SOME INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS

DİFERANSİYELLENEBİLEN KONVEKS FONKSİYONLAR İÇİN BAZI EŞİTSİZLİKLER

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ABSTRACT:

In the present paper, we establish some new inequalities for differentiable convex functions by using a fairly elementary analysis.

Key words. Convexity, differentiable functions, inequalities, special means.

ÖZET:

Bu makalede biz temel analiz işlemlerini kullanarak diferansiyellenebilen konveks fonksiyonlar için bazı yeni eşitsizlikler kurduk.

Key words. Konvekslik, diferansiyellenebilen fonksiyonlar, eşitsizlikler, özel anlamlar.

1. INTRODUCTION

In this paper, we obtain some theorems; in Theorem 1, we shall offer a new integral inequality for products of differentiable convex functions; in Theorem2, we obtain a new inequality which is connected with the Euler-β function and L_s mean. Finally, an application to special means of real numbers is given.

The following definitions are well known in literature.

Definition 1 [see, Mitrinonović and Vasić, (1970)]. A function f is called convex on a segment I if and only if

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$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y)$ holds for all $x,y\in \bar{I}$ and all real numbers $\lambda\in[0,1]$.

The following inequalities are well known in the literature.

A differentiable function $f: I\to R$ is convex if and only if

$$f(y)\ge f(x)+f'(x)(y-x)$$

for all x and y in **dom** f. Also $f: R\to R$ is convex for $a,b\in \boldsymbol{domf}$ with $a<b$, then we have

$$f(x)\le \frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)$$

for all $x\in[a,b]$.

Dragomir and Pearce proved that the following inequality hold for differentiable functions (Dragomir and Pearce, 2000).

Theorem A. Let $f: R\to R$ be a differentiable mapping on I°, $a,b\in I^{\circ}$, with $a<b$ and $p>1$. If $|f'|$ is q-integrable on $[a,b]$ where $q=\frac{p}{p-1}$, then we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le \frac{1}{2}\frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^b|f'(x)|^q dx\right)^{\frac{1}{q}} \tag{1.1}$$

We also will need the following usual definition:

$$L_p(a,b):=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & if \quad b\ne a \ ,p\in R\backslash\{-1,0\} \\ a \ , & if \quad b=a \ ,\end{cases} \quad a,b>0,$$

and

$$\beta(x,y):=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ \ x>0,y>0 \tag{1.2}$$

see (Dragomir et.al., 2000, p.108).

The main purpose of this note is to establish new inequalities for differentiable convex functions.

2. MAIN RESULTS

We start with the following theorem.

Theorem 2.1. Let $f, g: R \to R_+$ be differentiable convex functions. If $h(x) = f(x)g(x)$ and $x < y < a < b$, then

$$\int_a^b h'(x)dx \leq \left[1 + \frac{(y-b)}{(b-a)} ln \frac{(y-b)}{(y-a)}\right]\mu - \left[1 + \frac{(y-a)}{(b-a)} ln \frac{(y-b)}{(y-a)}\right]\eta \qquad (2.1)$$

where $\mu = f(y)g(a) + g(y)f(a)$ and $\eta = f(y)g(b) + g(y)f(b)$.

Proof. Since f and g are differentiable convex function we have

$$f'(x) \leq \frac{f(y)-f(x)}{y-x} \text{ and } g'(x) \leq \frac{g(y)-g(x)}{y-x} \qquad (2.2)$$

If $h(x) = f(x)g(x)$, using (2.2) we obtain

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

$$\leq \frac{f(y)-f(x)}{y-x} g(x) + \frac{g(y)-g(x)}{y-x} f(x)$$

$$= f(y)\frac{g(x)}{y-x} - \frac{f(x)g(x)}{y-x} + g(y)\frac{f(x)}{y-x} - \frac{f(x)g(x)}{y-x}$$

$$= f(y)\frac{g(x)}{y-x} + g(y)\frac{f(x)}{y-x} - 2\frac{f(x)g(x)}{y-x} \qquad (2.3)$$

On the other hand, from convexity of f and g, we have

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \quad \text{and}$$

$$g(x) \leq \frac{b-x}{b-a} g(a) + \frac{x-a}{b-a} g(b) \qquad (2.4)$$

From (2.3) and (2.4) we obtain

$$h'(x) \leq f(y)\frac{g(x)}{y-x} + g(y)\frac{f(x)}{y-x} - 2\frac{f(x)g(x)}{y-x}$$

$$\leq f(y)\frac{1}{y-x}\left[\frac{b-x}{b-a}g(a)+\frac{x-a}{b-a}g(b)\right]$$

$$+g(y)\frac{1}{y-x}\left[\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)\right]-2\frac{f(x)g(x)}{y-x}$$

Since $2\frac{f(x)g(x)}{y-x}\geq 0$, we get

$$h'(x)\leq f(y)\frac{1}{y-x}\left[\frac{b-x}{b-a}g(a)+\frac{x-a}{b-a}g(b)\right]$$

$$+g(y)\frac{1}{y-x}\left[\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)\right]$$

Integrating the resulting inequality with respect to x over $[a,b]$, we obtain

$$\int_a^b h'(x)dx\leq f(y)\frac{g(a)}{b-a}\int_a^b\frac{b-x}{y-x}dx+f(y)\frac{g(b)}{b-a}\int_a^b\frac{x-a}{y-x}dx$$

$$+g(y)\frac{f(a)}{b-a}\int_a^b\frac{b-x}{y-x}dx+g(y)\frac{f(b)}{b-a}\int_a^b\frac{x-a}{y-x}dx$$

$$=f(y)\frac{g(a)}{b-a}\left[(b-a)+(y-b)ln\frac{(y-b)}{(y-a)}\right]$$

$$+f(y)\frac{g(b)}{b-a}\left[(a-b)+(a-y)ln\frac{(y-b)}{(y-a)}\right]$$

$$+g(y)\frac{f(a)}{b-a}\left[(b-a)+(y-b)ln\frac{(y-b)}{(y-a)}\right]$$

$$+g(y)\frac{f(b)}{b-a}\left[(a-b)+(a-y)ln\frac{(y-b)}{(y-a)}\right]$$

which completes the proof.

The second result is embodied in the following theorem.

Theorem 2. 2. Let f be differentiable mapping on the interval of real numbers I° (the interior of I) $a<b$ and $a,b\in I^{\circ}$, with $f'(a)\neq f'(b)$

and $s > 1$. If $f^{'}:[a,b] \to R_+$ is convex functions and $s-$integrable on $[a,b]$ where $r = \frac{s}{s-1}$, then we have the inequality:

$$\int_a^b (x-a)^p (b-x)^q f^{'}(x)dx \le (b-a)^{p+q+1} [\beta(pr+1, qr+1)]^{\frac{1}{r}} L_s\left(f^{'}(a), f^{'}(b)\right) \quad (2.5)$$

where $p,q \ge 0$, $L_s(.,.)$ is $s-$logarithmic mean and β is beta function of Euler type.

Proof. Using Hölder's integral inequality we get

$$\int_a^b (x-a)^p (b-x)^q f^{'}(x)dx \le \left(\int_a^b [(x-a)^p (b-x)^q]^r dx\right)^{\frac{1}{r}} \left(\int_a^b [f^{'}(x)]^s dx\right)^{\frac{1}{s}} \quad (2.6)$$

Let us put $x = tb + (1-t)a$ with $t \in [0,1]$ on the right side of (2.6). Then we get

$$\int_a^b (x-a)^p (b-x)^q f^{'}(x)dx \le \left[\int_0^1 [t^p (b-a)^p (1-t)^q (b-a)^q]^r (b-a)dt\right]^{\frac{1}{r}}$$

$$\times (b-a)^{\frac{1}{s}} \left(\int_0^1 [f^{'}(tb+(1-t)a)]^s dt\right)^{\frac{1}{s}}$$

$$= (b-a)^{p+q+\frac{1}{r}+\frac{1}{s}} \left(\int_0^1 [t^p (1-t)^q]^r dt\right)^{\frac{1}{r}}$$

$$\times \left(\int_0^1 [f^{'}(tb+(1-t)a)]^s dt\right)^{\frac{1}{s}}$$

$$= (b-a)^{p+q+1} \left(\int_0^1 [t^p (1-t)^q]^r dt\right)^{\frac{1}{r}}$$

$$\times \left(\int_0^1 [f^{'}(tb+(1-t)a)]^s dt\right)^{\frac{1}{s}} \quad (2.7)$$

Since f' is convex function we have that:

$$f'(tb+(1-t)a) \le tf'(b)+(1-t)f'(a) \tag{2.8}$$

Using (2.8) on the right side of (2.7), we have

$$\int_a^b (x-a)^p(b-x)^q f'(x)dx \le (b-a)^{p+q+1}\left(\int_0^1 [t^p(1-t)^q]^r dt\right)^{\frac{1}{r}}$$

$$\times\left(\left(\int_0^1 [tf'(b)+(1-t)f'(a)]^s dt\right)^{\frac{1}{s}}\right) \tag{2.9}$$

However, from (1.4) we get

$$\int_0^1 [t^p(1-t)^q]^r dt = \int_0^1 t^{pr}(1-t)^{qr} dt = \beta(pr+1, qr+1) \tag{2.10}$$

And detoning $u := tf'(b)+(1-t)f'(a),\ t\in[0,1]$, we also get that:

$$\int_0^1 [tf'(b)+(1-t)f'(a)]^s dt = L_s^s(f'(a), f'(b)) \tag{2.11}$$

Combining the inequalities (2.9), (2.10) and (2.11) we get the required inequality in (2.1).

The following theorem is a result of a special condition of (1.1) proven by using the Buniakowski-Schwarz inequality (Mitrinonović and Vasić, 1970, p.43).

Theorem 2.3. Let $f: R \to R$ be a differentiable mapping on I°, $a, b \in I^\circ$; with $a < b$. If f' is integrable on $[a,b]$, then we have the inequality:

$$\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right]^2 \le \frac{b-a}{12}\int_a^b [f'(x)]^2 dx \tag{2.12}$$

Proof. Using Buniakowski-Schwarz inequality, we can state that:

$$\left[\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)dx\right]^2 \le \left[\int_a^b \left(x-\frac{a+b}{2}\right)^2 dx\right]\left[\int_a^b \left(f'(x)\right)^2 dx\right] \tag{2.13}$$

However,

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2 dx = \frac{(b-a)^3}{12} \tag{2.14}$$

and

$$\int_a^b \left(x-\frac{a+b}{2}\right) f'(x)dx = (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(x)dx \tag{2.15}$$

Thus, using (2.13) and (2.14) on the (2.12) we get the required inequality in (2.12).The proof is complete.

Let us put p=2 and $q = 2$ on (1.1) we obtain (2.12).

3. APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means:

a)The arithmetic mean

$$A(a,b) := \frac{a+b}{2}; \qquad a,b > 0,$$

b)The geometric mean

$$G(a,b) := \sqrt{ab}; \qquad a,b > 0,$$

c)The harmonic mean

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; \qquad a,b > 0,$$

d)The logarithmic mean

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & if \quad b \neq a \\ a & if \quad b = a \end{cases}; \qquad a,b > 0,$$

e)The p-logarithmic mean

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & if \quad b \neq a, p \in R\backslash\{-1,0\} \\ a & if \quad b = a \end{cases}; a,b > 0,$$

See (Dragomir and Wang, 1998).

The following proposition holds:

Proposition 3.1. Let $0 < a < b < \infty$ and $p > 1$. Then we have the inequality:

$$\left[A(a^p, b^p) - L_p^p(a,b)\right]^2 \leq \frac{p^2(b-a)^2}{12} L_{2p-2}^{2p-2}(a,b) \tag{3.1}$$

The proof follows by Theorem 2.3 on choosing $f:[a,b] \to (0,\infty),\ f(x) = x^p$ and we omit the details.

Proposition 3.2. Let $0 < a < b$. Then we have the inequality:

$$0 \leq [H^{-1}(a,b) - L^{-1}(a,b)]^2 \leq \left(\frac{a-b}{2}\right)^2 \left(\frac{4G^2H^{-2}-1}{G^4}\right) \tag{3.2}$$

The proof follows by Theorem 2.3 on choosing $f:[a,b] \to (0,\infty),\ f(x) = \frac{1}{x}$ and the details are omitted.

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