{H}_Y$ an $\mathcal{H}_Y$-filter on Y. Note that all prime $\mathcal{H}_Y$-filters and all $\mathcal{H}_Y$-ultrafilters are assumed to be proper filters. Now suppose that $\mathscr{F}$ is an $\mathcal{H}_Y$-filter on $Y \subseteq \mathrm{Spec}(R)$ and I is an ideal of R. We denote $\{h_Y(S) : S \in \mathbf{F}(I)\}$ and $\{a \in R : h_Y(a) \in \mathscr{F}\}$ by $\mathcal{H}_Y(I)$ and $\mathcal{H}_Y^{-1}(\mathscr{F})$, respectively.

Lemma 3.1. *Let I be an ideal of a ring R, $\mathscr{F}$ be an $\mathcal{H}_Y$-filter on $Y \subseteq Spec(R)$ and F be a finite subset of R. The following statements hold.*

(a) *$h_Y(F) \in \mathscr{F}$ if and only if $F \subseteq \mathcal{H}_Y^{-1}(\mathscr{F})$.*
(b) *$\mathcal{H}_Y^{-1}(\mathscr{F})$ is an ideal of R.*
(c) *$\mathcal{H}_Y(I)$ is an $\mathcal{H}_Y$-filter on Y.*

Proof. (a $\Rightarrow$). For every $s \in F$, $h_Y(F) \subseteq h_Y(s)$, thus $h_Y(s) \in \mathscr{F}$ and therefore $s \in \mathcal{H}_Y^{-1}(\mathscr{F})$, for every $s \in F$. Hence, $F \subseteq \mathcal{H}_Y^{-1}(\mathscr{F})$.

(a $\Leftarrow$). For every $s \in S$, $h_Y(s) \in \mathscr{F}$. Since F is finite, $h_Y(F) = \bigcap_{s \in F} h_Y(s) \in \mathscr{F}$.

(b). Let $a, b \in \mathcal{H}_Y^{-1}(\mathscr{F})$ and $r \in R$. Clearly since $h_Y(a) \cap h_Y(b) \subseteq h_Y(a+b)$ and $h_Y(a) \subseteq h_Y(ra)$, we have $a+b, ra \in \mathcal{H}_Y^{-1}(\mathscr{F})$.

(c). Suppose that $h_Y(F_1), h_Y(F_2) \in \mathcal{H}_Y(I)$, where F_1 and F_2 are finite subsets of I. Clearly $F_1 \cup F_2$ is a finite subset of I and consequently $h_Y(F_1) \cap h_Y(F_2) = h_Y(F_1 \cup F_2) \in \mathcal{H}_Y(I)$. Suppose now that F_1 is a finite subset of I, $h_Y(F_1) \subseteq h_Y(F_2)$ where F_2 is a finite subset of R. Clearly, $F_1F_2 = \{s_1s_2 : s_1 \in F_1 \text{ and } s_2 \in F_2\}$ is a finite subset of I and $h_Y(F_2) = h_Y(F_1) \cup h_Y(F_2) = h_Y(F_1F_2)$. Consequently $h_Y(F_2) \in \mathcal{H}_Y(I)$. □

Note that for a proper ideal I, $\mathcal{H}_Y(I)$ is not necessarily a proper $\mathcal{H}_Y$-filter; for example if $Y = \mathrm{Min}(R)$ and a proper ideal I contains a non zero-divisor then $\mathcal{H}_Y(I) = \mathcal{H}_Y$. By the way, it is easy to see that if $\mathrm{Max}(R) \subseteq Y$ then $\mathcal{H}_Y(I)$ is a proper $\mathcal{H}_Y$-filter, for every proper ideal I. Now by the two next propositions we define and characterize $\mathcal{H}_Y$-ideals and strong $\mathcal{H}_Y$-ideals.

Proposition 3.2. *Let R be a ring, $Y \subseteq Spec(R)$ and I be an ideal of R. Then the following are equivalent:*

(a) *For every $a \in I$ and $S \subseteq R$, it follows from $h_Y(a) \subseteq h_Y(S)$ that $S \subseteq I$.*
(b) *For every $a \in I$ and $S \subseteq R$, it follows from $h_Y(a) = h_Y(S)$ that $S \subseteq I$.*
(c) *For every $a \in I$ and $b \in R$, it follows from $h_Y(a) = h_Y(b)$ that $b \in I$.*
(d) *For every $a \in I$ and $b \in R$, it follows from $h_Y(a) \subseteq h_Y(b)$ that $b \in I$.*
(e) *If $a \in I$, then $kh_Y(a) \subseteq I$.*
(f) *For every $a \in I$ and $S \subseteq R$, it follows from $kh_Y(S) \subseteq kh_Y(a)$ that $S \subseteq I$.*
(g) *For every $a \in I$ and $S \subseteq R$, it follows from $kh_Y(S) = kh_Y(a)$ that $S \subseteq I$.*
(h) *For every $a \in I$ and $b \in R$, it follows from $kh_Y(b) = kh_Y(a)$ that $b \in I$.*
(k) *For every $a \in I$ and $b \in R$, it follows from $kh_Y(b) \subseteq kh_Y(a)$ that $b \in I$.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c). They are trivial.

(c) $\Rightarrow$ (d). We know that $h_Y(a) \cup h_Y(b) = h_Y(ab)$, so if $h_Y(a) \subseteq h_Y(b)$, then $h_Y(ab) = h_Y(b)$ and $ab \in I$, so $b \in I$.

(d) $\Rightarrow$ (e). It is readily seen that $b \in kh_Y(a)$ if and only if $h_Y(a) \subseteq h_Y(b)$, thus if $a \in I$, then by the assumption, we have $kh_Y(a) \subseteq I$.

(e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h). They are trivial.

(h) $\Rightarrow$ (k). Knowing this fact that $kh_Y(a) \cap kh_Y(b) = k(h_Y(a) \cup h_Y(b)) = kh_Y(ab)$, it is follows, by using the same technique as (c $\Rightarrow$ d).

(k) $\Rightarrow$ (a). If $h_Y(a) \subseteq h_Y(S)$, then $h_Y(a) \subseteq h_Y(s)$, for every $s \in S$. Whence $kh_Y(s) \subseteq kh_Y(a)$, for every $s \in S$ and consequently $S \subseteq I$. □

Definition 3.3. Let R be a ring and $Y \subseteq \text{Spec}(R)$. An ideal I of R is said to be an $\mathcal{H}_Y$-ideal if it satisfies in the equivalent conditions of Proposition 3.2.

Proposition 3.4. *Let R be a ring, $Y \subseteq Spec(R)$ and I be an ideal of R. Then the following are equivalent:*

(a) *For every finite subset F of I and every $S \subseteq R$, it follows from $h_Y(F) = h_Y(S)$ that $S \subseteq I$.*
(b) *For every finite subset F of I and every finite subset G of R, it follows from $h_Y(F) = h_Y(G)$ that $G \subseteq I$.*
(c) *For every finite subset F of I and every finite subset G of R, it follows from $h_Y(F) \subseteq h_Y(G)$ that $G \subseteq I$.*
(d) *It follows from $h_Y(a) \in \mathcal{H}_Y(I)$ that $a \in I$.*
(e) *For every finite subset F of R, it follows from $h_Y(F) \in \mathcal{H}_Y(I)$ that $F \subseteq I$.*
(f) *For every finite subset F of I and $a \in R$, it follows from $h_Y(F) = h_Y(a)$ that $a \in I$.*
(g) *For every finite subset F of I and $a \in R$, it follows from $h_Y(F) \subseteq h_Y(a)$ that $a \in I$.*
(k) *For every finite subset $F \subseteq I$, we have $kh_Y(F) \subseteq I$.*
(l) *For every finite subset F of I and $a \in R$, it follows from $kh_Y(a) = kh_Y(F)$ that $a \in I$.*
(m) *For every finite subset F of I and $a \in R$, it follows from $kh_Y(a) \subseteq kh_Y(F)$ that $a \in I$.*
(n) *For every finite subset F of I and any $S \subseteq R$, it follows from $kh_Y(S) = kh_Y(F)$ that $S \subseteq I$.*
(o) *For every finite subset F of I and any $S \subseteq R$, it follows from $kh_Y(S) \subseteq kh_Y(F)$ that $S \subseteq I$.*

Proof. By these facts that if A and B are arbitrary subsets of R, then $h_Y(AB) = h_Y(A) \cup h_Y(B)$ and $B \subseteq kh_Y(A)$ if and only if $h_Y(B) \supseteq h_Y(A)$, it has a similar proof to the previous proposition. □

Definition 3.5. Let R be a ring and $Y \subseteq \mathrm{Spec}(R)$. An ideal I of R is said to be a strong $\mathcal{H}_Y$-ideal if it satisfies in the equivalent conditions in Proposition 3.4.

Definition 3.6. Suppose $Y \subseteq \mathrm{Spec}(R)$. An ideal I of R is called a Y-Hilbert ideal, if I is an intersection of elements of some subfamily of Y; i.e., $I = kh_Y(I)$.

Obviously, if $Y = \mathrm{Max}(R)$, then the concepts of $\mathcal{H}_Y$-ideal, strong $\mathcal{H}_Y$-ideal and Y-Hilbert ideal coincide with the concepts of z-ideal, sz-ideal and Hilbert ideal in the literature, respectively, see [3] and [19]. Also, if $Y = \mathrm{Min}(R)$, then the concepts of $\mathcal{H}_Y$-ideal and strong $\mathcal{H}_Y$-ideal coincide with the concepts of z°-ideal (also known as d-ideal) and sz°-ideal (also known as ξ-ideal), respectively, see [3, 7, 9, 10, 15, 18]. Finally if $Y = \mathrm{Spec}(R)$, then the concepts of $\mathcal{H}_Y$-ideal, strong $\mathcal{H}_Y$-ideal, Y-Hilbert ideals and semiprime ideal coincide. It is clear that every Y-Hilbert ideal is a strong $\mathcal{H}_Y$-ideal and every strong $\mathcal{H}_Y$-ideal is an $\mathcal{H}_Y$-ideal. By the way, their converse does not hold generally even if $k(Y) = \langle 0\rangle$. If we set $Y = \mathrm{Max}(C(X))$ then the ideal O_0 in $C(\mathbb{R})$ is a strong $\mathcal{H}_Y$-ideal which is not intersection of maximal ideals. Moreover in [3, Example 4.1] an example of a reduced ring is given which contains a z°-ideal which is not a sz°-ideal.

Clearly $kh_Y(F)$ is a strong $\mathcal{H}_Y$-ideal, for every finite set $F \subseteq R$, in fact, it is the smallest strong $\mathcal{H}_Y$-ideal containing F. In addition an ideal I is a strong $\mathcal{H}_Y$-ideal if and only if $I = \bigcup_{F\in \mathbf{F}(I)} kh_Y(F) = \sum_{F\in\mathbf{F}(I)} kh_Y(F)$. Also it is easy to see that if $X, Y \subseteq \mathrm{Spec}(R)$, then the family of strong $\mathcal{H}_X$-ideals and strong $\mathcal{H}_Y$-ideals coincide if and only if $kh_X(F) = kh_Y(F)$, for every finite subset $F \subseteq R$. Note that in this case $k(X) = kh_X(0) = kh_Y(0) = k(Y)$, but the converse does not hold generally. For example $\mathrm{Jac}(C(X)) = \langle 0\rangle = \mathrm{Rad}(C(X))$ and the z-ideals and the z°-ideals need not be coincide. Moreover, since $k(Y) = kh_Y(0)$, it follows that $k(Y)$ is the smallest strong $\mathcal{H}_Y$-ideal ($\mathcal{H}_Y$-ideal, Y-Hilbert ideal) in R.

Naturally, in this paper we were about to study other classes of ideals, close to $\mathcal{H}_Y$-ideals and strong $\mathcal{H}_Y$-ideals, using interior in the right-hand side of the inclusion in their definitions. For example, for $\mathcal{H}_Y$-ideal (strong $\mathcal{H}_Y$-ideal) case, it springs to mind to consider the ideals that it follows from $(h_Y(x))^\circ \subseteq (h_Y(a))^\circ$ $((h_Y(F))^\circ \subseteq (h_Y(a))^\circ)$ and $x \in I$ $(F \subseteq I)$ that $a \in I$. But as one can observe below, we realized that if $Y \subseteq \mathrm{Spec}(R)$ and $k(Y) = \langle 0\rangle$, then these kind of ideals coincide with the z°-ideals (resp., sz°-ideals). The following lemma is an improvement of [7, Proposition 1.1], without the redundant condition $\mathrm{Min}(R) \subseteq Y$.

Lemma 3.7. *Let $Y \subseteq Spec(R)$ and $k(Y) = \langle 0\rangle$. Then $(h_Y(S))^\circ = h^c_Y(Ann(S))$, for every $S \subseteq R$.*

Proof. By Lemma 2.1, we have

$$h_Y(\mathrm{Ann}(S)) = h_Y(k(h^c_Y(S))) = \overline{(h^c_Y(S))} = ((h_Y(S))^\circ)^c.$$

Consequently $(h_Y(S))^\circ = h^c_Y(\mathrm{Ann}(S))$. □

Suppose that $X, Y \subseteq \mathrm{Spec}(R)$. Clearly, $k(X) = k(Y)$ if and only if $hk(X) = hk(Y)$; in the other words $\bigcap X = \bigcap Y$ if and only if $\overline{X} = \overline{Y}$. Also, assume that X is a topological space and dense in T. We know that if W is an open subset of T, then $cl_T(W \cap X) = cl_T W$; equivalently, if A is a closed subset of T, then $int_X(A \cap X) = (int_T A) \cap X$. By these facts we have the following lemma which is an improvement of [7, Theorem 2.3].

Lemma 3.8. *Let $X, Y \subseteq Spec(R)$ and $k(X) = Rad(R)$. Then the following are equivalent:*

(a) *$k(Y) = Rad(R)$.*
(b) *$(h_Y(S))^\circ \subseteq h_Y(T)$ if and only if $(h_X(S))^\circ \subseteq h_X(T)$, for every $T, S \subseteq R$.*
(c) *$(h_Y(S))^\circ = (h_Y(T))^\circ$ if and only if $(h_X(S))^\circ = (h_X(T))^\circ$, for every $T, S \subseteq R$.*
If $k(Y) = \langle 0\rangle$, then the above statements are equivalent to the following statement.
(d) *$kh_Y(S) \subseteq Ann^2(S)$, for every $S \subseteq R$.*

Proof. Without loss of generality we can suppose that $X = \mathrm{Spec}(R)$.

(a) $\Rightarrow$ (b). Since $k(Y) = \mathrm{Rad}(R)$, it follows that Y is dense in X and so for every $S, T \subseteq Y$, we have

$$\begin{aligned} &(h_X(S))^\circ \cap Y = (h_Y(S))^\circ \subseteq h_Y(T) \subseteq h_X(T) \\ \Rightarrow \quad &(h_X(S))^\circ \subseteq \overline{(h_X(S))^\circ} \subseteq \overline{h_X(T)} = h_X(T). \end{aligned}$$

The converse is clear.

(b) $\Rightarrow$ (c). It is evident.

(c) $\Rightarrow$ (a). Suppose that $a \in k(Y)$. Since $(h_Y(a))^\circ = Y = (h_Y(0))^\circ$, it follows that $(h_X(a))^\circ = (h_X(0))^\circ = X$. Therefore, $h_X(a) = X$ and so $a \in k(X) = \mathrm{Rad}(R)$.

(a) $\Leftrightarrow$ (d). Since $k(Y) = \langle 0 \rangle$, it is sufficient to show that (a) implies (d). For every $S \subseteq R$,

$$\begin{aligned} \mathrm{Ann}^2(S) &= kh_Y^c kh_Y^c(S) = k(Y \setminus h_Y kh_Y^c(S)) \\ &= k(Y \setminus \overline{h_Y^c(S)}) \supseteq k(Y \setminus h_Y^c(S)) = kh_Y(S). \end{aligned}$$

□

Lemma 3.9. *For every finite subset F of R, we have $h_m(F) = (h_m(F))^\circ$.*

Proof. Suppose that $P \in \mathrm{Min}(R)$, it easy to show that $F \subseteq P$ if and only if $b \notin P$ exists such that $bF \subseteq \mathrm{Rad}(R)$. Then

$$\begin{aligned} P \in h_m(F) \Leftrightarrow \quad & F \subseteq P \quad \Leftrightarrow \quad \exists b \notin P \quad bF \subseteq \mathrm{Rad}(R) \\ \Leftrightarrow \quad & \exists b \in (\mathrm{Rad}(R) : F) \setminus P \quad \Leftrightarrow \quad (\mathrm{Rad}(R) : F) \not\subseteq P \\ \Leftrightarrow \quad & P \notin h_m\left(\mathrm{Rad}(R) : F\right) \end{aligned}$$

Hence $h_m(F) = h_m^c\left(\mathrm{Rad}(R) : F\right)$. Now with a method similar to Lemma 3.7, $(h_m(F))^\circ = h_m^c\left(\mathrm{Rad}(R) : F\right)$, hence $(h_m(F))^\circ = h_m(F)$. □

By the above lemmas we give new characterizations of z°-ideals and sz°-ideals in the following proposition.

Proposition 3.10. *Let $Y \subseteq Spec(R)$ and $k(Y) = Rad(R)$. Then the following statements hold:*

(a) *I is a z°-ideal if and only if it follows from $(h_Y(b))^\circ \subseteq h_Y(a)$ and $b \in I$ that $a \in I$; if and only if it follows from $(h_Y(b))^\circ \subseteq h_Y(S)$ and $b \in I$ that $S \subseteq I$.*

(b) *I is a sz°-ideal if and only if for every finite subset F of I, it follows from $(h_Y(F))^\circ \subseteq h_Y(a)$ that $a \in I$; if and only if for every finite subset F of I, it follows from $(h_Y(F))^\circ \subseteq h_Y(S)$ that $S \subseteq I$.*

Proof. We prove one part and the other parts have similar proofs. By Proposition 3.4, I is a sz°-ideal if and only if for every finite subset F of I, $h_m(F) \subseteq h_m(a)$ implies that $a \in I$; if and only if for every finite subset F of I, $(h_m(F))^\circ \subseteq h_m(a)$ implies that $a \in I$, by Lemma 3.9. Now Lemma 3.8 concludes that this is equivalent to say, for every finite subset F of I, it follows from $(h_Y(F))^\circ \subseteq h_Y(a)$ that $a \in I$. □

Finally in the following improvement of [3, Proposition 2.9], [7, Theorem 2.3] and [18, Proposition 2.12], we see the conditions under which every z°-ideal (sz°-ideal) is an $\mathcal{H}_Y$-ideal (a strong $\mathcal{H}_Y$-ideal).

Proposition 3.11. *If $Y \subseteq Spec(R)$, then the following statements are equivalent:*

(a) *$k(Y) = Rad(R)$.*

(b) *Every z°-ideal is an $\mathcal{H}_Y$-ideal.*

(c) *Every sz°-ideal is a strong $\mathcal{H}_Y$-ideal.*

(d) *$kh_Y(F) \subseteq kh_m(F)$, for every finite set $F \subseteq R$.*

(e) *$kh_Y(a) \subseteq kh_m(a)$, for every $a \in R$.*

Proof. It has a same proof as [3, Proposition 2.9]. □

We use the following lemma frequently.

Lemma 3.12. *Let $Y \subseteq Spec(R)$. Every $\mathcal{H}_Y$-ideal is a semiprime ideal.*

Proof. Suppose $x^n \in I$, so $h_Y(x) = h_Y(x^n) \in \mathcal{H}_Y(I)$. Thus $x \in I$. □

The following theorem and corollary show that the prime (strong) $\mathcal{H}_Y$-ideals play a vital role in the study of the (strong) $\mathcal{H}_Y$-ideals.

Theorem 3.13. *Let $Y \subseteq Spec(R)$ and I be a (strong) $\mathcal{H}_Y$-ideal. If $P \in Min(I)$, then P is a (strong) $\mathcal{H}_Y$-ideal, too.*

Proof. From Lemma 3.12, it follows that I is a semiprime ideal. Now suppose that F is a finite subset of P, so there is some $b \notin P$ such that $bF \subseteq I$, thus $kh_Y(S) \cap kh_Y(b) = k(h_Y(F) \cup h_Y(b)) = kh_Y(bF) \subseteq I \subseteq P$. Since $kh_Y(b) \nsubseteq P$, it follows that $kh_Y(F) \subseteq P$. Consequently, by Proposition 3.4, P is a strong $\mathcal{H}_Y$-ideal. The other part has a similar proof. □

The above theorem concludes the following corollary, immediately.

Corollary 3.14. *If $Y \subseteq Spec(R)$, then the following statements hold:*

(a) *An ideal I is a (strong) $\mathcal{H}_Y$-ideal if and only if it is an intersection of minimal prime (strong) $\mathcal{H}_Y$-ideals over I.*
(b) *Every proper maximal (strong) $\mathcal{H}_Y$-ideal is a prime (strong) $\mathcal{H}_Y$-ideal.*

We turn our attention now to considering the situations under which strong $\mathcal{H}_Y$-ideals and $\mathcal{H}_Y$-ideals coincide. A ring R is said to have the h_Y-property if for every $a, b \in R$, there is some $c \in R$ such that $h_Y(a) \cap h_Y(b) = h_Y(c)$. Clearly, this is equivalent to saying that for any finite subset F of R, there is some $c \in R$ such that $h_Y(F) = h_Y(c)$. Clearly, if $Y \subseteq \mathrm{Spec}(R)$ and R satisfies h_Y-property (for example if R is Bézout domain), then the family of all $\mathcal{H}_Y$-ideals and the family of all strong $\mathcal{H}_Y$-ideals coincide. Also, the same fact is true in $C(X)$, since for every prime ideal P of $C(X)$, we have$f^2 + g^2 \in P$ if and only if $f, g \in P$ and consequently $h_Y(f) \cap h_Y(g) = h_Y(f^2 + g^2)$, for every $Y \subseteq \mathrm{Spec}(C(X))$ and every $f, g \in C(X)$. However, in Example 3.17, we show that the converse of this fact is not true.

Recall that a ring R is said to satisfy annihilator condition (is called an a.c. ring), if for each finite set $F \subseteq R$ there is some $c \in R$ such that $\mathrm{Ann}(F) = \mathrm{Ann}(c)$. If $k(Y) = \langle 0 \rangle$ and R has h_Y-property, then R is an a.c. ring. To see this, suppose $a, b \in R$ are given, then there exists some $c \in R$ such that $h_Y(a) \cap h_Y(b) = h_Y(c)$. Therefore using Lemma 2.1 we have,

$$h^c_Y(a) \cup h^c_Y(b) = (h_Y(a) \cap h_Y(b))^c = h^c_Y(c) \Rightarrow$$
$$kh^c_Y(a) \cap kh^c_Y(b) = k(h^c_Y(a) \cup h^c_Y(b)) = kh^c_Y(c) \Rightarrow Ann(a) \cap Ann(b) = Ann(c).$$

One can easily see that if $h^c_Y(a)$ is a closed set, for every $a \in R$, then the converse is also true, for example $\mathrm{Min}(R)$ has this property, see [14, Theorem 2.3].

Suppose that $Y \subseteq \mathrm{Min}(R)$, then [14, Theroems 2.2 and 2.3] imply that $h^c_Y(F)$ is closed in Y, for every finite subset F of R. Now, clearly, if I is an arbitrary ideal of R, then the mapping $a \to a + I$ induces a homeomorphism from $\mathrm{Min}(R/I)$ to $\mathrm{Min}(I)$. Consequently, if $Y \subseteq \mathrm{Min}(I)$, then $h^c_Y(F)$ is closed in Y, for every finite subset F of R. Using this fact, we have the following proposition, which characterizes the h_Y-property when $I = k(Y)$ and $Y \subseteq \mathrm{Min}(I)$.

Proposition 3.15. *Let $Y \subseteq Spec(R)$ and $I = k(Y)$. If $Y \subseteq Min(I)$, then the following statements are equivalent.*

(a) *R has h_Y-property.*

(b) *For all finite sets $F \subseteq R$, there is some $c \in R$ such that $(I : F) = (I : c)$.*
(c) *For every $a, b \in R$, there is some $c \in R$ such that $(I : a) \cap (I : b) = (I : c)$.*
(d) *R/I is an a.c. ring.*

Proof. (a) $\Rightarrow$ (b). Let F be a finite subset of R and $h_Y(F) = h_Y(c)$ for some $c \in R$, then by Lemma 2.1, $(I : F) = kh^c_Y(F) = kh^c_Y(c) = (I : c)$.

(b) $\Rightarrow$ (a). Let F be a finite subset of R. Since $Y \subseteq \mathrm{Min}(I)$, $h^c_Y(F)$ and $h^c_Y(c)$ are closed sets, using Lemma 2.1 and the assumption we have

$$h^c_Y(F) = h_Y kh^c_Y(F) = h_Y((I : F)) = h_Y((I : c)) = h_Y kh^c_Y(c) = h^c_Y(c).$$

Consequently $h_Y(F) = h_Y(c)$.

(b) $\Leftrightarrow$ (c). Since $(I : A) \cap (I : B) = (I : A \cup B)$, for every $A, B \subseteq R$, it is evident.

(c) $\Leftrightarrow$ (d). Clearly, for every $x \in R$, we have $\mathrm{Ann}(x + I) = \frac{(I:x)}{I}$. Therefore, we can write

$$\mathrm{Ann}(a+I) \cap \mathrm{Ann}(b+I) = \mathrm{Ann}(c+I) \quad \Leftrightarrow \quad \frac{(I:a)}{I} \cap \frac{(I:b)}{I} = \frac{(I:c)}{I}$$
$$\Leftrightarrow \quad \frac{(I:a) \cap (I:b)}{I} = \frac{(I:c)}{I} \quad \Leftrightarrow \quad (I:a) \cap (I:b) = (I:c).$$

□

Corollary 3.16. *Let $Y \subseteq Spec(R)$ and $I = k(Y)$. If one of the following conditions holds, then the family of all $\mathcal{H}_Y$-ideals and the family of all strong $\mathcal{H}_Y$-ideals coincide.*

(a) *Y is a fixed-place family and R/I is an a.c. ring.*
(b) *Y is a strong fixed-place family.*

Proof. (a). Since Y is a fixed-place family, $Y = \mathcal{B}(I) \subseteq \mathrm{Min}(R)$, by [1, Theorem 2.1]. Now Proposition 3.15, completes the proof.

(b). It follows immediately form [2, Theorem 2.10, Corollary 2.11 and Theorem 2.10] and part (a). □

Example 3.17. Suppose that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Min}(R)$. In [14, Example 3.3], a ring R is given which does not satisfy annihilator condition, so R does not satisfy h_Y-property, by Proposition 3.15. Thus R does not satisfy h_X-property, whereas the family of all $\mathcal{H}_X$-ideals coincides with the family all strong $\mathcal{H}_X$-ideals.

4. Correspondence between $\mathcal{H}_Y$-filters and strong $\mathcal{H}_Y$-ideals

In this section we study the relation and the correspondence between the strong $\mathcal{H}_Y$-ideals, the $\mathcal{H}_Y$-ideals and the H_Y-filters. First recall that if E and F are two partially ordered sets, then an order preserving mapping $f : E \to F$ is said to be residuated whenever there exists an order preserving mapping $g : F \to E$ such that $id_E \leq gf$ and $id_F \geq fg$; moreover, g is unique and it is called the residual of f. The set of all $\mathcal{H}_Y$-filters on $Y \subseteq \mathrm{Spec}(R)$ is denoted by $\mathcal{F}_Y$.

In the following proposition we state the properties of the mappings $\mathcal{H}_Y$ and $\mathcal{H}_Y^{-1}$ and the image and preimage of ideals and filters under them, respectively.

Proposition 4.1. *Let $Y \subseteq Spec(R)$, $I \in \mathcal{I}(R)$ and $\mathscr{F} \in \mathcal{F}_Y$. The following statements hold.*

(a) *$\mathcal{H}_Y^{-1}(\mathscr{F}) = R$ if and only if $\mathscr{F} = \mathcal{H}_Y$.*
(b) *$I \subseteq \mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$ and $\mathcal{H}_Y\mathcal{H}_Y^{-1}(\mathscr{F}) = \mathscr{F}$.*
(c) *$\mathcal{H}_Y$ is a residuated mapping from $\mathcal{I}(R)$ to $\mathcal{F}_Y$, and $\mathcal{H}_Y^{-1}$ is the residual of $\mathcal{H}_Y$. Consequently, $\mathcal{H}_Y\mathcal{H}_Y^{-1}\mathcal{H}_Y = \mathcal{H}_Y$ and $\mathcal{H}_Y^{-1}\mathcal{H}_Y\mathcal{H}_Y^{-1} = \mathcal{H}_Y^{-1}$.*
(d) *I is a strong $\mathcal{H}_Y$-ideal if and only if $I = \mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$.*
(e) *$\mathcal{H}_Y^{-1}(\mathscr{F})$ is a strong $\mathcal{H}_Y$-ideal of R.*

(f) *If $Max(R) \subseteq Y$, then $\mathcal{H}_Y(I)$ is a proper $\mathcal{H}_Y$-filter on Y, for every proper ideal I of R.*

(g) *If I is a proper strong $\mathcal{H}_Y$-ideal of R, then $\mathcal{H}_Y(I)$ is a proper $\mathcal{H}_Y$-filter on Y.*

Proof. (a). $\mathcal{H}_Y^{-1}(\mathscr{F}) = R \;\Leftrightarrow\; 1 \in \mathcal{H}_Y^{-1}(\mathscr{F}) \;\Leftrightarrow\; h_Y(1) \in \mathscr{F} \;\Leftrightarrow\; \emptyset \in \mathscr{F} \;\Leftrightarrow\; \mathscr{F} = \mathcal{H}_Y$.

(b). The first part is readily verified. Using Lemma 3.1(a), for every finite subset F of R we have

$$h_Y(F) \in \mathscr{F} \Leftrightarrow F \subseteq \mathcal{H}_Y^{-1}(\mathscr{F}) \Leftrightarrow h_Y(F) \in \mathcal{H}_Y\mathcal{H}_Y^{-1}(\mathscr{F}).$$

(c). By part (b) and Lemma 3.1, the first part is trivial. For the second part see [11, Theorem1.5].

(d). Let I be a strong $\mathcal{H}_Y$-ideal. If $a \in \mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$, then $h_Y(a) \in \mathcal{H}_Y(I)$ and so by Proposition 3.4, $a \in I$. Now by part (b), $I = \mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$. Conversely, suppose that $h_Y(a) \in \mathcal{H}_Y(I)$, whence $a \in \mathcal{H}_Y^{-1}\mathcal{H}_Y(I) = I$, therefore, by Proposition 3.4, I is a strong $\mathcal{H}_Y$-ideal.

(e). Clearly, by part (c) we have $\mathcal{H}_Y^{-1}\mathcal{H}_Y\mathcal{H}_Y^{-1}(\mathscr{F}) = \mathcal{H}_Y^{-1}(\mathscr{F})$, for every $\mathcal{H}_Y$-filter $\mathscr{F}$ on Y and thus by part (d), $\mathcal{H}_Y^{-1}(\mathscr{F})$ is a strong $\mathcal{H}_Y$-ideal of R.

(f). On the contrary, let $\emptyset \in \mathcal{H}_Y(I)$, then $\emptyset = h_Y(F)$, for some finite set $F \subseteq I$, now by the hypothesis $\langle F\rangle = R$, which is a contradiction.

(g). Since $R \neq I = \mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$, by part (a), it follows that $\mathcal{H}_Y(I)$ is a proper $\mathcal{H}_Y$-filter. □

The following corollary is an immediate consequence of the above proposition which gives a correspondence between the strong $\mathcal{H}_Y$-ideals and the $\mathcal{H}_Y$-filters.

Corollary 4.2. *The following facts hold.*

(a) *Suppose $\mathscr{F}$ and $\mathscr{G}$ are two $\mathcal{H}_Y$-filters on $Y \subseteq Spec(R)$. Then $\mathscr{F} = \mathscr{G}$ if and only if $\mathcal{H}_Y^{-1}(\mathscr{F}) = \mathcal{H}_Y^{-1}(\mathscr{G})$.*

(b) *If I and J are two strong $\mathcal{H}_Y$-ideals then $\mathcal{H}_Y(I) = H_Y(J)$ if and only if $I = J$.*

(c) *The mapping $\mathcal{H}_Y$ is an order isomorphism from the set of all strong $\mathcal{H}_Y$-ideals onto the set of all $\mathcal{H}_Y$-filters on Y.*

In the following theorem we try to present a correspondence between the prime (maximal) strong $\mathcal{H}_Y$-ideals and the prime (maximal) $\mathcal{H}_Y$-filters.

Theorem 4.3. *Let $Y \subseteq Spec(R)$, $I \in \mathcal{I}(R)$ and $\mathscr{F} \in \mathfrak{F}_Y$. The following statements hold.*

(a) *$\mathcal{H}_Y^{-1}(\mathscr{F})$ is a prime strong $\mathcal{H}_Y$-ideal if and only if $\mathscr{F}$ is a prime $\mathcal{H}_Y$-filter.*

(b) *If I is a strong $\mathcal{H}_Y$-ideal, then I is a prime ideal of R if and only if $\mathcal{H}_Y(I)$ is a prime $\mathcal{H}_Y$-filter.*

(c) *The mapping $\mathcal{H}_Y$ is one-to-one from the set of all prime strong $\mathcal{H}_Y$-ideals onto the set of all prime $\mathcal{H}_Y$-filters.*

(d) *An ideal I of R is a maximal proper strong $\mathcal{H}_Y$-ideal if and only if there exists an $\mathcal{H}_Y$-ultrafilter $\mathscr{F}$ such that $I = \mathcal{H}_Y^{-1}(\mathscr{F})$. In addition the mapping $\mathcal{H}_Y$ is one-to-one from the set of all maximal proper strong $\mathcal{H}_Y$-ideals onto the set of all $\mathcal{H}_Y$-ultrafilters.*

(e) *Assume that $Max(R) \subseteq Y$. If I is a maximal ideal, then $\mathcal{H}_Y(I)$ is an $\mathcal{H}_Y$-ultrafilter. Supposing I is a strong $\mathcal{H}_Y$-ideal, the converse is also true.*

(f) *If $Max(R) \subseteq Y$, then $\mathscr{F}$ is an $\mathcal{H}_Y$-ultrafilter if and only if $\mathcal{H}_Y^{-1}(\mathscr{F})$ is a maximal ideal.*

Proof. (a $\Rightarrow$). Clearly, by Proposition 4.1, $\mathcal{H}_Y^{-1}(\mathscr{F})$ is a proper ideal if and only if $\mathscr{F}$ is a proper $\mathcal{H}_Y$-filter. Now, suppose that F_1 and F_2 are two finite subsets of R and $h_Y(F_1) \cup h_Y(F_2) \in \mathscr{F}$, then $h_Y(F_1F_2) \in \mathscr{F}$, so $F_1F_2 \subseteq \mathcal{H}_Y^{-1}(\mathscr{F})$, by Lemma 3.1. Thus, either $F_1 \subseteq \mathcal{H}_Y^{-1}(\mathscr{F})$ or $F_2 \subseteq \mathcal{H}_Y^{-1}(\mathscr{F})$ and therefore either $h_Y(F_1) \in \mathscr{F}$ or $h_Y(F_2) \in \mathscr{F}$. Hence $\mathscr{F}$ is a prime $\mathcal{H}_Y$-filter.

(a $\Leftarrow$). Suppose $ab \in \mathcal{H}_Y^{-1}(\mathscr{F})$, then $h_Y(ab) \in \mathscr{F}$, so $h_Y(a) \cup h_Y(b) \in \mathscr{F}$, thus either $h_Y(a) \in \mathscr{F}$ or $h_Y(b) \in \mathscr{F}$, and therefore either $a \in \mathcal{H}_Y^{-1}(\mathscr{F})$ or $b \in \mathcal{H}_Y^{-1}(\mathscr{F})$. Hence, $\mathcal{H}_Y^{-1}(\mathscr{F})$ is a prime ideal.

(b). It can be obtained easily by the previous part and Proposition 4.1.

(c). It is straightforward by using parts (a) and (b) as well as Corollary 4.2.

(d $\Rightarrow$). Assume that I is a maximal proper strong $\mathcal{H}_Y$-ideal. Clearly by Proposition 4.1, $\mathcal{H}_Y(I)$ is a proper $\mathcal{H}_Y$-filter and so there exists an $\mathcal{H}_Y$-ultrafilter $\mathscr{F}$ such that $\mathcal{H}_Y(I) \subseteq \mathscr{F}$. Thus, $I = \mathcal{H}_Y^{-1}\mathcal{H}_Y(I) \subseteq \mathcal{H}_Y^{-1}(\mathscr{F})$ and since $\mathcal{H}_Y^{-1}(\mathscr{F})$ is a proper strong $\mathcal{H}_Y$-ideal, it follows that $I = \mathcal{H}_Y^{-1}(\mathscr{F})$.

(d $\Leftarrow$). Assume that $I = \mathcal{H}_Y^{-1}(\mathscr{F})$ where $\mathscr{F}$ is an $\mathcal{H}_Y$-ultrafilter. Clearly by Proposition 4.1, I is a proper strong $\mathcal{H}_Y$-ideal. Now suppose that J is a proper strong $\mathcal{H}_Y$-ideal containing I. Thus by Proposition 4.1, $\mathcal{H}_Y(J)$ is a proper $\mathcal{H}_Y$-filter containing $\mathcal{H}_Y(I) = \mathscr{F}$, so $\mathcal{H}_Y(I) = \mathcal{H}_Y(J)$, whence by Corollary 4.2, we have $I = J$. The second part of (d) is straightforward.

(e). Knowing this fact that if $\mathrm{Max}(R) \subseteq Y$, then the maximal proper strong $\mathcal{H}_Y$-ideals are exactly the elements of $\mathrm{Max}(R)$, this part follows easily from the previous part.

(f). It is clear from the previous part. □

Since $\mathcal{H}_Y$ is a distributive lattice and a filter is a dual of an ideal, clearly, we have the following facts.

Proposition 4.4. *Suppose $Y \subseteq Spec(R)$, $\mathscr{F}$ is an $\mathcal{H}_Y$-filter on Y and $\mathcal{S}$ is a $\cup$-closed subset of $\mathcal{H}_Y$. If $\mathscr{F} \cap \mathcal{S} = \emptyset$, then there is a prime $\mathcal{H}_Y$-filter $\mathscr{P}$ containing $\mathscr{F}$ such that $\mathscr{P} \cap \mathcal{S} = \emptyset$.*

Definition 4.5. An $\mathcal{H}_Y$-filter $\mathscr{P}$ is called a minimal prime $\mathcal{H}_Y$-filter over a $\mathcal{H}_Y$-filter $\mathscr{F}$, if there are no prime $\mathcal{H}_Y$-filter strictly contained in $\mathscr{P}$ that contains $\mathscr{F}$. By $\mathrm{Min}(\mathscr{F})$ we mean the set of all minimal prime $\mathcal{H}_Y$-filters over $\mathscr{F}$.

The following corollary is an immediate consequence of Lemma 3.1 and Theorem 4.3.

Corollary 4.6. *Let $Y \subseteq Spec(R)$. Every $\mathcal{H}_Y$-filter $\mathscr{F}$ is the intersection of all minimal prime $\mathcal{H}_Y$-filters over $\mathscr{F}$*

By this fact that for each semiprime ideal I, $P \in \mathrm{Min}(I)$ if and only if for each $a \in P$, there is some $b \notin P$ such that $ab \in I$, the following proposition and corollary conclude from Theorem 3.1 and the previous corollary.

Proposition 4.7. *Let $\mathscr{F}$ be an $\mathcal{H}_Y$-filter. $\mathscr{P} \in Min(\mathscr{F})$ if and only if for every $A \in \mathscr{P}$ there is some $B \in \mathcal{H}_Y \setminus \mathscr{P}$ such that $A \cup B \in \mathscr{F}$.*

Corollary 4.8. *$\mathscr{P} \in Min(\{Y\})$ if and only if for every $A \in \mathscr{P}$ there is a $B \notin \mathscr{P}$ such that $A \cup B = Y$.*

Proposition 4.9. *Let $Y \subseteq Spec(R)$ and $\mathscr{F}$ and $\mathscr{P}$ are two $\mathcal{H}_Y$-filters. Then the following statements hold*

(a) *$\mathscr{P} \in Min(\mathscr{F})$ if and only if $\mathcal{H}_Y^{-1}(\mathscr{P}) \in Min(\mathcal{H}_Y^{-1}(\mathscr{F}))$.*

(b) *If I is a strong $\mathcal{H}_Y$-ideal, then $P \in Min(I)$ if and only if $\mathcal{H}_Y(P) \in Min(\mathcal{H}_Y(I))$.*

Proof. (a $\Rightarrow$). Let $P_\circ$ be a minimal prime ideal over the strong $\mathcal{H}_Y$-ideal $\mathcal{H}_Y^{-1}(\mathscr{F})$ such that $\mathcal{H}_Y^{-1}(\mathscr{F}) \subseteq P_\circ \subseteq \mathcal{H}_Y^{-1}(\mathscr{P})$. By Theorem 3.13, $P_\circ$ is a strong $\mathcal{H}_Y$-ideal and hence by Theorem 4.3, $\mathcal{H}_Y(P_\circ)$ is a prime $\mathcal{H}_Y$-filter such that $\mathscr{F} \subseteq \mathcal{H}_Y(P_\circ) \subseteq \mathscr{P}$. Therefore, $\mathcal{H}_Y(P_\circ) = \mathscr{P}$ and so $\mathcal{H}_Y^{-1}(\mathscr{P}) = P_\circ$, by Corollary 4.2.

(a $\Leftarrow$). Assume that $\mathscr{P}_\circ$ is a prime $\mathcal{H}_Y$-filter such that $\mathscr{F} \subseteq \mathscr{P}_\circ \subseteq \mathscr{P}$. By Theorem 4.3, $\mathcal{H}_Y^{-1}(\mathscr{P}_\circ)$ is a prime strong $\mathcal{H}_Y$-ideal and $\mathcal{H}_Y^{-1}(\mathscr{F}) \subseteq \mathcal{H}_Y^{-1}(\mathscr{P}_\circ) \subseteq \mathcal{H}_Y^{-1}(\mathscr{P})$. Therefore, $\mathcal{H}_Y^{-1}(\mathscr{P}_\circ) = \mathcal{H}_Y^{-1}(\mathscr{P})$ and so $\mathscr{P}_\circ = \mathscr{P}$, by Corollary 4.2.

(b $\Rightarrow$). Assume that I is a strong $\mathcal{H}_Y$-ideal and $P \in \mathrm{Min}(I)$, then by Theorem 3.13, P is a strong $\mathcal{H}_Y$-ideal, so $\mathcal{H}_Y^{-1}\mathcal{H}_Y(P) \in \mathrm{Min}(\mathcal{H}_Y^{-1}\mathcal{H}_Y(I))$. Hence by part (a) and Theorem 4.3, $\mathcal{H}_Y(P) \in \mathrm{Min}(\mathcal{H}_Y(I))$.

(b $\Leftarrow$). Let $Q \in \mathrm{Min}(I)$ and $I \subseteq Q \subseteq P$. Hence, $\mathcal{H}_Y(Q)$ is a prime $\mathcal{H}_Y$-filter, by Theorem 4.3, and $\mathcal{H}_Y(I) \subseteq \mathcal{H}_Y(Q) \subseteq \mathcal{H}_Y(P)$, so $\mathcal{H}_Y(Q) = \mathcal{H}_Y(P)$. By Theorem 3.13, Q is a strong $\mathcal{H}_Y$ -ideal, thus $P \subseteq \mathcal{H}_Y^{-1}\mathcal{H}_Y(P) = \mathcal{H}_Y^{-1}\mathcal{H}_Y(Q) = Q$ and so $Q = P$. □

Proposition 4.10. *Let $Y \subseteq Spec(R)$, $k(Y) = I$ and $\mathscr{F}$ be an $\mathcal{H}_Y$-filter on Y. Set $T = \{P/I : P \in Y\}$ and for every $A \in \mathscr{F}$ define $A' = \{P/I : P \in A\}$ and $\mathscr{F}' = \{A' : A \in \mathscr{F}\}$. The following statements hold.*

(a) *$\mathscr{F}'$ is an $\mathcal{H}_T$-filter on T.*
(b) *$\mathscr{F}$ is a prime $\mathcal{H}_Y$-filter on Y if and only if $\mathscr{F}'$ is a prime $\mathcal{H}_T$-filter on T.*
(c) *$\mathscr{F}$ is an $\mathcal{H}_Y$-ultrafilter on Y if and only if $\mathscr{F}'$ is an $\mathcal{H}_T$-ultrafilter on T.*
(d) $\frac{\mathcal{H}_Y^{-1}(\mathscr{F})}{I} = \mathcal{H}_T^{-1}(\mathscr{F}')$.

Proof. It is straightforward. □

Proposition 4.11. *Suppose R' is a subring of a ring R and $Y \subseteq Spec(R)$, then $Y' = \{P \cap R' : P \in Y\} \subseteq Spec(R')$. Set*

$$\mathscr{F}' = \{h_{Y'}(F) : h_Y(F) \in \mathscr{F} \text{ and } F \text{ is a finite subset of } R'\}$$

for every $\mathcal{H}_Y$-filter $\mathscr{F}$. The following statements hold.

(a) *$h_{Y'}(S) = \{P \cap R' : P \in h_Y(S)\}$, for every $S \subseteq R'$.*
(b) *If $\mathscr{F}$ is an $\mathcal{H}_Y$-filter, then $\mathscr{F}'$ is an $\mathcal{H}_{Y'}$-filter .*
(c) *For every $\mathcal{H}_{Y'}$-filter $\mathscr{G}$, there is some $\mathcal{H}_Y$-filter $\mathscr{F}$ such that $\mathscr{G} = \mathscr{F}'$.*
(d) *$\mathcal{H}_Y^{-1}(\mathscr{F}) \cap R' = \mathcal{H}_{Y'}^{-1}(\mathscr{F}')$, for every $\mathcal{H}_Y$-filter $\mathscr{F}$.*
(e) *If I is a (strong) $\mathcal{H}_Y$-ideal, then $I \cap R'$ is a (strong) $\mathcal{H}_{Y'}$-ideal.*
(f) *M' is a maximal (strong) $\mathcal{H}_{Y'}$-ideal if and only if there is some maximal (strong) $\mathcal{H}_Y$-ideal such that $M' = M \cap R'$.*

Proof. The proof is straightforward. □

5. Some important classes of $\mathcal{H}_Y$-ideals, strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals

In this section, we give propositions which generate a numerous class of $\mathcal{H}_Y$-ideals, strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals. Recall that, associated with each ideal I, there exists the ideal $m(I) = \{a \in R : a = ai \text{ for some } i \in I\} = \bigcup_{i\in I} \mathrm{Ann}(1-i)$ and associated with each prime ideal P, there is the ideal $O_P = \{a \in R : ab = 0 \text{ for some } b \notin P\} = \bigcup_{a\notin P} \mathrm{Ann}(a)$. $m(I)$ and O_P are called the quasi-regular part of I and the P component of the zero, respectively. Also an ideal I of R is called pure if $I = m(I)$. It is easy to check that when a union of (strong) $\mathcal{H}_Y$-ideals is an ideal, then the union is also a (strong) $\mathcal{H}_Y$-ideal. We refer to [4,5,18] for more detailed information about these classes of ideals. The following facts show that if the zero ideal is a (strong) $\mathcal{H}_Y$-ideal, then $Ann(I)$, $m(I)$ and O_P are (strong) $\mathcal{H}_Y$-ideals, where I and P are an arbitrary ideal and a prime ideal of R, respectively.

Proposition 5.1. *Suppose that $Y \subseteq Spec(R)$. If J is a strong $\mathcal{H}_Y$-ideal, then $(J : I)$ is a strong $\mathcal{H}_Y$-ideal, for every ideal I of R. The same assertions hold for $\mathcal{H}_Y$-ideals and Y-Hilbert ideals.*

Proof. Suppose that F is a finite subset of $(J : I)$ and $h_Y(F) \subseteq h_Y(a)$. For each $i \in I$

$$h_Y(Fi) = h_Y(F) \cup h_Y(i) = h_Y(a) \cup h_Y(i) = h_Y(ai)$$

since Fi is a finite subset of J and J is a strong $\mathcal{H}_Y$-ideal, it follows that $ai \in J$, thus $a \in (J : I)$. Using Proposition 3.4, concludes that $(I : J)$ is a strong $\mathcal{H}_Y$-ideal. □

Definition 5.2. Let $Y \subseteq \mathrm{Spec}(R)$ and $k(Y) = \langle 0 \rangle$. By a minimal (strong) $\mathcal{H}_Y$-ideal we mean a non-zero (strong) $\mathcal{H}_Y$-ideal which contains no (strong) $\mathcal{H}_Y$-ideal except $\langle 0 \rangle$.

Recall that a ring R is called Gelfand, if every prime ideal is contained in a unique maximal ideal. Also, a ring R is called weakly regular, if every non-zero ideal contains a non-zero idempotent element.

Proposition 5.3. *Let $Y \subseteq Spec(R)$ and $k(Y) = \langle 0 \rangle$. If R is either a semiprimitive Gelfand ring or a weakly regular ring then the minimal $\mathcal{H}_Y$-ideals, the minimal strong $\mathcal{H}_Y$-ideals and the minimal Y-Hilbert ideals coincide.*

Proof. Let I be a minimal $\mathcal{H}_Y$-ideal, minimal strong $\mathcal{H}_Y$-ideal or minimal $\mathcal{H}_Y$-ideal. If R is a semiprimitive Gelfand ring R, then since I is a non-zero ideal,by [5, Thereom 3.2] $m(I)$ is non-zero ideal. By [5, Propostion 2.1] $m(I) = \bigcup_{i \in I} \mathrm{Ann}(1-i)$. Therefore, $\langle 0 \rangle \neq \mathrm{Ann}(1-i) \subseteq m(I) \subseteq I$, for some $i \in I$. Also if R is a weakly regular ring, then the non-zero ideal I, contains an ideal of the form $\langle e \rangle = \mathrm{Ann}(1-e)$, where e is the non-zero idempotent element of I. So in the both of these rings we have $\mathrm{Ann}(x) \subseteq I$, for some $x \in R$. By Lemma 2.1, $\mathrm{Ann}(x) = kh_Y^c(x)$, hence $\mathrm{Ann}(I)$ is a Y-Hilbert ideal and so is (strong) $\mathcal{H}_Y$-ideal. Consequently, by the minimality of I, $I = \mathrm{Ann}(x)$ and we are done. □

The following proposition shows that a considerable class of ideals are strong $\mathcal{H}_Y$-ideal. Recall that for every multiplicatively closed subset A and any ideal I of a ring R with $A \cap I = \emptyset$, we can define the ideal $I_A = \{r \in R : ra \in I \text{ for some } a \in A\} = \bigcup_{a \in A}(I : a) = \sum_{a \in A}(I : a)$.

Proposition 5.4. *Suppose that $Y \subseteq Spec(R)$. If A is multiplicatively closed set and I is a (strong) $\mathcal{H}_Y$-ideal of R with $A \cap I = \emptyset$, then I_A is a (strong) $\mathcal{H}_Y$-ideal.*

Proof. Since $A \cap I = \emptyset$, $1 \notin \bigcup_{a \in A}(I : a) = I_A$ is a proper ideal. By Proposition 5.1, $(I : a)$ is a (strong) $\mathcal{H}_Y$-ideal, for every $a \in A$. Clearly, $\{(I : a)\}_{a \in A}$ is a directed family of (strong) $\mathcal{H}_Y$-ideals and since the union of a directed family of (strong) $\mathcal{H}_Y$-ideals is also a (strong) $\mathcal{H}_Y$-ideal, it follows that $I_A = \bigcup_{a \in A}(I : a)$ is a (strong) $\mathcal{H}_Y$-ideal. □

Remark 5.5. Suppose that $Y \subseteq \mathrm{Spec}(R)$, $k(Y) = \langle 0 \rangle$ and A is a multiplicatively closed subset of R. Then we can define the ideal $\langle 0 \rangle_A = 0_A = \{r \in R : ra = 0 \text{ for some } a \in A\} = \bigcup_{a \in A} \mathrm{Ann}(a) = \sum_{a \in A} \mathrm{Ann}(a)$. Since in this case $\langle 0 \rangle$ is a (strong) $\mathcal{H}_Y$-ideal, 0_A is always a (strong) $\mathcal{H}_Y$-ideal, by Proposition 5.4. Some of the most important cases are the ideals $O_P = 0_{R \setminus P}$ and $m(I) = 0_{1+I}$ where $1 + I = \{1 + i : i \in I\}$. On the other words, if $Y \subseteq \mathrm{Spec}(R)$ and $k(Y) = \langle 0 \rangle$, then the quasi-pure part (the zero-component) of every ideal (prime ideal) of R is a strong $\mathcal{H}_Y$-ideal. Consequently every pure ideal is a strong $\mathcal{H}_Y$-ideal. Recall that an element a of R is called (Von Neumann) regular if $a = a^2 b$, for some $b \in R$; an ideal I is said to be regular, if every element of I is regular and R is called regular if each elements of R is regular. It is easy to see that every regular ideal is a pure ideal. Thus every minimal ideal, every summand of any ring and the socle of a reduced ring are pure (for example see [4]), hence they all are strong $\mathcal{H}_Y$-ideal.

Remark 5.6. In $C(X)$ if either $\mathrm{Max}(C(X)) \subseteq Y$ or $\mathrm{Min}(C(X)) \subseteq Y$, then $k(Y) = \langle 0 \rangle$ is a strong $\mathcal{H}_Y$-ideal. Hence every minimal prime ideal is a strong $\mathcal{H}_Y$-ideal. Thus for every $A \subseteq \beta X$, O^A which is an intersection of minimal prime ideals is a strong $\mathcal{H}_Y$-ideal. Thus, if A is a round subset of βX (i.e., from $A \subseteq cl_{\beta X} Z(f)$, it follows that $A \subseteq int_{\beta X} cl_{\beta X} Z(f)$), then M^A is a strong $\mathcal{H}_Y$-ideal, too. By [12, 7E], $C_K(X) = O^{\beta X \setminus X}$ and by [16, Theorem 3.1], $C_\psi(X) = O^{\beta X \setminus \upsilon X}$, so $C_K(X)$ and $C_\psi(X)$ are strong $\mathcal{H}_Y$-ideals.

In the sequel we focus on answering this question that "What happens when all the ideals of a ring are either strong $\mathcal{H}_Y$-ideals or $\mathcal{H}_Y$-ideals?", which gives another characterizations of regular rings. First we give the following lemma which is easy to prove.

Lemma 5.7. *Suppose that $Y \subseteq Spec(R)$. Then every finitely generated strong $\mathcal{H}_Y$-ideal of R is a Y-Hilbert ideal. Also, if $a \in R$, then the following are equivalent.*

(a) *$\langle a \rangle$ is an $\mathcal{H}_Y$-ideal.*
(b) *$\langle a \rangle$ is a strong $\mathcal{H}_Y$-ideal.*
(c) *$\langle a \rangle$ is Y-Hilbert ideal.*

Proposition 5.8. *Let $Y \subseteq Spec(R)$, then the following statements are equivalent:*

(a) *Every ideal of R is a strong $\mathcal{H}_Y$-ideal.*
(b) *Every finitely generated ideal of R is a strong $\mathcal{H}_Y$-ideal.*
(c) *Every finitely generated ideal of R is a Y-Hilbert ideal.*
(d) *Every ideal of R is an $\mathcal{H}_Y$-ideal.*
(e) *Every principal ideal of R is an $\mathcal{H}_Y$-ideal.*
(f) *Every principal ideal of R is a strong $\mathcal{H}_Y$-ideal.*
(g) *Every principal ideal of R is a Y-Hilbert ideal.*
(h) *$k(Y) = \langle 0 \rangle$ and R is a regular ring.*
(k) *$k(Y) = \langle 0 \rangle$ and every essential ideal of R is a strong $\mathcal{H}_Y$-ideal.*
(l) *$k(Y) = \langle 0 \rangle$ and every essential ideal of R is an $\mathcal{H}_Y$-ideal.*

Proof. (a) $\Rightarrow$ (b). It is clear.

The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are straightforward.

(d) $\Rightarrow$ (e). It is evident.

(e) $\Rightarrow$ (f) $\Rightarrow$ (g). They follow from Lemma 5.7.

(g) $\Rightarrow$ (h). By the hypothesis the zero ideal is a Y-Hilbert and this implies that $k(Y) = \langle 0 \rangle$. Also by the assumption, every ideal of R is semiprime and consequently R is a regular ring.

(h) $\Rightarrow$ (a). Sine R is regular, for every ideal I of R we have $I = m(I)$ and so by Remark 5.5, the result follows.

(a) $\Rightarrow$ (k) $\Rightarrow$ (l). They are trivial.

(l) $\Rightarrow$ (h). It is well-known and easy to be verified that if in a reduced ring every essential ideal is a semiprime ideal then it is a regular ring. So by Lemma 3.12, we are done. □

In the above proposition if we take either $\mathrm{Max}(R) \subseteq Y$ or $\mathrm{Min}(R) \subseteq Y$, then we can add the assertions: "every ideal of R is a Y-Hilbert ideal" and " Every essential ideal of R is a Y-Hilbert ideal". In the following example we show that this is not true in general.

Example 5.9. Suppose that $R = C(\mathbb{N})$, $Y = \mathcal{B}(R)$ and M is a maximal ideal of R. Since the zero ideal of R is a fixed-place ideal, by [2, Theorem 4.7], it follows that there is an ultrafilter $\mathscr{U}$ on Y such that $M = \mathbf{J}(\mathscr{U}) = \{a \in R : h_Y(a) \in \mathscr{U}\}$. Set $\mathscr{F} = \mathscr{U} \cap \mathcal{H}_Y$, then $\mathscr{F}$ is a $\mathcal{H}_Y$-filter on Y and

$$\begin{aligned}
\mathcal{H}_Y^{-1}(\mathscr{F}) &= \{a \in R : h_Y(a) \in \mathscr{F}\} \\
&= \{a \in R : h_Y(a) \in \mathscr{U} \cap \mathcal{H}_Y\} \\
&= \{a \in R : h_Y(a) \in \mathscr{U}\} \\
&= \mathbf{J}(\mathscr{U}) = M.
\end{aligned}$$

Thus M is a strong $\mathcal{H}_Y$-ideal. Since R is regular ring, every ideal of R is an intersection of maximal ideals and therefore every ideal is a strong $\mathcal{H}_Y$-ideal. But, if M is a free maximal ideal, then M is not a Y-Hilbert ideal.

Corollary 5.10. *Let Y be a finite subset of $Spec(R)$. If I is an ideal of R, then the following statements are equivalent.*

(a) *I is an $\mathcal{H}_Y$-ideal.*
(b) *I is a strong $\mathcal{H}_Y$-ideal.*

(c) *I is a Y-Hilbert ideal.*

Proof. It suffices to show (a) $\Rightarrow$ (c). To see this, suppose that I is an $\mathcal{H}_Y$-ideal. Since Y is finite, by prime avoidance theorem, there exists some $x \in I \setminus \cup_{Q \in h^c_Y(I)} Q$. Thus, clearly, $h_Y(x) \subseteq h_Y(I)$ and so we have $kh_Y(I) \subseteq kh_Y(x) \subseteq I \subseteq kh_Y(I)$. Therefore, $I = kh_Y(I)$ and so I is a Y-Hilbert ideal. □

6. Operations on $\mathcal{H}_Y$-ideals, strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals

As the title of this section shows, it is devoted to considering quotients, products, homomorphic images of $\mathcal{H}_Y$-ideals, strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals.

We shall note that, a product of $\mathcal{H}_Y$-ideals (resp., strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals) is not necessarily an $\mathcal{H}_Y$-ideal (resp., a strong $\mathcal{H}_Y$-ideal and a Y-Hilbert ideal). For instance, if we set $R = \mathbb{Z}$ and $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$ then for every prime number p, the ideal $J = p\mathbb{Z}$ is a strong $\mathcal{H}_Y$-ideal while $J^2 = p^2\mathbb{Z}$ is not even a semiprime ideal. In general we have the following proposition.

Proposition 6.1. *Let R be a ring and $\{J_i\}_{i=1}^n$ be a finite family of strong $\mathcal{H}_Y$-ideals of R, then $\prod_{i=1}^n J_i$ is a strong $\mathcal{H}_Y$-ideal if and only if $\bigcap_{i=1}^n J_i = \prod_{i=1}^n J_i$. The same statements hold for $\mathcal{H}_Y$-ideals and Y-Hilbert ideals.*

Proof. By Lemma 3.12, it is clear. □

Let $f : R \to R'$ be a ring homomorphism and I and J be ideals of R and R', respectively. Then I^e and J^c denote the extension and the contraction of the ideals I and J, (i.e., $\langle f(I) \rangle$ and $f^{-1}(J)$), respectively. In the following proposition we study the contraction of (strong) $\mathcal{H}_Y$-ideals and Y-Hilbert ideals under a ring homomorphism.

Proposition 6.2. *Let $f : R \to R'$ be a ring homomorphism, $X \subseteq Spec(R)$ and $Y \subseteq Spec(R')$. Every strong $\mathcal{H}_Y$-ideal of R' contracts to a strong $\mathcal{H}_X$-ideal of R if and only if every $P \in Y$ contracts to a strong $\mathcal{H}_X$-ideal. The same statements hold for $\mathcal{H}_Y$-ideals and Y-Hilbert ideals.*

Proof. ($\Rightarrow$). Suppose that J is a strong $\mathcal{H}_Y$-ideal of R'. If F_1 and F_2 are two arbitrary subsets of R which $h_X(F_1) = h_X(F_2)$ and $F_1 \subseteq J^c$, then

$$\begin{aligned} P \in h_Y(f(F_1)) &\Leftrightarrow \quad f(F_1) \subseteq P \quad \Leftrightarrow \quad F_1 \subseteq P_c \\ &\Leftrightarrow \quad P^c \in h_X(F_1) \quad \Leftrightarrow \quad P^c \in h_X(F_2) \\ &\Leftrightarrow \quad F_2 \subseteq P^c \quad \Leftrightarrow \quad f(F_2) \subseteq P \\ &\Leftrightarrow \quad P \in h_Y(f(F_2)). \end{aligned}$$

So $h_Y(f(F_1)) = h_Y(f(F_2))$ and $f(F_1) \subseteq J$, hence $f(F_2) \subseteq J$, by Proposition 3.4. Thus $F_2 \subseteq J^c$ and this implies that J^c is a strong $\mathcal{H}_Y$-ideal, by Proposition 3.4.

($\Leftarrow$). It is clear. □

Corollary 6.3. *Let $I \subseteq J$ be a pair of ideals of R, $Y \subseteq Spec(R)$ and $Y/I = \{P/I : P \in h_Y(I)\}$. Then J/I is a strong $\mathcal{H}_{\frac{Y}{I}}$-ideal if and only if J is a strong $\mathcal{H}_Y$-ideal. Also, supposing that I_λ is an ideal of R, for every $\lambda \in \Lambda$, if $\sum_{\lambda \in \Lambda} I_\lambda$ is a direct sum and a strong $\mathcal{H}_Y$-ideal, then I_λ is a strong $\mathcal{H}_Y$-ideal, for every $\lambda \in \Lambda$. The same statements hold for $\mathcal{H}_Y$-ideals and Y-Hilbert ideals.*

The following corollaries show the relation between the strong $\mathcal{H}_Y$-ideals of two different subspaces of $\mathrm{Spec}(R)$. Note that the same statements hold for the $\mathcal{H}_Y$-ideals and the Y-Hilbert ideals.

Corollary 6.4. *Let $X, Y \subseteq Spec(R)$. Then we have the following facts:*

(a) *Every element of X is a strong $\mathcal{H}_Y$-ideal if and only if every strong $\mathcal{H}_X$-ideal is a strong $\mathcal{H}_Y$-ideal.*
(b) *If $X \subseteq Y$, then every strong $\mathcal{H}_X$-ideal is a strong $\mathcal{H}_Y$-ideal.*
(c) *If $X \subseteq Y$ and every element of Y is a strong $\mathcal{H}_X$-ideal, then the strong $\mathcal{H}_X$-ideal and the strong $\mathcal{H}_Y$-ideals coincide.*

Proof. If we take the identity mapping from (R, Y) to (R, X) and apply Proposition 6.2, then they conclude. □

Corollary 6.5. *Let $X, Y \subseteq Spec(R)$, $I_\circ = k(X) \subseteq k(Y)$ and $X \subseteq Min(I_\circ)$. Every strong $\mathcal{H}_X$-ideal ($\mathcal{H}_X$-ideal) is a strong $\mathcal{H}_Y$-ideal ($\mathcal{H}_Y$-ideal) if and only if $k(X) = k(Y)$.*

Proof. We just prove the part concerned with the strong $\mathcal{H}_Y$-ideal. The part concerned with the $\mathcal{H}_X$-ideal has a same proof.

$\Rightarrow$) Clearly, $I_\circ$ is a strong $\mathcal{H}_X$-ideal and therefore $I_\circ$ is a strong $\mathcal{H}_Y$-ideal. Again, $k(Y)$ is the smallest strong $\mathcal{H}_Y$-ideal, since $I_\circ \subseteq k(Y)$, we conclude that $k(Y) = I_\circ = k(X)$.

$\Leftarrow$) Since $I_\circ = k(Y)$, $I_\circ$ is a strong $\mathcal{H}_Y$-ideal and therefore every element of $\mathrm{Min}(I_\circ)$ is a strong H_Y-ideal, hence every element of X is a strong $\mathcal{H}_Y$-ideal and therefore each strong $\mathcal{H}_X$-ideal is a strong $\mathcal{H}_Y$-ideal, by Corollary 6.4. □

Proposition 6.6. *Let A be a multiplicatively closed subset of R and $f : R \to A^{-1}R$ be the natural ring homomorphism. If I is a (strong) $\mathcal{H}_Y$-ideal, then I^{ec} is a (strong) $\mathcal{H}_Y$-ideal, too.*

Proof. It is easy to see that $I^{ec} = I_A$ and so by Proposition 5.4 we are done. □

7. Certain (strong) $\mathcal{H}_Y$-ideals over or contained in an ideal

This section is about the particular (strong) $\mathcal{H}_Y$-ideals related to an ideal. First we study the maximal (strong) $\mathcal{H}_Y$-ideals, then the smallest (strong) $\mathcal{H}_Y$-ideal containing an ideal are characterized. As we will see that some of the results hold for Y-Hilbert ideals too. For convenience we use some notations. Let E be a partially ordered set. By $maxl(E)$, we mean the set of all maximal elements of E. Also if R is a ring, $Y \subseteq \mathrm{Spec}(R)$ and $\mathcal{A} \subseteq \mathfrak{I}(R)$, we denote by $S\mathcal{H}_Y(\mathcal{A})$ ($PS\mathcal{H}_Y(\mathcal{A})$) the set of all strong H_Y-ideals (proper strong H_Y-ideals) of A. For $\mathcal{H}_Y$-ideals we use the notations $\mathcal{H}_Y(\mathcal{A})$ and $P\mathcal{H}_Y(\mathcal{A})$, respectively. By $[I, J]$ we mean $\{K \in \mathfrak{I}(R) :\ I \subseteq K \subseteq J\}$; and by $\downarrow I$ and $\uparrow I$ we mean $\{K \in \mathfrak{I}(R) :\ K \subseteq I\}$ and $\{K \in \mathfrak{I}(R) :\ I \subseteq K\}$, respectively. It is straightforward to observe that the union of a chain of proper (strong) $\mathcal{H}_Y$-ideals is a proper (strong) $\mathcal{H}_Y$-ideal.

Proposition 7.1. *Let R be a ring and $Y \subseteq Spec(R)$ and I is a proper (strong) $\mathcal{H}_Y$-ideal of R. Then the following statements hold.*

(a) *For every ideal $J \supseteq I$, $maxl(P\mathcal{H}_Y[I, J]) \neq \emptyset$ ($maxl(PS\mathcal{H}_Y[I, J]) \neq \emptyset$). In the particular, $maxl(P\mathcal{H}_Y(\uparrow I)) \neq \emptyset$ ($maxl(PS\mathcal{H}_Y(\uparrow I)) \neq \emptyset$) and for every ideal $J \supseteq k(Y)$, $maxl(P\mathcal{H}_Y(\downarrow J)) \neq \emptyset$ ($maxl(PS\mathcal{H}_Y(\downarrow J)) \neq \emptyset$).*
(b) *Let $Y \subseteq Spec(R)$ and P be a prime ideal containing $k(Y)$. Then $maxl(\mathcal{H}_Y(\downarrow P))$ and $maxl(S\mathcal{H}_Y(\downarrow P))$ are contained in $Spec(R)$.*
(c) *If $k(Y) = \langle 0\rangle$, then every prime ideal of R is either a (strong) $\mathcal{H}_Y$-ideal or contains a maximal (strong) $\mathcal{H}_Y$-ideal which is a prime (strong) $\mathcal{H}_Y$-ideal.*

Proof. We just prove the part concerned with the strong $\mathcal{H}_Y$-ideal. The part concerned with the $\mathcal{H}_X$-ideal has a same proof.

(a). By using Zorn's lemma, it implies immediately.

(b). If P is an $\mathcal{H}_Y$-ideal, then it is clear. Now suppose that P is not an $\mathcal{H}_Y$-ideal, by part (a), $Q \in maxl(\mathcal{H}_Y(\downarrow P))$. Since P is not an $\mathcal{H}_Y$-ideal, by Corollary 3.14, $P \notin \mathrm{Min}(Q)$, so $Q' \in \mathrm{Min}(Q)$ exists such that $Q' \subseteq P$. Now Corollary 3.14, deduces that Q' is an $\mathcal{H}_Y$-ideal, so $Q = Q'$ is prime, by maximality of Q.

(c). It is clear by part (b). □

In the following example we show that $maxl(PS\mathcal{H}_Y[I, J])$ need not be a proper maximal strong H_Y-ideal, even if J is a maximal ideal.

Example 7.2. Suppose that $R = \mathbb{R}[x, y]$, $I = \langle x-1\rangle$, $J = \langle y\rangle$, $K = \langle x-1, y\rangle$, $M = \langle x, y\rangle$ and $Y = \{J, K\}$. It is clear that $Y \subseteq \mathrm{Spec}(R)$ and it is easy to show that $maxl(PS\mathcal{H}_Y[I\cap J, M]) = \{J\}$, whereas J is not a proper maximal strong $\mathcal{H}_Y$-ideal, because K is a strong $\mathcal{H}_Y$-ideal that properly contains J.

Using Theorem 3.13, Proposition 4.1 and Proposition 4.9, one can obtain the following corollary straightforward.

Corollary 7.3. *Let $Y \subseteq Spec(R)$ and $Rad(R) = k(Y)$. Every $P \in Min(R)$ is a strong $\mathcal{H}_Y$-ideal and therefore there is some minimal prime $\mathcal{H}_Y$-filter $\mathscr{P}$ such that $\mathcal{H}_Y^{-1}(\mathscr{P}) = P$.*

Definition 7.4. Let $Y \subseteq \mathrm{Spec}(R)$. It is obvious that the intersection of any family of $\mathcal{H}_Y$-ideals (resp., strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals) is an $\mathcal{H}_Y$-ideal (resp., a strong $\mathcal{H}_Y$-ideal and Y-Hilbert ideals). According to this fact, the smallest $\mathcal{H}_Y$-ideal (resp., strong $\mathcal{H}_Y$-ideal and Y-Hilbert ideal) containing an arbitrary ideal I exists. We denote it by $I_{\mathcal{H}_Y}$ (resp., $I_{S\mathcal{H}_Y}$ and $kh_Y(I)$) which is the intersection of $\mathcal{H}_Y$-ideals (resp., strong $\mathcal{H}_Y$-ideals and Y-Hilbert ideals) containing I. If there is not any ambiguity we use $I_{\mathcal{H}}$ (resp., $I_{S\mathcal{H}}$) instead of $I_{\mathcal{H}_Y}$ (resp., $I_{S\mathcal{H}_Y}$).

Clearly, if $Y = \mathrm{Max}(R)$, then the concepts of $I_{\mathcal{H}}$ and $I_{S\mathcal{H}}$ coincide with the concepts of I_z and I_{sz}, respectively. See [3] and [19] for more detailed information about these concepts. Also, if $Y = \mathrm{Min}(R)$, then the concepts of $I_{\mathcal{H}}$ and $I_{S\mathcal{H}}$ coincide with the concepts of I_{z° (also known as $I_\circ$ and I°) and I_{sz° (also known as $\zeta(I)$-ideal), respectively. We refer to [3, 9, 10, 18], for more information about these concepts. Finally if $Y = \mathrm{Spec}(R)$, then the concepts of $I_{\mathcal{H}}$ and $I_{S\mathcal{H}}$ and $\sqrt{I}$ coincide. It is clear that $I_{\mathcal{H}} \subseteq I_{S\mathcal{H}}$.

Proposition 7.5. *Let $Y \subseteq Spec(R)$ and I and J be two ideals of R. Then the following statements hold:*

- (a) $I_{S\mathcal{H}} = \mathcal{H}_Y^{-1}\mathcal{H}_Y(I) = \{a \in R : \exists F \in \mathbf{F}(I),\ h_Y(F) \subseteq h_Y(a)\} = \{a \in R : \exists F \in \mathbf{F}(I),\ kh_Y(a) \subseteq kh_Y(F)\}$ *and if R satisfies h_Y-property, then we have* $I_{S\mathcal{H}} = I_{\mathcal{H}} = \{a \in R : \exists\, b \in I$ *such that* $h_Y(b) \subseteq h_Y(a)\}$.
- (b) $I_{S\mathcal{H}} = \sum_{F\in\mathbf{F}(I)} kh_Y(F) = \bigcup_{F\in\mathbf{F}(I)} kh_Y(F)$.
- (c) $(IJ)_{\mathcal{H}} = I_{\mathcal{H}} \cap J_{\mathcal{H}} = (I\cap J)_{\mathcal{H}}$ *(resp.,* $(IJ)_{S\mathcal{H}} = I_{S\mathcal{H}} \cap J_{S\mathcal{H}} = (I\cap J)_{S\mathcal{H}}$*).*
- (d) $kh_Y(I) = \{a \in R : \exists S \subseteq I,\ h_Y(S) \subseteq h_Y(a)\}$. *Also* $kh_Y(IJ) = kh_Y(I)\cap kh_Y(J) = kh_Y(I\cap J)$.

Proof. (a). By Proposition 4.1, $\mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$ is a strong $\mathcal{H}_Y$-ideal containing I. Now, assume that J is a strong $\mathcal{H}_Y$-ideal containing I, then $\mathcal{H}_Y^{-1}\mathcal{H}_Y(I) \subseteq \mathcal{H}_Y^{-1}\mathcal{H}_Y(J) = J$, so the first equality holds. On the other hand, since $I_{S\mathcal{H}}$ is a strong $\mathcal{H}_Y$-ideal, the set $H = \{a \in R : \exists F \in \mathbf{F}(I),\ h_Y(F) \subseteq h_Y(a)\}$ is a subset of $I_{S\mathcal{H}}$. Since H contains I, to show the second equality it is enough to prove that H is a strong $\mathcal{H}_Y$-ideal. Let $a, b \in H$, so there exist finite subsets F_1 and F_2 of I such that $h_Y(F_1) \subseteq h_Y(a)$ and $h_Y(F_2) \subseteq h_Y(b)$, so

$$h_Y(F_1 \cup F_2) = h_Y(F_1) \cap h_Y(F_2) \subseteq h_Y(a) \cap h_Y(b) \subseteq h_Y(a+b).$$

Since $F_1 \cup F_2$ is finite, $a+b \in H$. Also, since $h_Y(a) \subseteq h_Y(ra)$, for each $r \in R$, H is an ideal. Now it is enough to show that H is a strong $\mathcal{H}_Y$-ideal. Let $h_Y(F) \subseteq h_Y(a)$, where

$F = \{x_1, x_2, \ldots, x_n\}$ is a finite subset of H, then for each $1 \leqslant i \leqslant n$, there exists a finite set $F_i \subseteq I$ such that $h_Y(F_i) \subseteq h_Y(x_i)$. Now we have that

$$h_Y\Big(\bigcup_{i=1}^{n} F_i\Big) = \bigcap_{i=1}^{n} h_Y(F_i) \subseteq \bigcap_{i=1}^{n} h_Y(x_i) = h_Y(F) \subseteq h_Y(a).$$

Now since $\bigcup_{i=1}^{n} F_i$ is a finite subset of I, we are done. It is clear that if R satisfies in h_Y-property, then $I_{S\mathcal{H}} = I_{\mathcal{H}} = \{a \in R : \exists\, b \in I \text{ such that } h_Y(b) \subseteq h_Y(a)\}$.

(b). Since $\{kh_Y(F)\}_{F\in\mathbf{F}(I)}$ is a directed set, it follows that $\bigcup_{F\in\mathbf{F}(I)} kh_Y(F)$ is an ideal and so $\sum_{F\in\mathbf{F}(I)} kh_Y(F) = \bigcup_{F\in\mathbf{F}(I)} kh_Y(F)$. Also, it is clear that $\bigcup_{F\in\mathbf{F}(I)} kh_Y(F)$ is a strong $\mathcal{H}_Y$-ideal containing I, so $\bigcup_{F\in\mathbf{F}(I)} kh_Y(F) = I_{S\mathcal{H}}$.

(c). Obviously, since $IJ \subseteq I \cap J$, we have $(IJ)_{\mathcal{H}} \subseteq (I \cap J)_{\mathcal{H}} \subseteq I_{\mathcal{H}} \cap J_{\mathcal{H}}$ (resp., $(IJ)_{S\mathcal{H}} \subseteq (I \cap J)_{S\mathcal{H}} \subseteq I_{S\mathcal{H}} \cap J_{S\mathcal{H}}$). Now, suppose that P is a prime $\mathcal{H}_Y$-ideal containing $(IJ)_{\mathcal{H}}$ (resp., a prime strong $\mathcal{H}_Y$-ideal containing $(IJ)_{S\mathcal{H}}$). Then clearly $I \subseteq P$ or $J \subseteq P$ and so $I_{\mathcal{H}} \subseteq P$ or $J_{\mathcal{H}} \subseteq P$ (resp., $I_{S\mathcal{H}} \subseteq P$ or $J_{S\mathcal{H}} \subseteq P$) and consequently, $I_{\mathcal{H}} \cap J_{\mathcal{H}} \subseteq P$ (resp., $I_{S\mathcal{H}} \cap J_{S\mathcal{H}} \subseteq P$).

(d). It is clear. □

Recall that a ring R satisfies property A, if each finitely generated ideal of R consisting of zero-divisors has a non-zero annihilator (equivalently every finitely generated ideal with a zero annihilator contains a non zero-divisor, known as condition C in [20]). As it is stated in [10], Noetherian rings, $C(X)$, regular rings satisfy property A. Clearly a proper ideal I is contained in a proper (strong) $\mathcal{H}_Y$-ideal if and only if $I_{\mathcal{H}}$ ($I_{S\mathcal{H}}$) is a proper ideal. Also according to Proposition 4.1, $\mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$ is a proper ideal of R if and only if $\emptyset \notin \mathcal{H}_Y(I)$ (equivalently, $\mathcal{H}_Y(I)$ is a proper $\mathcal{H}_Y$-filter). It is also clear that every maximal ideal is a (strong) $\mathcal{H}_Y$-ideal if and only if every proper ideal is contained in a proper (strong) $\mathcal{H}_Y$-ideal. For any $Y \subseteq \operatorname{Spec}(R)$ we have the following corollary of the above proposition which is an improvement of [10, Theorem 1.21] with a totally different proof.

Corollary 7.6. *Let R be a reduced ring, $Y \subseteq Spec(R)$, $k(Y) = \langle 0\rangle$ and R satisfies property A. Then any singular ideal I (i.e., every element of I is a zero-divisor) is contained in a proper strong $\mathcal{H}_Y$-ideal and therefore is contained in a proper $\mathcal{H}_Y$-ideal.*

Proof. It is sufficient to show that every element of $I_{S\mathcal{H}}$ is a zero-divisor. Let $a \in I_{S\mathcal{H}}$, thus $h_Y(F) \subseteq h_Y(a)$, for some finite set $F \subseteq I$, thus $kh_Y^{c}(F) \subseteq kh_Y^{c}(a)$, therefore by Lemma 2.1, $\operatorname{Ann}(F) \subseteq \operatorname{Ann}(a)$. Since R satisfies property A and F consists of zero-divisors, it follows that $\operatorname{Ann}(a) \neq \langle 0\rangle$, that is, a is a zero-divisor. □

If $k(Y) = \langle 0\rangle$, then according to Lemma 3.8, Lemma 3.9 and Proposition 7.5, we have the following characterization of I_{sz°.

Corollary 7.7. *Let R be a ring, $Y \subseteq Spec(R)$, $k(Y) = \langle 0\rangle$. Then $I_{sz^\circ} = \{a \in R : (h_Y(F))^\circ \subseteq h_Y(a)$ for some finite $F \subseteq I\}$.*

Corollary 7.8. *Let R be a ring, $Y \subseteq Spec(R)$, $k(Y) = \langle 0\rangle$ and R satisfies property A. Then the following facts hold:*

(a) *Every maximal ideal consisting of zero-divisors is a (strong) $\mathcal{H}_Y$-ideal.*
(b) *Every ideal consisting of zero-divisors is contained in a maximal (strong) $\mathcal{H}_Y$-ideal which is a prime ideal.*

Remark 7.9. With a method similar to [10, Thoerem 1.21], we can observe that if I is an ideal of R and we set $I_0 = I$, $I_1 = \sum_{a\in I_0} kh_Y(a)$, $I_\alpha = \sum_{a\in I_\beta} kh_Y(a)$ for a nonlimit ordinal $\alpha = \beta + 1$ and $I_\alpha = \bigcup_{\beta\leq\alpha} I_\beta$, for a limit ordinal α, then the smallest ordinal α that $I_\alpha = I_\gamma$, for every $\gamma \geq \alpha$, is exactly $I_{\mathcal{H}}$.

Proposition 7.10. *Let R be a ring, I is an arbitrary ideal of R and $Y \subseteq Spec(R)$. The following statements hold.*

(a) *If* $k(Y) = \langle 0\rangle$, *then* $(m(I))_{\mathcal{H}} = (m(I))_{S\mathcal{H}} = m(I)$.
(b) *If* $Max(R) \subseteq Y$, *then* $m(I) = m(I_{\mathcal{H}}) = m(I_{S\mathcal{H}}) = m(kh_Y(I))$.

Proof. (a). It is clear from Remark 5.5.

(b). It is shown in [5, Remark 2.6] that when $\mathrm{Max}(R) \subseteq Y$, then $m(kh_Y(I)) = m(I)$, for every ideal I and since clearly $I \subseteq I_{\mathcal{H}} \subseteq I_{S\mathcal{H}} \subseteq kh_Y(I)$, it follows that $m(I) = m(I_{\mathcal{H}}) = m(I_{S\mathcal{H}}) = m(kh_Y(I))$. □

The condition $\mathrm{Max}(R) \subseteq Y$ is necessary for the equalities of part (b) of the above proposition. For instance, if $Y = \mathrm{Min}(R)$ and M is a maximal ideal containing a non zero-divisor, then $m(M) \subseteq M \neq R = m(M_{\mathcal{H}_Y}) = m(M_{S\mathcal{H}_Y})$. Also, as we see in Example 5.9, there exists a ring R, $Y \subseteq \mathrm{Spec}(R)$ and a maximal ideal $M \notin Y$ such that M is a strong $\mathcal{H}_Y$-ideal. Hence, in this case $m(M_{S\mathcal{H}_Y}) = m(M) \neq R = m(R) = m(kh_Y(M))$.

As a corollary of the above proposition we have the following proposition which gives more facts about $I_{\mathcal{H}}$, $I_{S\mathcal{H}}$ and $kh_Y(I)$.

Proposition 7.11. *Let* $X, Y \subseteq Spec(R)$. *If* I *is an ideal of* R *and* $n \in \mathbb{N}$, *then*

(a) $I^n \subseteq I \subseteq \sqrt{I} \subseteq I_{\mathcal{H}_Y} \subseteq I_{S\mathcal{H}_Y} \subseteq kh_Y(I)$.
(b) $(I^n)_{\mathcal{H}_Y} = (\sqrt{I})_{\mathcal{H}_Y} = I_{\mathcal{H}_Y}$, $(I^n)_{S\mathcal{H}_Y} = (\sqrt{I})_{S\mathcal{H}_Y} = I_{S\mathcal{H}_Y}$ *and* $kh_Y(I^n) = kh_Y(\sqrt{I}) = kh_Y(I)$.
(c) *If every element of* Y *is an* $\mathcal{H}_X$*-ideal (resp., strong* $\mathcal{H}_X$*-ideal and* X*-Hilbert ideal), then* $I_{\mathcal{H}_X} \subseteq I_{\mathcal{H}_Y}$ *(resp.,* $I_{S\mathcal{H}_Y} \subseteq \mathcal{H}_Y^{-1}\mathcal{H}_Y(I)$ *and* $kh_X(I) \subseteq kh_Y(I)$*).*

Proof. (a). It follows form Lemma 3.12.

(b). By Proposition 7.5 and part (a) it is straightforward.

(c). It follows from Corollary 6.4. □

Supposing $R = \mathbb{R}[x, y]$, $Y = \{\langle x\rangle, \langle y\rangle\}$, then $\langle x, y\rangle = \langle x\rangle + \langle y\rangle$ is not an $\mathcal{H}_Y$-ideal, so the sum of a two strong $\mathcal{H}_Y$-ideals, need not be an $\mathcal{H}_Y$-ideal. One can easily see that the sum of a family of strong $\mathcal{H}_Y$-ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ is a strong $\mathcal{H}_Y$-ideal if and only if $(\sum_{\lambda\in\Lambda} I_\lambda)_{S\mathcal{H}} = \sum_{\lambda\in\Lambda}(I_\lambda)_{S\mathcal{H}}$. In addition, we can see that $(\sum_{\lambda\in\Lambda} I_\lambda)_{S\mathcal{H}} = (\sum_{\lambda\in\Lambda}(I_\lambda)_{S\mathcal{H}})_{S\mathcal{H}}$. Also, if $\mathrm{Max}(R) \subseteq Y$, then $\sum_{\lambda\in\Lambda} I_\lambda = R$ if and only if $\sum_{\lambda\in\Lambda}(I_\lambda)_{S\mathcal{H}} = R$. To see this, suppose that $\sum_{\lambda\in\Lambda} I_\lambda \neq R$, then by the hypothesis there is a strong $\mathcal{H}_Y$-ideal containing $\sum_{\lambda\in\Lambda} I_\lambda$ and so $(\sum_{\lambda\in\Lambda} I_\lambda)_{S\mathcal{H}_Y} \neq R$. Therefore, $\sum_{\lambda\in\Lambda}(I_\lambda)_{S\mathcal{H}_Y} \subseteq (\sum_{\lambda\in\Lambda}(I_\lambda)_{S\mathcal{H}_Y})_{S\mathcal{H}_Y} = (\sum_{\lambda\in\Lambda}(I_\lambda))_{S\mathcal{H}_Y} \neq R$. We shall note that the same statements hold for the case $\mathcal{H}_Y$-ideals.

References

[1] A.R. Aliabad and M. Badie, *Fixed-place ideals in commutative rings*, Comment. Math. Univ. Carolin. **54** (1), 53–68, 2013.

[2] A.R. Aliabad and M. Badie, *On Bourbaki associated prime divisors of an ideal*, Quaest. Math. **42** (4), 479-500, 2019.

[3] A.R. Aliabad and R. Mohamadian, *On sz°-ideals in polynomial rings*, Comm. Algebra **39** (2) (2011), 701–717, 2011.

[4] A.R. Aliabad, R. Mohamadian, and S. Nazari, *On regular ideals in reduced rings*, Filomat **31** (12), 3715–3726, 2017.

[5] A.R. Aliabad and S. Nazari, *On the spectrum of a commutative ring via* $C(X)$, (to appear).

[6] A.R. Aliabad, A. Taherifar, and N. Tayarzadeh, *α-Baer rings and some related concepts via* $C(X)$, Quaest. Math. **39** (3), 401–419, 2016.

[7] G. Artico, U. Marconi, and R. Moresco, *A subspace of Spec(A) and its connexions with the maximal ring of quotients*, Rend. Sem. Mat. Univ., Padova **64**, 93–107, 1981.

[8] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, vol. 2, Addison-Wesley Reading, 1969.

[9] F. Azarpanah, O.A.S. Karamzadeh, and A.R. Aliabad, *On z°-ideals in C(X)*, Fund. Math. **160** (1), 15-25,1999.

[10] F. Azarpanah, O.A.S. Karamzadeh, and A.R. Aliabad, *On ideals consisting entirely of zero-divisors*, Comm. Algebra **28** (2), 1061–1073, 2000.

[11] T.S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer-Verlag, London, 2005.

[12] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand Reinhold, New York, 1960.

[13] G. Grätzer, *General Lattice Theory*, Birkhäuser, Basel, 1998.

[14] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115**, 110–130, 1965.

[15] C.B. Huijsmans and B. De Patger, *On z-ideals and d-ideals in Riesz spaces. I*, Indag. Math. (Proceeding) A **83** (2), 183–195, 1980.

[16] D.G. Johnson and M. Mandelker, *Functions with pseudocompact support*, Topology Appl. **3** (4), 331–338, 1973.

[17] J. Kist, *Minimal prime ideals in commutative semigroups*, Proc. Lond. Math. Soc. **3** (1), 31–50, 1963.

[18] G. Mason, *z-ideals and prime ideals*, J. Algebra **26** (2), 280–297, 1973.

[19] G. Mason, *Prime ideals and quotient rings of reduced rings*, Math. Jpn. **34** (6), 941–956, 1989.

[20] Y. Quentel, *Sur la compacité du spectre minimal d'un anneau*, Bull. Soc. Math. France **99**, 265–272, 1971.

[21] R.Y. Sharp, *Steps in Commutative Algebra*, Cambridge university press, London, 1990.

[22] S. Willard, *General Topology*, Addison Wesley, Reading Mass., New York, 1970.