



Characterizing local rings via complete intersection homological dimensions

Fatemeh Mohammadi Aghjeh Mashhad 

Parand Branch, Islamic Azad University, Tehran, Iran

Abstract

Let (R, \mathfrak{m}) be a commutative Noetherian local ring. It is known that R is Cohen-Macaulay if there exists either a nonzero Cohen-Macaulay R -module of finite projective dimension or a nonzero finitely generated R -module of finite injective dimension. In this article, we will prove the complete intersection analogues of these facts. Also, by using complete intersection homological dimensions, we will characterize local rings which are either regular, complete intersection or Gorenstein.

Mathematics Subject Classification (2010). 13D05

Keywords. Complete intersection homological dimensions, complete intersection ring, CI-regular ring, Gorenstein ring, regular ring

1. Introduction and prerequisites

Throughout this paper, (R, \mathfrak{m}) is a local ring and all rings are commutative and Noetherian with identity. The projective dimension is a familiar and famous numerical invariant in classical homological algebra. One of the fascinating theorems which is related to this dimension, is Auslander-Buchsbaum-Serre Theorem [1, 15] which asserts that R is a regular ring if every finitely generated R -module has finite projective dimension. Motivated by this, Auslander and Bridger [2], introduced the Gorenstein dimension (abbr. G-dimension) for any finitely generated R -module and they proved that R is Gorenstein when every finitely generated R -module has finite G-dimension. The G-dimension has a very essential role for studying Gorenstein homological algebra and it was studied in more details in [2, 7]. Let us recall the definition of G-dimension. Let M be a nonzero finitely generated R -module. The G-dimension of M is zero, $\text{G-dim}_R M = 0$, if and only if the natural homomorphism $M \rightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)$ is an isomorphism, and $\text{Ext}_R^i(M, R) = 0 = \text{Ext}_R^i(\text{Hom}_R(M, R), R)$ for any $i > 0$. We set $\text{G-dim}_R 0 = -\infty$. Also, for an integer n , $\text{G-dim}_R M \leq n$ if and only if there exists an exact sequence

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

of R -modules such that $\text{G-dim}_R X_i = 0$ for any $0 \leq i \leq n$. More recently, Avramov, Gasharov and Peeva [3] introduced the concept of complete intersection dimension for finitely generated R -modules as a generalization of projective dimension. They proved that R is complete intersection when every finitely generated R -module has finite complete intersection dimension. For defining complete intersection dimension, we need the

Email addresses: mohammadifh@ipm.ir (F.M.A. Mashhad)

Received: 28.11.2015; Accepted: 12.05.2016

definition of quasi-deformation of R . A quasi-deformation of R is a diagram of local ring homomorphisms $R \longrightarrow R' \longleftarrow Q$ such that $R \longrightarrow R'$ is faithfully flat and $R' \longleftarrow Q$ is surjective with the kernel which is generated by a Q -regular sequence. The complete intersection dimension of a finitely generated R -module M , $\text{CI-dim}_R M$ is defined as follows:

$$\text{CI-dim}_R M := \inf\{\text{pd}_Q(M \otimes_R R') - \text{pd}_Q R' \mid R \longrightarrow R' \longleftarrow Q \text{ is a quasi-deformation}\}.$$

These homological dimensions satisfy in the following inequalities

$$\text{G-dim}_R M \leq \text{CI-dim}_R M \leq \text{pd}_R M, \quad (1)$$

with equality to the left of any finite quantity, see [3].

Wagstaff [20] extended the definition of complete intersection dimension for any R -module N . He defined the complete intersection projective dimension of N , $\text{CI-pd}_R N$, the complete intersection flat dimension of N , $\text{CI-fd}_R N$, and the complete intersection injective dimension of N , $\text{CI-id}_R N$, as follows:

$$\text{CI-pd}_R N := \inf\{\text{pd}_Q(N \otimes_R R') - \text{pd}_Q R' \mid R \longrightarrow R' \longleftarrow Q \text{ is a quasi-deformation}\},$$

$$\text{CI-fd}_R N := \inf\{\text{fd}_Q(N \otimes_R R') - \text{pd}_Q R' \mid R \longrightarrow R' \longleftarrow Q \text{ is a quasi-deformation}\},$$

$$\text{CI-id}_R N := \inf\{\text{id}_Q(N \otimes_R R') - \text{pd}_Q R' \mid R \longrightarrow R' \longleftarrow Q \text{ is a quasi-deformation}\}.$$

One can see that if N is finitely generated, then $\text{CI-pd}_R N = \text{CI-dim}_R N$.

We denote the category of finitely generated R -modules by $\text{mod}(R)$, the subcategory of $\text{mod}(R)$ consisting of all free R -modules by $F(R)$, the subcategory of $\text{mod}(R)$ consisting of zero module and all R -modules M such that $\text{G-dim}_R M = 0$ (resp. $\text{CI-dim}_R M = 0$) by $G(R)$ (resp. $CI(R)$). By using (1), we have the following inclusion relations between the subcategories of $\text{mod}(R)$,

$$F(R) \subseteq CI(R) \subseteq G(R).$$

Takahashi [18] defined R to be G-regular if $G(R) = F(R)$. We define R to be CI-regular if $CI(R) = F(R)$. The first goal of this paper is a characterization of local rings by using complete intersection homological dimensions. We prove that R is regular if and only if R is complete intersection and CI-regular. Also, we prove that R is complete intersection if and only if R is Gorenstein and $CI(R) = G(R)$. Let M be an R -module of finite depth, that is, $\text{Ext}_R^i(k, M) \neq 0$ for some $i \in \mathbb{Z}$. In Theorem 2.5 below, we show that if either

- i) $\text{CI-pd}_R M < \infty$ and $\text{id}_R M < \infty$ or
- ii) $\text{CI-id}_R M < \infty$ and $\text{fd}_R M < \infty$,

then R is Gorenstein.

In the classical homological algebra, there exist two celebrated and important facts which are obtained by virtue of the (Peskin-Szpiro) intersection theorem [12] as follow:

- i) If there exists a nonzero Cohen-Macaulay R -module of finite projective dimension, then R is Cohen-Macaulay,
- ii) If there exists a nonzero R -module of finite injective dimension, then R is Cohen-Macaulay,

where the second assertion is known as Bass's Theorem. Now, it is natural to ask the following questions:

Question 1.1. If there exists a nonzero Cohen-Macaulay R -module with finite complete intersection dimension, is then R Cohen-Macaulay?

Question 1.2. If there exists a nonzero finitely generated R -module with finite complete intersection injective dimension, is then R Cohen-Macaulay?

As a second goal of this article, we give the positive answers to the above questions, see Theorem 2.7. Also, for the interested reader, we recall that the Gorenstein analogues of i) and ii) are still open questions and many people have tried to prove them; see [8, 16, 17, 21].

2. The results

We start this section by the following definition.

Definition 2.1. We say that a local ring (R, \mathfrak{m}, k) is CI-regular if $CI(R)$ coincides with $F(R)$.

Proposition 2.2. *Let R be a local ring.*

- i) R is CI-regular if and only if $CI\text{-dim}_R M = \text{pd}_R M$ for any finitely generated R -module M .
- ii) A normal local ring R is CI-regular if and only if $CI\text{-dim}_R R/I = \text{pd}_R R/I$ for every ideal I of R .

Proof. i) (\Rightarrow) Let M be a finitely generated R -module. It suffices to show that $\text{pd}_R M \leq CI\text{-dim}_R M$. Without loss of generality, we assume that $CI\text{-dim}_R M$ is finite. Set $n := CI\text{-dim}_R M$. By [19, Corollary 3.9], there exists an exact sequence

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

such that $CI\text{-dim}_R F_i = 0$ for any $0 \leq i \leq n$, and so by the assumption, F_i is projective for any $0 \leq i \leq n$ which implies that $\text{pd}_R M \leq n$.

(\Leftarrow) is trivial.

ii) (\Rightarrow) implies from i).

(\Leftarrow) Let M be a finitely generated R -module such that $CI\text{-dim}_R M = 0$. Then $G\text{-dim}_R M = 0$ and so M is torsion-free R -module by [7, Lemma 1.1.8]. By [4, Theorem 6 in Chapter VII §4], there exists an exact sequence $0 \longrightarrow R^n \longrightarrow M \longrightarrow I \longrightarrow 0$, where I is an ideal of R . By [19, Lemma 3.6], one can see that $CI\text{-dim}_R I$ is finite and so using again of [19, Lemma 3.6] for the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$, implies that $CI\text{-dim}_R R/I$ is finite. So by the assumption, $\text{pd}_R R/I$ is finite and then from the exact sequence

$$0 \longrightarrow R^n \longrightarrow M \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

one can deduce that $\text{pd}_R M$ is finite. Hence $\text{pd}_R M = CI\text{-dim}_R M = 0$, and so M is free. \square

Next, we present a criterion for specification regular local rings.

Proposition 2.3. *Let (R, \mathfrak{m}, k) be a local ring. The following are equivalent:*

- i) R is regular,
- ii) R is complete intersection and CI-regular.

Proof. i) \Rightarrow ii) By [5, Proposition 3.1.20 and Theorem 2.2.7], R is complete intersection and for any finitely generated R -module M , $\text{pd}_R M$ is finite, and so by [3, Theorem 1.4], $\text{pd}_R M = CI\text{-dim}_R M$ which implies that R is CI-regular by Proposition 2.2 i).

ii) \Rightarrow i) By [3, Theorem 1.3] and Proposition 2.2 i), $\text{pd}_R k$ is finite, and so R is regular by [5, Theorem 2.2.7]. \square

Theorem 1.3 in [3] characterizes complete intersection local rings with complete intersection dimension. In the following, we will characterize these rings with complete intersection homological dimensions. Before doing this, we recall that an R -module N is said to be Gorenstein injective if there exists an exact complex I of injective R -modules such that $N \cong \text{im}(I_1 \longrightarrow I_0)$ and $\text{Hom}_R(E, I)$ is exact for all injective R -modules E . Any injective R -module is Gorenstein injective. Let N be an R -module. We say that Gorenstein injective dimension of N , $\text{Gid}_R N$, is finite if it has a finite Gorenstein injective resolution. It is easy to see that when the usual injective dimension of N , $\text{id}_R N$, is finite, then $\text{Gid}_R N$ is also finite.

Proposition 2.4. *Let (R, \mathfrak{m}, k) be a local ring. The following are equivalent:*

- i) R is complete intersection,
- ii) $\text{CI-id}_R M < \infty$ for any R -module M ,
- iii) $\text{CI-id}_R k < \infty$,
- iv) $\text{CI-dim}_R k < \infty$,
- v) R is Gorenstein and $\text{CI}(R) = G(R)$.

Proof. i) \Rightarrow ii) Since R is complete intersection, then there exists a regular ring Q such that $\Lambda^m(R) \cong Q/xQ$, where $x = x_1, x_2, \dots, x_n$ is a Q -regular sequence and $\Lambda^m(R)$ is an \mathfrak{m} -adic completion of R . Consider a quasi-deformation $R \rightarrow \Lambda^m(R) \leftarrow Q$. Then $\text{id}_Q(M \otimes_R \Lambda^m(R)) < \infty$ for any R -module M and this completes the proof.

ii) \Rightarrow iii) is clear.

iii) \Rightarrow iv) There exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $R' \cong Q/I$ for some ideal I of Q and $\text{id}_Q(k \otimes_R R') = \text{id}_Q(R'/\mathfrak{m}R') < \infty$. Then, we can conclude that for some ideal J of Q , we have $\text{id}_Q(Q/J) < \infty$ and so $\text{Gid}_Q(Q/J) < \infty$ which implies that Q should be a Gorenstein ring by [9, Theorem 4.5]. So $\text{pd}_Q(k \otimes_R R') < \infty$ by [5, Exercise 3.1.25] which implies that $\text{CI-dim}_R k < \infty$.

iv) \Rightarrow i) It follows from [3, Theorem 1.3].

i) \Rightarrow v) R is a Gorenstein ring by [5, Proposition 3.1.20], and also for any finitely generated R -module M , we have $\text{CI-dim}_R M = G\text{-dim}_R M$ by [3, Theorems 1.3 and 1.4].

v) \Rightarrow i) Since R is Gorenstein, then $G\text{-dim}_R k$ is finite by [7, Theorem 1.4.9]. Set $n := G\text{-dim}_R k$. Let P be a projective resolution of k , as follow:

$$\dots \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow k \rightarrow 0.$$

Then $\ker(P_{n-1} \rightarrow P_{n-2}) \in G(R)$ by [7, Theorem 2.3.16], and so by the assumption, $\ker(P_{n-1} \rightarrow P_{n-2}) \in \text{CI}(R)$. Hence, $\text{CI-dim}_R k$ is finite by [19, Corollary 3.8] and the assertion follows from [3, Theorem 1.3]. □

We recall that for any R -module M , $\text{depth}_R M$ is defined as follow:

$$\text{depth}_R M := \inf\{i \in \mathbb{Z} \mid \text{Ext}_R^i(k, M) \neq 0\}.$$

If M is a non-zero finitely generated R -module, then M has finite depth.

Theorem 2.5. *Let (R, \mathfrak{m}, k) be a local ring. The following are equivalent:*

- i) R is Gorenstein,
- ii) There exists a nonzero R -module M with finite depth such that $\text{CI-pd}_R M < \infty$ and $\text{id}_R M < \infty$,
- iii) There exists a nonzero R -module M with finite depth such that $\text{CI-fd}_R M < \infty$ and $\text{id}_R M < \infty$,
- iv) There exists a nonzero R -module M with finite depth such that $\text{CI-id}_R M < \infty$ and $\text{fd}_R M < \infty$.

Proof. i) \Rightarrow iii) and i) \Rightarrow iv) Set $M := R$.

ii) \Leftrightarrow iii) For any R -module M , by [20, Remark 2.5] one has that $\text{CI-pd}_R M < \infty$ if and only if $\text{CI-fd}_R M < \infty$.

iii) \Rightarrow i) Since $\text{CI-fd}_R M < \infty$, [14, Theorem 4.5] yields that $\text{CI-fd}_R M = G\text{fd}_R M$ and so, the assertion follows from [10, Corollary 3.3].

iv) \Rightarrow i) By the definition of $\text{CI-id}_R M$, there exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\text{id}_Q(M \otimes_R R') < \infty$ and so $\text{Gid}_Q(M \otimes_R R') < \infty$. Assume that $x = x_1, x_2, \dots, x_n$ is a Q -regular sequence such that $R' \cong Q/xQ$. Then $\text{Gid}_Q(M \otimes_R R') = \text{Gid}_{R'}(M \otimes_R R') + n$ by [6, Theorem 4.2], and so we have $\text{Gid}_{R'}(M \otimes_R R') < \infty$. Since $R \rightarrow R'$ is a flat extension and $\text{depth}_R M < \infty$, we deduce that $\text{depth}_{R'}(M \otimes_R R') < \infty$ by [11, Corollary 2.6]. Also, we have $\text{fd}_{R'}(M \otimes_R R') < \infty$. Hence, [10, Corollary 3.3] yields that R' is Gorenstein, and so R is Gorenstein by [5, Corollary 3.3.15]. □

Finally, we characterize Cohen-Macaulay local rings with complete intersection homological dimensions which also gives positive answers to questions 1.1 and 1.2. Before doing that, we mention the following theorem which is obtained by virtue of the (Peskin-Szpiro) intersection theorem. The implication $iii) \Rightarrow i)$ in Theorem 2.6, is called Bass'conjecture which was proven by Peskin-Szpiro [12, Theorem 5.1] by using the theorem of intersection [12, Theorem 2.1] for any Noetherian local ring of characteristic $p > 0$. After some years, Roberts [13] proved that the theorem of intersection holds for every Noetherian local ring and he called it "New intersection theorem", see [13, Theorem 13.4.1].

Theorem 2.6. *Let (R, \mathfrak{m}, k) be a local ring. The following are equivalent:*

- i) R is Cohen-Macaulay,
- ii) There exists a nonzero finite length R -module of finite projective dimension,
- iii) There exists a nonzero finitely generated R -module of finite injective dimension.

Proof. $i) \Rightarrow ii)$ Assume that $n := \dim R$. Since R is Cohen-Macaulay, there is a maximal R -regular sequence such as $x = x_1, x_2, \dots, x_n$. Then $R/(x)$ is a nonzero finite length R -module and by [5, Exercise 1.3.6], its projective dimension is finite.

$ii) \Rightarrow iii)$ Assume that M is a nonzero finite length R -module of finite projective dimension. Set $N := \text{Hom}_R(M, E_R(k))$, where $E_R(k)$ is an injective envelope of k . By [5, Proposition 3.2.12 b)], N is a finite length R -module. Also, it is easy to see that for any projective R -module Q , R -module $\text{Hom}_R(Q, E_R(k))$ is an injective R -module. By using this and our assumption, we conclude that injective dimension of N is finite.

$iii) \Rightarrow i)$ The same proof of [12, Theorem 5.1] works by using [13, Theorem 13.4.1] instead of the theorem of intersection [12, Theorem 2.1]. \square

Theorem 2.7. *Let (R, \mathfrak{m}, k) be a local ring. The following are equivalent:*

- i) R is Cohen-Macaulay,
- ii) There exists a nonzero Cohen-Macaulay R -module M such that $\text{CI-dim}_R M < \infty$,
- iii) There exists a nonzero finitely generated R -module M such that $\text{CI-id}_R M < \infty$.

Proof. $i) \Rightarrow ii)$ Set $M := R$.

$ii) \Rightarrow i)$ By [20, Remark 2.5 and Theorem 3.2], there exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that Q is complete, the closed fiber $R'/\mathfrak{m}R'$ is Artinian and Gorenstein and $\text{fd}_Q(M \otimes_R R')$ is finite. So $\text{pd}_Q(M \otimes_R R')$ is finite, too. [5, Theorem 2.1.7] yields that $M \otimes_R R'$ is a Cohen-Macaulay R' -module. Since $\text{depth}_Q(M \otimes_R R') = \text{depth}_{R'}(M \otimes_R R')$ by [5, Exercise 1.2.26 b)] and $\dim_Q(M \otimes_R R') = \dim_{R'}(M \otimes_R R')$, then $M \otimes_R R'$ is a Cohen-Macaulay Q -module. Hence, Q is a Cohen-Macaulay ring, and so by [5, Theorem 2.1.3 a)], R' is a Cohen-Macaulay ring which implies that R should be a Cohen-Macaulay ring by [5, Exercise 2.1.23].

$i) \Rightarrow iii)$ The assertion follows from Theorem 2.6, $i) \Rightarrow iii)$. Note that $\text{CI-id}_R M \leq \text{id}_R M$ for any R -module M .

$iii) \Rightarrow i)$ Since $\text{CI-id}_R M$ is finite, there exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that the finitely generated Q -module $M \otimes_R R'$ has finite injective dimension. Hence, Q is a Cohen-Macaulay ring by Theorem 2.6 and so R' is a Cohen-Macaulay ring by [5, Theorem 2.1.3 a)] which implies that R should be a Cohen-Macaulay ring by [5, Exercise 2.1.23]. \square

Acknowledgment. I would like to thank the referee for his/her detailed reading of this paper and thoughtful suggestions.

References

[1] M. Auslander and D.A. Buchsbaum, *Homological dimension in Noetherian rings*, Proc. Nat. Acad. Sci. U.S.A., **42**, 1956.

- [2] M. Auslander and M. Bridger, *Stable module theory*, American Mathematical Society, Providence, R.L., **94**, 1969.
- [3] L.L. Avramov, V.N. Gasharov and I.V. Peeva, *Complete intersection dimension*, Inst. Hautes Etudes Sci. **86**, 67–114, 1997.
- [4] N. Bourbaki, *Commutative algebra*, Chapter 1-7, Springer-Verlag, Berlin, 1998.
- [5] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, in: Cambridge Studies in Advanced Math. **39**, 1993.
- [6] D. Bennis and N. Mahdou, *First, second, and third change of rings theorems for Gorenstein Homological dimensions*, Comm. Algebra, **38** (10), 3837–3850, 2010.
- [7] L.W. Christensen, *Gorenstein dimensions*, in: Lecture Notes in Mathematics, **1747**, Springer-Verlag, Berlin, 2000.
- [8] K. Divaani-Aazar, F. Mohammadi Aghjeh Mashhad and M. Tousi, *On the existence of certain modules of finite Gorenstein homological dimensions*, Comm. Algebra, **42**, 1630–1643, 2014.
- [9] H-B. Foxby and A. Frankild, *Cyclic modules of finite Gorenstein injective dimension and Gorenstein rings*, Illinois J. Math. **51** (1), 67–82, 2007.
- [10] H. Holm, *Rings with finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **132**, 1279–1283, 2004.
- [11] S. Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230**, 545–567, 1999.
- [12] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Publ. Math. Inst. Hautes Études Sci., **42**, 47–119, 1973.
- [13] P. Roberts, *Multiplicities and chern classes in local algebra*, in: Cambridge Tracts in Math. **133**, Cambridge University Press, Cambridge, 1998.
- [14] P. Sahandi, T. Sharif and S. Yassemi, *Homological flat dimensions*, arXiv:0709.4078v2.
- [15] J.P. Serre, *Sur la dimension homologique des anneaux et des modules Noetheriens*, Proc. Intern. Symp., Tokyo-Nikko, 1955, Science Council of Japan, 175–189, 1956.
- [16] R. Takahashi, *Some characterizations of Gorenstein local rings in terms of G-dimension*, Acta Math. Hungar. **104** (4), 315–322, 2004.
- [17] R. Takahashi, *The existence of finitely generated modules of finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **134** (11), 3115–3121, 2006.
- [18] R. Takahashi, *On G-regular local rings*, Comm. Algebra, **36** (12), 4472–4491, 2008.
- [19] S.S. Wagstaff, *Complete intersection dimensions for complexes*, J. Pure Appl. Algebra, **190**, 267–290, 2004.
- [20] S.S. Wagstaff, *Complete intersection dimensions and Foxby classes*, J. Pure Appl. Algebra, **212**, 2594–2611, 2008.
- [21] S. Yassemi, *A generalization of a theorem of Bass*, Comm. Algebra, **35**, 249–251, 2007.