



An inequality for warped product submanifolds of a locally product Riemannian manifold

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Abstract

Recently, Sahin studied the warped product semi-slant submanifolds of locally product Riemannian manifolds. In this paper, we obtain some geometric properties of such submanifolds with an example. Also, we establish a sharp relationship between the squared norm of the second fundamental form and the warping function in terms of the slant angle. The equality case is also considered.

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1. Introduction

The idea of slant immersions is an increasing development in differential geometry which is given by Chen [6] in almost Hermitian setting. Recently, Sahin defined and studied these immersions for locally product Riemannian manifolds [10]. In [9], Papaghuic has extended this idea to semi-slant submanifolds of almost Hermitian manifolds which are the generalization of holomorphic, totally real and slant submanifolds. Recently, these submanifolds of locally product Riemannian manifolds were studied by Li and Li [7].

On the other hand, the idea of warped product manifolds is given by Bishop and O'Neill in [5]. They defined these manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2 , respectively, and a positive differentiable function f on M_1 . Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y)$$

for any vector field X, Y tangent to M , where \star is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply *Riemannian product manifold* if the warping function f is constant. Let X be an unit vector field

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tangent to M_1 and U be an another unit vector field on M_2 , then from Lemma 7.3 of [5], we have

$$\nabla_X U = \nabla_U X = \left(\frac{Xf}{f}\right)U, \quad (1.1)$$

where ∇ is the Levi-Civita connection on M . If $M = M_1 \times_f M_2$ be a warped product manifold then M_1 is a totally geodesic submanifold of M and M_2 is a totally umbilical submanifold of M [5].

Recently, warped product submanifolds of locally product Riemannian manifolds were studied in [1–3, 11, 12, 15]. Furthermore, the warped products of almost contact metric and almost Hermitian manifolds were appeared in [13, 14, 16, 17]. In this paper, we study warped product semi-slant submanifolds of a locally product Riemannian manifold. We discuss some geometric properties of such submanifolds and give an example, our example fulfil the definition of a proper semi-slant submanifold i.e., the slant angle lies in the first quadrant. Also, we establish an inequality for the squared norm of second fundamental form in terms of the warping function and the slant angle. The equality case is also discussed.

2. Preliminaries

Let \widetilde{M} be a m -dimensional differentiable manifold with a tensor field F of type (1,1) such that $F^2 = I$ and $F \neq \pm I$, then we say that \widetilde{M} is an almost product manifold with almost product structure F . If we set

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F)$$

then we can easily see that

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad \text{and } P - Q = F.$$

Thus P and Q define two orthogonal complementary distributions \mathcal{D}_1 and \mathcal{D}_2 on \widetilde{M} . If \widetilde{M} admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \quad (2.1)$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then \widetilde{M} is called an almost product Riemannian manifold [18], where $\Gamma(T\widetilde{M})$ denotes the set all vector fields of \widetilde{M} . Since $F^2 = I$, we can easily see that the eigenvalues of F are 1 or -1 . An eigenvector corresponding to the eigenvalue 1 associates with P and the eigenvector corresponding to the eigenvalue -1 is associated with Q . Let $\widetilde{\nabla}$ denotes the Levi-Civita connection on \widetilde{M} with respect to the Riemannian metric g . Then the covariant derivative of F is defined by

$$(\widetilde{\nabla}_X F)Y = \widetilde{\nabla}_X FY - F\widetilde{\nabla}_X Y$$

for any $X, Y \in \Gamma(T\widetilde{M})$. If $(\widetilde{\nabla}_X F)Y = 0$, the almost product Riemannian manifold \widetilde{M} is said to be a *locally product Riemannian manifold*.

Let M be a Riemannian manifold isometrically immersed in \widetilde{M} and we denote by the same symbol g the Riemannian metric induced on M . Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M . Let ∇ be the Levi-Civita connection on M . Then the Gauss and Weingarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \widetilde{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (2.2)$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where ∇^\perp is the normal connection in the normal bundle $\Gamma(T^\perp M)$ and A_N is the shape operator of M with respect to N . Moreover, $\sigma : TM \times TM \rightarrow T^\perp M$ is the second fundamental form of M in \widetilde{M} . Furthermore, A_N and σ are related by

$$g(\sigma(X, Y), N) = g(A_N X, Y) \quad (2.3)$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

For any X tangent to M , we write

$$FX = TX + \omega X, \tag{2.4}$$

where TX (resp. ωX) denotes the tangential (resp. normal) component of FX . Then T is an endomorphism of tangent bundle TM and ω is a normal bundle valued 1-form on TM .

A submanifold M is said to be *F-invariant* if ω is identically zero, i.e., $FX \in \Gamma(TM)$, for any $X \in \Gamma(TM)$. On the other hand, M is said to be *F-anti-invariant* if T is identically zero i.e., $FX \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$. Moreover, from (2.1) and (2.4), we have $g(TX, Y) = g(X, TY)$, for any $X, Y \in \Gamma(TM)$.

A submanifold M of a locally product Riemannian manifold \widetilde{M} is said to be *totally umbilical submanifold* if $\sigma(X, Y) = g(X, Y)H$, for any $X, Y \in \Gamma(TM)$, where $H = \frac{1}{n} \sum_{i=1}^n \sigma(e_i, e_i)$, the mean curvature vector of M . A submanifold M is said to be *totally geodesic* if $\sigma(X, Y) = 0$. Also, we set

$$\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r), \quad i, j = 1, \dots, n; \quad r = n + 1, \dots, m,$$

and

$$\|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(e_i, e_j), \sigma(e_i, e_j)) \tag{2.5}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space $T_p M$, for any $p \in M$.

For a differentiable function f on a m -dimensional manifold \widetilde{M} , the gradient $\vec{\nabla} f$ of f is defined as $g(\vec{\nabla} f, X) = Xf$, for any X tangent to \widetilde{M} . As a consequence, we have

$$\|\vec{\nabla} f\|^2 = \sum_{i=1}^m (e_i(f))^2 \tag{2.6}$$

for an orthonormal frame $\{e_1, \dots, e_m\}$ on \widetilde{M} .

By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a locally product Riemannian manifold were considered.

Let \widetilde{M} be a locally product Riemannian manifold and M a submanifold of \widetilde{M} , then M is said to be *semi-invariant submanifold* of \widetilde{M} [4] if there exist a differentiable distribution $\mathcal{D} : p \rightarrow \mathcal{D}_p \subset T_p M$ such that \mathcal{D} is invariant with respect to F and the complementary distribution \mathcal{D}^\perp is anti-invariant with respect to F [8].

Let M be a Riemannian submanifold of a locally product Riemannian manifold \widetilde{M} . For each nonzero vector field $X \in \Gamma(TM)$ at $p \in M$, the angle $\theta(X)$, $0 \leq \theta(X) \leq \frac{\pi}{2}$ between FX and $T_p M$ is called the Wirtinger angle of X . If the angle $\theta(X)$ is constant, which is independent of the choice $p \in M$ and $X \in \Gamma(TM)$, then M is called a *slant submanifold* of \widetilde{M} and the angle θ is called the *slant angle* of the immersion. Thus, *F*-invariant immersion and *F*-anti-invariant immersion are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither *F*-invariant nor *F*-anti-invariant is called *proper slant immersion*. It is easy to see that M is slant submanifold of a locally product Riemannian manifold \widetilde{M} if and only if

$$T^2 = \lambda I \tag{2.7}$$

for some real number $\lambda \in [0, 1]$ [10], where I denotes the identity transformation of the tangent bundle TM of the submanifold M . Moreover, if M is a slant submanifold and θ is the slant angle of M , then $\lambda = \cos^2 \theta$. The following relations are consequences of (2.7)

$$\begin{aligned} g(TX, TY) &= \cos^2 \theta g(X, Y) \\ g(\omega X, \omega Y) &= \sin^2 \theta g(X, Y) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

Moreover, we say that M is a semi-slant submanifold of \widetilde{M} , if there exist two orthogonal distributions \mathcal{D} and \mathcal{D}^θ such that

- (i) $TM = \mathcal{D} \oplus \mathcal{D}^\theta$,
- (ii) the distribution \mathcal{D} is invariant, i.e. $F(\mathcal{D}) = \mathcal{D}$,
- (iii) the distribution \mathcal{D}^θ is slant with slant angle $\theta \neq 0$.

We will call \mathcal{D}^θ as a slant distribution. In particular, if $\theta = \frac{\pi}{2}$, then the semi-slant submanifold is a semi-invariant submanifold [3]. On the other hand, If we denote the dimension of \mathcal{D} and \mathcal{D}^θ by d_1 and d_2 , respectively. It is clear that if $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then M is a proper slant submanifold with slant angle θ . Also, if $d_2 = 0$ then M is F -invariant submanifold and if $d_1 = 0$ and $\theta = \frac{\pi}{2}$ then M is F -anti-invariant submanifold. Furthermore, if neither $d_1 = 0$ nor $\theta = \frac{\pi}{2}$, then M is a proper semi-slant submanifold.

3. Warped product semi-slant submanifolds

Warped product semi-slant submanifolds of locally product Riemannian manifolds were studied by Atceken and Sahin in [1] and [12]. They proved that the warped products of the type $M_T \times_f M_\theta$ do not exist. On the other hand, they provided some examples and a characterization on the existence of warped products $M_\theta \times_f M_T$, where M_T and M_θ are invariant and proper slant submanifolds of a locally product Riemannian manifold \widetilde{M} , respectively. In this section, we study the warped product semi-slant submanifolds of type $M_\theta \times_f M_T$ of locally product Riemannian manifolds which have not been considered in earlier studies. First, we prove the following results for later use.

Lemma 3.1. *Let $M = M_\theta \times_f M_T$ be a warped product semi-slant submanifold of a locally product Riemannian manifold \widetilde{M} . Then*

- (i) $g(\sigma(X, U), \omega V) = -g(\sigma(X, V), \omega U)$;
- (ii) $g(\sigma(X, Y), \omega U) = -(U \ln f)g(X, FY) + (TU \ln f)g(X, Y)$;
- (iii) $g(\sigma(X, Y), \omega TU) = -(TU \ln f)g(X, FY) + \cos^2 \theta (U \ln f)g(X, Y)$

for any $X, Y \in \Gamma(TM_T)$ and $U, V \in \Gamma(TM_\theta)$.

Proof. For any $X \in \Gamma(TM_T)$ and $U, V \in \Gamma(TM_\theta)$, we have

$$\begin{aligned} g(\sigma(X, U), \omega V) &= g(\widetilde{\nabla}_X U, FV) - g(\widetilde{\nabla}_X U, TV) \\ &= g((\widetilde{\nabla}_X F)U, V) - g(\widetilde{\nabla}_X FU, V) - (U \ln f)g(X, TV). \end{aligned}$$

First and last terms in the right hand side are identically zero by using the structure of a locally product Riemannian manifold and the orthogonality of vector fields, then from (2.4), we obtain

$$g(\sigma(X, U), \omega V) = g(\widetilde{\nabla}_X TU, V) + g(\widetilde{\nabla}_X \omega U, V).$$

By using (2.2)-(2.3) and (1.1) in above relation, we get (i). For the second part, consider $X, Y \in \Gamma(TM_T)$ and $U \in \Gamma(TM_\theta)$, then we have

$$\begin{aligned} g(\sigma(X, Y), \omega U) &= g(\widetilde{\nabla}_X Y, \omega U) \\ &= g(\widetilde{\nabla}_X Y, FU) - g(\widetilde{\nabla}_X Y, TU) \\ &= g((F\widetilde{\nabla}_X Y, U) + g(Y, \widetilde{\nabla}_X TU) \\ &= g(\widetilde{\nabla}_X FY, U) + (TU \ln f)g(X, Y) \\ &= -g(FY, \widetilde{\nabla}_X U) + (TU \ln f)g(X, Y). \end{aligned}$$

Using (2.2) and (1.1), we obtain

$$g(\sigma(X, Y), \omega U) = -(U \ln f)g(X, FY) + (TU \ln f)g(X, Y),$$

which is (ii). If we replace U by TU in (ii), and then using (2.7), we get (iii), which proves the lemma completely. \square

From the above lemma we can easily find the following relations by interchanging X by FX and Y by FY , for any $X, Y \in \Gamma(TM_T)$ in Lemma 3.1 (ii)-(iii)

$$g(\sigma(X, FY), \omega U) = -(U \ln f)g(X, Y) + (TU \ln f)g(X, FY), \tag{3.1}$$

$$g(\sigma(FX, Y), \omega U) = -(U \ln f)g(X, Y) + (TU \ln f)g(FX, Y) \tag{3.2}$$

and

$$g(\sigma(FX, Y), \omega TU) = -(TU \ln f)g(X, Y) + \cos^2 \theta (U \ln f)g(FX, Y) \tag{3.3}$$

$$g(\sigma(X, FY), \omega TU) = -(TU \ln f)g(X, Y) + \cos^2 \theta (U \ln f)g(X, FY). \tag{3.4}$$

Then using (2.1) in (3.1) and (3.2), we find

$$g(\sigma(X, FY), \omega U) = g(\sigma(FX, Y), \omega U). \tag{3.5}$$

Similarly, from (3.3) and (3.4), we obtain

$$g(\sigma(FX, Y), \omega TU) = g(\sigma(X, FY), \omega TU). \tag{3.6}$$

Also, if we interchange X by FX in (3.1) and Y by FY in (3.3), we arrive at

$$g(\sigma(FX, FY), \omega U) = -(U \ln f)g(FX, Y) + (TU \ln f)g(X, Y), \tag{3.7}$$

$$g(\sigma(FX, FY), \omega TU) = -(TU \ln f)g(FX, Y) + \cos^2 \theta (U \ln f)g(X, Y). \tag{3.8}$$

Then from (3.7) and Lemma 3.1 (ii), we get

$$g(\sigma(FX, FY), \omega U) = g(\sigma(X, Y), \omega U) \tag{3.9}$$

and by Lemma 3.1 (iii) and (3.8), we derive

$$g(\sigma(FX, FY), \omega TU) = g(\sigma(X, Y), \omega TU). \tag{3.10}$$

Now, we construct the following example of warped product semi-slant submanifolds of a locally product Riemannian manifold.

Example 3.2. Let us consider the almost product manifold $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$ and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial y_j}, \quad i = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3, 4.$$

Let M be a submanifold of \mathbb{R}^7 given by

$$f(\theta, \varphi, v, u) = (u \cos \theta, u \sin \theta, u + v, \sqrt{3}u - v, v, u \sin \varphi, u \cos \varphi)$$

$$\text{with } u \neq 0, \quad v \neq 0 \quad \text{and} \quad \theta, \varphi \in \left(0, \frac{\pi}{2}\right).$$

Then the tangent space TM of M is spanned by the following vector fields

$$Z_1 = -u \sin \theta \frac{\partial}{\partial x_1} + u \cos \theta \frac{\partial}{\partial x_2}, \quad Z_2 = u \cos \varphi \frac{\partial}{\partial y_3} - u \sin \varphi \frac{\partial}{\partial y_4}$$

$$Z_3 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}$$

$$Z_4 = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \sqrt{3} \frac{\partial}{\partial y_1} + \sin \varphi \frac{\partial}{\partial y_3} + \cos \varphi \frac{\partial}{\partial y_4}.$$

Then with respect to the Riemannian product structure F , we get

$$FZ_1 = -Z_1, \quad FZ_2 = Z_2, \quad FZ_3 = -\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}$$

$$FZ_4 = -\cos \theta \frac{\partial}{\partial x_1} - \sin \theta \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \sqrt{3} \frac{\partial}{\partial y_1} + \sin \varphi \frac{\partial}{\partial y_3} + \cos \varphi \frac{\partial}{\partial y_4}.$$

Then, it is easy to see that the invariant and slant distributions are spanned by

$$\mathcal{D} = \text{span}\{Z_1, Z_2\} \quad \text{and} \quad \mathcal{D}^{\theta_1} = \text{span}\{Z_3, Z_4\}$$

with the slant angle say θ_1 . Then

$$\theta_1 = \arccos \left(\frac{g(FZ_3, Z_3)}{\|FZ_3\| \|Z_3\|} \right) = \arccos \left(\frac{g(FZ_4, Z_4)}{\|FZ_4\| \|Z_4\|} \right) = \arccos \left(\frac{1}{3} \right) = 70^\circ 31'.$$

Thus, M is a proper semi-slant submanifold of \mathbb{R}^7 . It is also easy to check that \mathcal{D} and \mathcal{D}^{θ_1} are integrable. If we denote the integral manifolds of \mathcal{D} and \mathcal{D}^{θ_1} by M_T and M_{θ_1} , respectively. Then the induced metric tensor on M is given by

$$g = (6du^2 + 3dv^2) + u^2 (d\theta^2 + d\varphi^2) = g_{M_{\theta_1}} + u^2 g_{M_T}.$$

Thus M is a warped product submanifold of the form $M = M_{\theta_1} \times_{f_1} M_T$ with the warping function $f_1 = u$.

If M is a warped product semi-slant submanifold of the form $M = M_\theta \times_f M_T$ of a locally product Riemannian manifold \widetilde{M} and if there is no μ -components in the normal bundle of M , then M is *mixed totally geodesic* (Proposition 4.1 [12]) i.e., $\sigma(X, U) = 0$, for any $X \in \Gamma(TM_T)$ and $U \in \Gamma(TM_\theta)$.

Now, we construct the following frame field for an n -dimensional warped product semi-slant submanifold $M = M_\theta \times_f M_T$ of a m -dimensional locally product Riemannian manifold \widetilde{M} . Let us denote by \mathcal{D} and \mathcal{D}^θ the tangent bundles of M_T and M_θ , respectively instead of TM_T and TM_θ . Also, we consider the $\dim(M_T) = t$ and $\dim(M_\theta) = q$, then the orthonormal frames of \mathcal{D} and \mathcal{D}^θ , respectively are given by $\{e_1 = Fe_1, \dots, e_k = Fe_k, e_{k+1} = -Fe_{k+1}, \dots, e_t = -Fe_t\}$ and $\{e_{t+1} = e_1^* = \sec \theta Te_1^*, \dots, e_{t+q} = e_q^* = \sec \theta Te_q^*\}$. Then the orthonormal frame fields of the normal subbundles of $\omega\mathcal{D}^\theta$ and μ , respectively are $\{e_{n+1} = \tilde{e}_1 = \csc \theta \omega e_1^*, \dots, e_{n+q} = \tilde{e}_q = \csc \theta \omega e_q^*\}$ and $\{e_{n+q+1} = \tilde{e}_{q+1}, \dots, e_m = \tilde{e}_{m-n-q}\}$.

Now, we are able to construct the following inequality with the help of the above constructed frame fields and some previous formulas which we have obtained for warped product semi-slant submanifolds of a locally product Riemannian manifold.

Theorem 3.3. *Let $M = M_\theta \times_f M_T$ be a proper warped product semi-slant submanifold of a locally product Riemannian manifold \widetilde{M} , where M_T and M_θ are invariant and proper slant submanifolds of \widetilde{M} , respectively. Then*

- (i) *The squared norm of the second fundamental form of the warped product immersion satisfies*

$$\|\sigma\|^2 \geq t(\csc \theta - \cot \theta)^2 \|\nabla^\theta \ln f\|^2$$

where $t = \dim M_T$ and $\nabla^\theta \ln f$ is gradient of the function $\ln f$ along M_θ .

- (ii) *If equality sign in (i) holds identically, then M_θ is totally geodesic in \widetilde{M} and M_T is a totally umbilical submanifold of \widetilde{M} . Furthermore, $M_\theta \times_f M_T$ is a mixed totally geodesic submanifold of \widetilde{M}*

Proof. From (2.5), we have

$$\|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=n+1}^m \sum_{i,j=1}^n g(\sigma(e_i, e_j), e_r)^2.$$

Then from the assumed frame fields of \mathcal{D} and \mathcal{D}^θ , we derive

$$\begin{aligned} \|\sigma\|^2 &= \sum_{r=n+1}^m \sum_{i,j=1}^t g(\sigma(e_i, e_j), e_r)^2 + 2 \sum_{r=n+1}^m \sum_{i=1}^t \sum_{j=t+1}^n g(\sigma(e_i, e_j), e_r)^2 \\ &\quad + \sum_{r=n+1}^m \sum_{i,j=t+1}^n g(\sigma(e_i, e_j), e_r)^2. \end{aligned} \tag{3.11}$$

After leaving the second and third term in the right hand side of (3.11) and using the constructed frame fields, we find

$$\|\sigma\|^2 \geq \sum_{r=1}^q \sum_{i,j=1}^t g(\sigma(e_i, e_j), \csc \theta \omega e_r^*)^2 + \sum_{r=q+1}^{m-n-q} \sum_{i,j=1}^t g(\sigma(e_i, e_j), \tilde{e}_r)^2 \quad (3.12)$$

The second term in the right hand side of the above expression has the μ -components, therefore we shall leave this term and hence using Lemma 3.1, we get

$$\begin{aligned} \|\sigma\|^2 \geq \csc^2 \theta \sum_{r=1}^q \sum_{i,j=1}^t & [(e_r^* \ln f)^2 g(e_i, F e_j)^2 + (T e_r^* \ln f)^2 g(e_i, e_j)^2 \\ & - 2(T e_r^* \ln f) \cdot (e_r^* \ln f) g(e_i, e_j) g(e_i, F e_j)] \end{aligned} \quad (3.13)$$

Now, since $\nabla^\theta \ln f \in \Gamma(TM_\theta)$ and $e_r^* = \sec \theta T e_r^*$, then from (2.6) we have

$$\sum_{r=1}^q (T e_r^* \ln f) \cdot (e_r^* \ln f) = \sum_{r=1}^q \cos \theta (e_r^* \ln f)^2 = \cos \theta \|\nabla^\theta \ln f\|^2 \quad (3.14)$$

and

$$\sum_{r=1}^q (T e_r^* \ln f)^2 = \sum_{r=1}^q g(T e_r^*, \nabla^\theta \ln f)^2 = \cos^2 \theta \|\nabla^\theta \ln f\|^2. \quad (3.15)$$

Then, from (3.13), (3.14) and (3.15), we find

$$\|\sigma\|^2 \geq t \csc^2 \theta (1 - \cos \theta)^2 \|\nabla^\theta \ln f\|^2,$$

which is inequality (i). For the equality, from the remaining terms of (3.11), we find

$$\sigma(\mathcal{D}, \mathcal{D}^\theta) = 0, \quad \text{and} \quad \sigma(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0. \quad (3.16)$$

Also, from the remaining second term in the right hand side of (3.12), we observe that

$$\sigma(\mathcal{D}, \mathcal{D}) \perp \mu \quad \Rightarrow \quad \sigma(\mathcal{D}, \mathcal{D}) \in \Gamma(\omega \mathcal{D}^\theta) \quad (3.17)$$

The second condition of (3.16) implies that M_θ is totally geodesic in \widetilde{M} due to M_θ being totally geodesic in M [5]. On the other hand, (3.17) implies that M_T is totally umbilical in \widetilde{M} with the fact that M_T is totally umbilical in M [5]. Moreover, all conditions of (3.16) imply that M is a mixed totally geodesic submanifold of \widetilde{M} . Hence, the proof is complete. \square

From the above theorem, we have the following remark.

Remark 3.4. In Theorem 3.3, if we assume $\theta = \frac{\pi}{2}$, then the warped product becomes $M = M_\perp \times_f M_T$ in a locally product Riemannian manifold \widetilde{M} , where M_T and M_\perp are invariant and anti-invariant submanifolds of \widetilde{M} , respectively, which is a case of warped product semi-nivariant submanifolds which have been discussed in ([3], [11]). Thus, Theorem 4.2 of [11] and Theorem 4.1 of [3] are the special cases of Theorem 3.3.

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