Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 221 – 235
DOI : 10.15672/hujms.546981

Research Article

Computational results and analysis for a class of linear and nonlinear singularly perturbed convection delay problems on Shishkin mesh

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Abstract

This article presents a hybrid numerical scheme for a class of linear and nonlinear singularly perturbed convection delay problems on piecewise uniform. The proposed hybrid numerical scheme comprises with the tension spline scheme in the boundary layer region and the midpoint approximation in the outer region on piecewise uniform mesh. Error analysis of the proposed scheme is discussed and is shown ε-uniformly convergent. Numerical experiments for linear and nonlinear are performed to confirm the theoretical analysis.

Mathematics Subject Classification (2010). 65L10, 65L11

Keywords. nonlinear delay problems, differential-difference equations, singular perturbation problems, tension spline, Shishkin mesh

1. Introduction

Consider a class of singularly perturbed convection delay problem of the form:

$$\begin{cases} L_\varepsilon \equiv \varepsilon u''(x) + a(x)u'(x-\delta) + b(x)u(x) = f(x), \\ u(x) = \phi(x), \ -\delta \leqslant x \leqslant 0, \quad u(1) = \gamma \ \text{ on } \ \Omega = (0,1), \end{cases} \tag{1.1}$$

where δ is a small delay parameter and $\varepsilon (0 < \varepsilon \ll 1)$ is the singular perturbation parameter. Here $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are sufficiently smooth functions and γ is a constant. It is assumed that the solution $u(x)$ of the problem (1.1) is continuous on $[0,1]$ and differentiable on $(0,1)$. The corresponding problem has solution with a layer on the right side for $a(x) < 0$ and the left side for $a(x) > 0$ on $[0,\ 1]$ when $\delta = 0$. Singularly perturbed differential-difference equations arise in reaction-diffusion equations [3], thermo-elasticity [5], hydrodynamics of liquid helium [7], second-sound theory [8], diffusion in polymers [13], a variety of model for physical processes or diseases [12, 14] etc.

In general, the solution of the singularly perturbed delay problems exhibits boundary layers and has rapid changes in the boundary layer region and behaves evenly in the outer region. The classical methods fail to prevent the rapid change in the boundary layer region. Therefore, a special deliberation is required to construct an appropriate numerical methods for these problems whose accuracy does not depend on singular perturbation

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Received: 28.05.2018; Accepted: 25.10.2018

parameter (ε), *i.e.*, the methods that are ε-uniformly convergent [4,6,22]. A hybrid method with Shishkin mesh for solving singularly perturbed delay differential equations in [10]. Mohapatra and Natesan [16] constructed an adaptive grid method for singularly perturbed delay differential equation. Boundary value problems for linear second order singularly perturbed differential-difference equations of convection-diffusion type with a small shift in the convection term is considered in [17]. Sharma and Kadalbajoo [11] employed a parameter uniform numerical difference scheme based on finite difference. Ravi and Murali [18] studied an exponentially fitted spline method for solving singularly perturbed delay differential equations. A numerical technique is presented for solving nonlinear singularly perturbed delay differential equations in [20]. Kadalbajoo and Kumar [9] solved singularly perturbed nonlinear differential difference equations with negative shift using B'spline collocation method. A numerical treatment for a singularly perturbed convection delayed dominated diffusion equation via tension splines are presented in [19]. Recently, Ravi and Murali [21] proposed parametric cubic spline method for solving nonlinear singularly perturbed delay equations.

This article is organized in the following manner: In Sect. 2, recalls the continuous problem. Description of the method is given Sect. 3. Error analysis for the proposed method is discussed in Sect. 4. In Sect. 5, the non-linear problem is discussed. Computational results and discussion are discussed in Sect. 6. Finally, Sect. 7 ends with the conclusion.

2. Continuous problem

Equation (1.1) becomes after using Taylor's expansion for small delayed convection term

$$L_\varepsilon \equiv \varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \tag{2.1}$$

where

$$p(x) = \frac{a(x)}{1-\frac{\delta}{\varepsilon}a(x)}\ , \quad q(x) = \frac{b(x)}{1-\frac{\delta}{\varepsilon}a(x)} \ \text{and} \ \ r(x) = \frac{f(x)}{1-\frac{\delta}{\varepsilon}a(x)},$$

with

$$u(x) = \phi(0) = \phi_0, \quad u(1) = \gamma. \tag{2.2}$$

It is assumed that $p(x) \geqslant M > 0$ and $q(x) \leqslant -\theta < 0$, $\theta > 0$.

Lemma 2.1. *Assume that $\Theta(x)$ is any sufficiently smooth function satisfying $\Theta(0) \geqslant 0$ and $\Theta(1) \geqslant 0$. Then $L_\varepsilon\Theta(x) \leqslant 0$, $\forall\ x \in$ (0,1) implies $\Theta(x) \geqslant 0$, $\forall\ x \in [0,1]$.*

Proof. Let $x^* \in \overline{\Omega}$ be such that $\Theta(x^*) < 0$ and $\Theta(x^*) = \min_{x\in\overline{\Omega}} \Theta(x)$. Clearly $x^* \notin \{0,1\}$, therefore $\Theta'(x^*) = 0$ and $\Theta''(x^*) \geqslant 0$.
Therefore, we obtain

$$L_\varepsilon\Theta(x^*) = \varepsilon\Theta''(x^*) + p(x^*)\Theta'(x^*) + q(x^*)\Theta(x^*) > 0,$$

which is contradiction. Hence it is proved that $\Theta(x^*) \geqslant 0$ and thus $\Theta(x) \geqslant 0\ \forall\ x \in \overline{\Omega}$. □

Lemma 2.2. *Let $u(x)$ be the solution of the problem* (2.1) *and* (2.2) *then*

$$\|u\| \leqslant \|r\|\,\theta^{-1} + max\left(|\phi_0|\,,|\gamma|\right), \tag{2.3}$$

Proof. Let $\Theta^{\pm}(x)$ be two barrier functions defined by

$$\Theta^{\pm}(x) \quad = \quad \|r\|\,\theta^{-1} + \max\left(|\phi_0|\,,|\gamma|\right) \pm u(x).$$

Then this implies,

$$\begin{aligned}
\Theta^{\pm}(0) &= \|r\|\,\theta^{-1}+\max\left(|\phi_0|\,,|\gamma|\right)\pm u(0),\\
&= \|r\|\,\theta^{-1}+\max\left(|\phi_0|\,,|\gamma|\right)\pm \phi_0,\\
\Theta^{\pm}(0) &\geqslant 0,\\
\Theta^{\pm}(1) &= \|r\|\,\theta^{-1}+\max\left(|\phi_0|\,,|\gamma|\right)\pm u(1),\\
&= \|r\|\,\theta^{-1}+\max\left(|\phi_0|\,,|\gamma|\right)\pm \gamma,\\
\Theta^{\pm}(1) &\geqslant 0.
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
L_\varepsilon\Theta^{\pm}(x) &= \varepsilon(\Theta^{\pm}(x))''+p(x)(\Theta^{\pm}(x))'+q(x)\Theta^{\pm}(x),\\
&= q(x)\left\{\|r\|\,\theta^{-1}+\max(|\phi_0|\,,|\gamma|)\right\}\pm\ L_\varepsilon u(x),\\
&= q(x)\left\{\|r\|\,\theta^{-1}+\max(|\phi_0|\,,|\gamma|)\right\}\pm\ r(x).
\end{aligned}$$

Since $\|r\|\geqslant r(x)$, $q(x)\theta^{-1}\leqslant -1$ then $L_\varepsilon\Theta\leqslant 0$,
by using Lemma 2.1, we obtain $\Theta^{\pm}(x)\geqslant 0\ \forall\ x\in[0,1]$. □

Lemma 2.3. *Let $u(x)$ be the solution of (2.1) and (2.2), then*

$$\left|u^{(i)}(x)\right|\leqslant K\left[1+\varepsilon^{-i}exp\left(\frac{-Mx}{\varepsilon}\right)\right]\quad\text{for}\quad i=0,1,2,3.$$

Proof. For the proof of the lemma can refer to [15]. □

3. Description of the method

Let $\overline{\Omega}=[0,1]$ be

$$x_0=0,\ x_i=\sum_{k=0}^{i-1}h_k,\ h_k=x_{k+1}-x_k,\ x_N=1,\ i=1(1)N-1.$$

A function as $S(x,\nu)\in C^2[a,b]$ interpolates $u(x)$ at the mesh points x_i which depends on a parameter ν. When $\nu\to 0$, $S(x,\nu)$ reduces to cubic spline. The spline function $S(x,\nu)=S(x)$ satisfying in $[x_i,x_{i+1}]$, the differential equation

$$S''(x)+\nu S(x)=\left[S''(x_i)+\nu S(x_i)\right]\frac{(x_{i+1}-x)}{h_i}+\left[S''(x_{i+1})+\nu S(x_{i+1})\right]\frac{(x-x_i)}{h_i}\tag{3.1}$$

where $S(x_i)=u_i$ and $\nu>0$ is termed as tension spline. Following Aziz and Khan [1], we obtain the following system

$$\lambda_1 h_{i-1}M_{i-1}+\lambda_2(h_i+h_{i-1})M_i+\lambda_1 h_i M_{i+1}=\frac{(u_{i+1}-u_i)}{h_i}-\frac{(u_i-u_{i-1})}{h_{i-1}},\quad i=1,2,\cdots N-1\tag{3.2}$$

where

$$\lambda_1=\frac{1}{\lambda^2}\left(1-\lambda\ \operatorname{csc h}\ \lambda\right),\quad \lambda_2=\frac{1}{\lambda^2}\left(\lambda\ \coth\lambda-1\right),$$

and

$$\lambda=h\nu^{1/2},\quad M_j=u''(x_j),\quad j=i,i\pm 1.$$

We have the following condition by equating the coefficients of M_i from (3.2)

$$\lambda_1+\lambda_2=\frac{1}{2}.\tag{3.3}$$

Using (3.3) in (3.3), we get

$$\frac{\lambda}{2} = \tanh\left(\frac{\lambda}{2}\right). \tag{3.4}$$

Equation (3.4) has infinitely many roots and $\lambda = 0.001$ is the smallest positive non-zero root. Equation (2.1) can be written as,

$$\varepsilon M_j = r(x_j) - p(x_j)u'_j - q(x_j)u_j, \quad j = i, i \pm 1. \tag{3.5}$$

We have,

$$u'_i \cong \frac{1}{h_i h_{i-1}(h_{i-1}+h_i)}\left(h_{i-1}^2 u_{i+1} + (h_i^2 - h_{i-1}^2)u_i - h_i^2 u_{i-1}\right), \tag{3.6}$$

$$u'_{i-1} \cong \frac{1}{h_i h_{i-1}(h_{i-1}+h_i)}\left[-h_{i-1}^2 u_{i+1} + (h_{i-1}+h_i)^2 u_i - (h_i^2 + 2h_i h_{i-1})u_{i-1}\right], \tag{3.7}$$

$$u'_{i+1} \cong \frac{1}{h_i h_{i-1}(h_{i-1}+h_i)}\left[(h_{i-1}^2 + 2h_i h_{i-1})u_{i+1} - (h_{i-1}+h_i)^2 u_i + h_i^2 u_{i-1}\right]. \tag{3.8}$$

Using the equations (3.6)-(3.8) and (3.5) in the equation (3.2), we obtain

$$\begin{aligned}
&\left[\frac{\varepsilon}{h_{i-1}(h_i+h_{i-1})} - \lambda_1\frac{(h_i+2h_{i-1})}{(h_i+h_{i-1})^2}p_{i-1} - \lambda_2\frac{h_i}{h_{i-1}(h_i+h_{i-1})}p_i\right.\\
&\quad\left. + \lambda_1\frac{h_i^2}{h_{i-1}(h_i+h_{i-1})^2}p_{i+1} + \lambda_1\frac{h_{i-1}}{(h_i+h_{i-1})}q_{i-1}\right]u_{i-1}\\
&+\left[-\frac{\varepsilon}{h_i h_{i-1}} + \lambda_1\frac{1}{h_i}p_{i-1} + \lambda_2\frac{(h_i-h_{i-1})}{h_i h_{i-1}}p_i - \lambda_1\frac{1}{h_{i-1}}p_{i+1} + \lambda_2 q_i\right]u_i\\
&+\left[\frac{\varepsilon}{h_{i-1}(h_i+h_{i-1})} - \lambda_1\frac{h_{i-1}^2}{(h_i+h_{i-1})^2}p_{i-1} + \lambda_2\frac{h_{i-1}}{h_i(h_i+h_{i-1})}p_i\right.\\
&\quad\left. + \lambda_1\frac{(h_{i-1}+2h_i)}{(h_i+h_{i-1})^2}p_{i+1} + \lambda_1\frac{h_i}{(h_i+h_{i-1})}q_{i+1}\right]u_{i+1}\\
&= \lambda_1\frac{(h_i+h_{i-1})}{(h_i+h_{i-1})}r_{i-1} + \lambda_2 r_i + \lambda_1\frac{h_i}{(h_i+h_{i-1})}r_{i+1}, \; i = 1(1)N-1.
\end{aligned} \tag{3.9}$$

with

$$u_0(0) = \phi_0, \; u_N(1) = \gamma. \tag{3.10}$$

Equation (3.9) with (3.10) gives the system of linear equations.

3.1. Piecewise uniform mesh

In this section, we constructed a piecewise uniform mesh [23] on $\overline{\Omega} = [0,1]$. Since $p(x) > 0$, the problem (2.1) exhibits a boundary layer at $x = 0$. Divide the domain $\overline{\Omega} = [0,1]$ into two subintervals namely $\overline{\Omega}_1 = [0,\tau]$, $\overline{\Omega}_2 = [\tau, 1]$, where $\tau = \min(0.5, c\,\varepsilon\,\ln(N))$, where $c > 0$ is a constant to be chosen so that the resulting numerical scheme is of second order convergent. More precisely, we choose $c \geqslant 2/\sqrt{\theta} > 0$ and N is the number of mesh points. A uniform mesh with $N/2$ mesh intervals placed on $\overline{\Omega}_1$, while on $\overline{\Omega}_2$, a uniform mesh with $N/2$ mesh intervals placed. The mesh size is defined by $h^{(1)} = 2\tau/N$ in $\overline{\Omega}_1$, $h^{(2)} = 2(1-\tau)/N$ in $\overline{\Omega_2}$.

3.2. Hybrid numerical scheme

In this section, a hybrid numerical scheme is proposed for the problem (2.1)-(2.2) on piecewise uniform mesh. This hybrid numerical scheme is a combination of tension spline difference scheme in the boundary layer region and mid-point scheme [24] in the outer region. The resulting scheme yields the following system of equations:

$$L^N \equiv E_i u_{i-1} + F_i u_i + G_i u_{i+1} = H_i,\ i = 1(1)N-1. \tag{3.11}$$

For $i = 1, 2, \cdots, N/2$

$$\begin{aligned}
E_i &= \Big[\frac{\varepsilon}{h_{i-1}(h_i+h_{i-1})} - \lambda_1 \frac{(h_i+2h_{i-1})}{(h_i+h_{i-1})^2} p_{i-1} - \lambda_2 \frac{h_i}{h_{i-1}(h_i+h_{i-1})} p_i \\
&\quad + \lambda_1 \frac{h_i^2}{h_{i-1}(h_i+h_{i-1})^2} p_{i+1} + \lambda_1 \frac{h_{i-1}}{(h_i+h_{i-1})} q_{i-1}\Big], \\
F_i &= \Big[-\frac{\varepsilon}{h_i h_{i-1}} + \lambda_1 \frac{1}{h_i} p_{i-1} + \lambda_2 \frac{(h_i-h_{i-1})}{h_i h_{i-1}} p_i - \lambda_1 \frac{1}{h_{i-1}} p_{i+1} + \lambda_2 q_i\Big], \\
G_i &= \Big[\frac{\varepsilon}{h_i(h_i+h_{i-1})} - \lambda_1 \frac{h_{i-1}^2}{h_i(h_i+h_{i-1})^2} p_{i-1} + \lambda_2 \frac{h_{i-1}}{h_i(h_i+h_{i-1})} p_i \\
&\quad + \lambda_1 \frac{(h_{i-1}+2h_i)}{(h_i+h_{i-1})^2} p_{i+1} + \lambda_1 \frac{h_i}{(h_i+h_{i-1})} q_{i+1}\Big], \\
H_i &= \lambda_1 \frac{h_{i-1}}{(h_i+h_{i-1})} r_{i-1} + \lambda_2 r_i + \lambda_1 \frac{h_i}{(h_i+h_{i-1})}.
\end{aligned}$$

For $i = N/2+1, \cdots, N-1$

$$\begin{aligned}
E_i &= \frac{2\varepsilon}{h_{i-1}(h_i+h_{i-1})}, \\
F_i &= -\frac{2\varepsilon}{h_{i-1}(h_i+h_{i-1})} - \frac{2\varepsilon}{h_i(h_i+h_{i-1})} - \frac{1}{2h_i}(p_i+p_{i+1}) + \frac{1}{2}(q_i+q_{i+1}), \\
G_i &= \frac{2\varepsilon}{h_i(h_i+h_{i-1})} + \frac{1}{2h_i}(p_i+p_{i+1}) + \frac{1}{2}(q_i+q_{i+1}), \\
H_i &= 0.5r_i + 0.5r_{i+1}.
\end{aligned}$$

The above Eq. (3.11) gives $(N-1)$ system of equations with $(N+1)$ unknowns. These $(N-1)$ equations together with (3.10) leads the solution.

4. Error analysis

In this section, we derive the truncation error for the proposed hybrid numerical scheme (3.11). The truncation error for $i = 1, \cdots, N/2$ is given by

$$\begin{aligned}
\mathcal{T}_i &= E_i u(x_{i-1}) + F_i u(x_i) + G_i u(x_{i+1}) \\
&\quad - \Big[\lambda_1 \frac{h_{i-1}}{(h_i+h_{i-1})} r(x_{i-1}) + \lambda_2 r(x_i) + \lambda_1 \frac{h_i}{(h_i+h_{i-1})} r(x_{i+1})\Big].
\end{aligned} \tag{4.1}$$

Using the Eq. (2.1) for $r(x_{i-1}),\ r(x_i)$ and $r(x_{i+1})$ in the above expression, we get

$$\begin{aligned}
\mathcal{T}_i &= E_i u(x_{i-1}) + F_i u(x_i) + G_i u(x_{i+1}) \\
&\quad - \lambda_1 \frac{h_{i-1}}{(h_i+h_{i-1})} \left[\varepsilon u''(x_{i-1}) + p_{i-1} u'(x_{i-1}) + q_{i-1} u(x_{i-1})\right] \\
&\quad - \lambda_2 \left[\varepsilon u''(x_i) + p_i u'(x_i) + q_i u(x_i)\right] \\
&\quad - \lambda_1 \frac{h_i}{(h_i+h_{i-1})} \left[\varepsilon u''(x_{i+1}) + p_{i+1} u'(x_{i+1}) + q_{i+1} u(x_{i+1})\right].
\end{aligned} \tag{4.2}$$

An application of Taylor series expansion for $u(x_{i-1})$ and $u(x_{i+1})$, we have

$$u(x_{i-1}) \cong u(x_i) - h_{i-1}u'(x_i) + \frac{h_{i-1}^2}{2!}u''(x_i) - \frac{h_{i-1}^3}{3!}u^{(iii)}(x_i) + \frac{h_{i-1}^4}{4!}u^{(iv)}(x_i) + \cdots, \tag{4.3}$$

$$u(x_{i+1}) \cong u(x_i) + h_iu'(x_i) + \frac{h_i^2}{2!}u''(x_i) + \frac{h_i^3}{3!}u^{(iii)}(x_i) + \frac{h_i^4}{4!}u^{(iv)}(x_i) + \cdots. \tag{4.4}$$

Using these approximations $u(x_{i-1})$, $u(x_{i+1})$ in Eq. (4.2), we have

$$\mathcal{T}_i = \zeta_{0,i}u(x_i) + \zeta_{1,i}u'(x_i) + \zeta_{2,i}u''(x_i) + \zeta_{3,i}u^{(iii)}(x_i) + \zeta_{4,i}u^{(iv)}(x_i) + \text{h.o.t}, \tag{4.5}$$

where

$$\begin{aligned}
\zeta_{0,i} &= E_i + F_i + G_i - \lambda_1\frac{h_{i-1}}{(h_i+h_{i-1})}q_{i-1} - \lambda_2 q_i - \lambda_1\frac{h_i}{(h_i+h_{i-1})}q_{i+1},\\
\zeta_{1,i} &= -h_{i-1}E_i + h_iG_i - \lambda_1\frac{h_{i-1}}{(h_i+h_{i-1})}p_{i-1} - \lambda_2 p_i - \lambda_1\frac{h_i}{(h_i+h_{i-1})}p_{i+1}\\
&\quad + \lambda_1\frac{h_{i-1}^2}{(h_i+h_{i-1})}q_{i-1} - \lambda_1\frac{h_i^2}{(h_i+h_{i-1})}q_{i+1},\\
\zeta_{2,i} &= -\varepsilon\lambda_1\frac{h_{i-1}}{(h_i+h_{i-1})} - \varepsilon\lambda_2 - \varepsilon\lambda_1\frac{h_i}{(h_i+h_{i-1})}) + \frac{h_{i-1}^2}{2!}E_i + \frac{h_i^2}{2!}G_i\\
&\quad + \lambda_1\frac{h_{i-1}^2}{(h_i+h_{i-1})}p_{i-1} - \lambda_1\frac{h_i^2}{(h_i+h_{i-1})}p_{i+1}\\
&\quad - \lambda_1\frac{h_{i-1}^3}{2!(h_i+h_{i-1})}q_{i-1} - \lambda_1\frac{h_i^3}{2!(h_i+h_{i-1})}q_{i+1},\\
\zeta_{3,i} &= -\frac{h_{i-1}^3}{3!}E_i + \frac{h_i^3}{3!}G_i + \varepsilon\frac{\lambda_1}{(h_i+h_{i-1})}\left(h_{i-1}^2 - h_i^2\right)\\
&\quad - \frac{\lambda_1}{2!(h_i+h_{i-1})}(h_{i-1}^3p_{i-1} + h_i^3p_{i+1}) + \frac{\lambda_1}{3!(h_i+h_{i-1})}(h_{i-1}^4q_{i-1} - h_i^4q_{i+1}),\\
\zeta_{4,i} &= \frac{h_{i-1}^4}{4!}E_i + \frac{h_i^4}{4!}G_i - \varepsilon\lambda_1\frac{(h_{i-1}^3 + h_i^3)}{2!(h_i+h_{i-1})} + \lambda_1\frac{(h_{i-1}^4p_{i-1} + h_i^4p_{i+1})}{3!(h_i+h_{i-1})}\\
&\quad - \frac{\lambda_1}{4!(h_i+h_{i-1})}(h_{i-1}^5q_{i-1} + h_i^5q_{i+1}).
\end{aligned}$$

It can be easily seen that

$$\zeta_{0,i} = \zeta_{1,i} = 0,$$

$$\zeta_{2,i} = \varepsilon\left(\frac{1}{2} - (\lambda_1+\lambda_2)\right),$$

by using (3.3), we obtain

$$\zeta_{2,i} = 0,$$

$$\zeta_{3,i} = 0,$$

and

$$\zeta_{4,i} = \varepsilon\left(\frac{1}{4!} - \frac{\lambda_1}{2!}\right)\frac{(h_i^3 + h_{i-1}^3)}{(h_i+h_{i-1})}.$$

Thus, we have

$$\mathcal{T}_i = \varepsilon\left(\frac{1}{4!} - \frac{\lambda_1}{2!}\right)\frac{(h_i^3 + h_{i-1}^3)}{(h_i+h_{i-1})}u^{(iv)}(x_i) + O(N^{-3}),\ i = 1, \cdots, N/2. \tag{4.6}$$

In a similar manner, the truncation error for $i = N/2+1, \cdots, N-1$ is given by

$$\mathcal{T}_i = \frac{2\varepsilon}{4!}\left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right) u^{(iv)}(x_i) + O(N^{-3}). \tag{4.7}$$

Using the bounds of the solution stated in Lemma 2.3, we have the following theorem.

Theorem 4.1. *Let $u(x)$, $x \in \overline{\Omega}$ be the solution of Eq.* (1.1) *and let $U(x_i)$, $x \in \overline{\Omega}^N$ be the solution of* (3.11) *and* (3.10) *respectively. Then, the local truncation error satisfies the following error estimate:*

$$\sup_{0<\varepsilon\leqslant 1} \|U - u\|_{\Omega^N} \leqslant CN^{-2}\ln^3 N,$$

where C is a constant independent of ε.

Proof. There arise in two cases:

Case 4.2. When $\tau = 1/2$. In this case, the mesh is uniform with spacing $1/N$ and $c\varepsilon \ln N \geqslant 1/2$, this gives $\varepsilon^{-1} \leqslant C\ln N$. Now using the Lemma 2.3 in Eqs. (4.6)-(4.7), we get

$$|\mathcal{T}_i| \leqslant CN^{-2}\varepsilon\left(1 + \varepsilon^{-4} exp\left(-\frac{Mx}{\varepsilon}\right)\right), \text{ for } i = 1, 2, \cdots, N-1.$$

$$i.e., \quad |\mathcal{T}_i| \leqslant CN^{-2}\ln^3 N. \tag{4.8}$$

Case 4.3. When $\tau = c\,\varepsilon\,\ln N$. In this case, the mesh is piece-wise uniform with spacing $2\tau/N$ in $[0, \tau]$ and $2(1-\tau)/N$ in $[\tau, 1]$.
For $N/2 < i \leqslant N$, the region is regular region, *i.e.*, the interval in $[\tau, 1]$, then from Lemma 2.3

$$\left|u^{(k)}(x)\right| \leqslant C.$$

Therefore

$$|\mathcal{T}_i| \leqslant CN^{-2}. \tag{4.9}$$

For $1 \leqslant i \leqslant N/2$, the region is boundary layer region, *i.e.*, the interval in $[0, \tau]$. In this case, we have $h = 2\tau/N = 2(C\varepsilon \ln N)/N$.

$$i.e., \ \frac{h}{\varepsilon} = CN^{-1}\ln N$$

From Eq. (4.6), we get

$$|\mathcal{T}_i| \leqslant CN^{-2}\ln^2 N. \tag{4.10}$$

Combining the Eqs. (4.8) and (4.10), we obtain the required truncation error

$$|\mathcal{T}_i| \leqslant CN^{-2}\ln^3 N. \tag{4.11}$$

□

5. Non-linear problems

We consider a nonlinear singularly perturbed convection delay problem of the form:

$$\varepsilon u'' = F(x, u, u'(x-\delta)), \quad \Omega = (0, 1), \tag{5.1}$$

with

$$u(x) = \phi(x) \ \text{ on } \ -\delta \leqslant x \leqslant 0, \quad u(1) = \gamma, \tag{5.2}$$

where ε is a small singular perturbation parameter, $0 < \varepsilon \ll 1$ and δ is the delay parameter of $o(\varepsilon)$. It is assumed that the solution $u(x)$ of the problem (5.1)-(5.2) is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Also assume that $F(x, u, u'(x-\delta))$ as $F(x, u, z)$ is a smooth function satisfying

i. $F_u(x,u,z) \geqslant 0$,
ii. $F_z(x,u,z) \leqslant 0$,
iii. $(F_u - F_z)(x,u,z) \geqslant \alpha > 0, \alpha$ is a positive constant,
iv. The growth condition $F(x,u,z) = O(|z|^2)$ as $z \to \infty\ \forall\ x \in [0,1]$ and all real u and z.

For $\delta = 0$, under these conditions the problem (5.1)-(5.2) has a unique solution (see,[4]). In order to obtain a numerical solution for (5.1)-(5.2), firstly quasilinearization method [2] is used and then the problem is linearized. Consequently, we get a sequence $\left\{u^{(k)}\right\}_0^\infty$ of successive approximations with a proper choice of initial guess $u^{(0)}$. In fact, we define $u^{(k+1)}$, for each fixed non-negative integer k, to the solution of the following linear problem.

$$\varepsilon u''^{(k+1)}(x) + a^{(k)}(x)u'^{(k+1)}(x-\delta) + b^{(k)}(x)u^{(k+1)}(x) = c^{(k)}(x),\ k = 0,1,\cdots \tag{5.3}$$

where

$$a^{(k)}(x) = -\frac{\partial F^{(k)}}{\partial u'},$$
$$b^{(k)}(x) = -\frac{\partial F^{(k)}}{\partial u},$$

and
$$c^{(k)}(x) = F^{(k)} - u^{(k)}\frac{\partial F^{(k)}}{\partial u} - u'^{(k)}(x-\delta)\frac{\partial F^{(k)}}{\partial u'},$$

with

$$u^{(k+1)}(x) = \phi(x) \quad \text{on} \quad -\delta \leqslant x \leqslant 0, \quad u^{(k+1)}(1) = \gamma. \tag{5.4}$$

The problem (5.3)-(5.4), for each fixed k, is a linear singularly perturbed convection delay problem and is solved by the hybrid numerical scheme on piecewise uniform mesh as discussed in Sect. 3.

6. Computational results with discussion

In this section, we have solved linear and nonlinear problems to validate the efficiency and applicability of the proposed method. The exact solution for the following problems are not known so we use the double mesh principle [4] for the maximum point-wise error and rate of convergence are defined as

$$E_\varepsilon^N = \max_{0\leqslant i\leqslant N}\left|u_i^N - u_i^{2N}\right| \quad \text{and} \quad r_\varepsilon^N = \log_2\left(E_\varepsilon^N / E_\varepsilon^{2N}\right).$$

We compute the uniform error and rate of convergence as

$$E^N = \max_\varepsilon E_\varepsilon^N \quad \text{and} \quad r^N = \log_2\left(\frac{E^N}{E^{2N}}\right).$$

Example 6.1. (Variable Coefficient)

$$\begin{cases} \varepsilon u''(x) + exp(-x)u'(x-\delta) - u(x) = 0, \\ u(x) = 1,\ -\delta \leqslant x \leqslant 0,\quad u(1) = 1. \end{cases}$$

Table 1 presents the maximum point-wise error, rate of convergence and uniform error for different choices of ε and N. The numerical results clearly indicates that the proposed scheme is ε-uniformly convergent. It is also observed from the table that the maximum absolute error decreases as the mesh size decreases. The maximum point-wise error and rate of convergence for different choices of λ_1 and λ_2 are presented in Table 2. The numerical solution is plotted for different choices of δ in Figs. 1-2 for $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-2}$ respectively. Figure 3 represents the loglog plot of maximum point-wise error.

Table 1. Maximum point-wise error (E_ε^N), rate of convergence (r_ε^N), uniform error (E^N) for Example 6.1 with $\lambda_1 = 1/12,\ \lambda_2 = 5/12,\ \delta = 0.5\varepsilon$.

	Number of mesh points (N)				
ε	32	64	128	256	512
2^{-4}	1.73E-02	5.38E-03	1.45E-03	4.33E-04	1.80E-04
	1.682	1.8956	1.7418	1.2675	1.0006
2^{-8}	1.91E-02	7.26E-03	3.10E-03	1.41E-03	6.68E-04
	1.3981	1.229	1.1305	1.0825	1.0586
2^{-12}	2.06E-02	6.44E-03	2.13E-03	6.93E-04	2.19E-04
	1.6798	1.595	1.6224	1.6633	1.4679
2^{-16}	2.05E-02	6.45E-03	2.13E-03	6.93E-04	2.19E-04
	1.6665	1.595	1.6226	1.6635	1.6973
2^{-20}	2.04E-02	6.45E-03	2.13E-03	6.93E-04	2.19E-04
	1.6655	1.595	1.6226	1.6635	1.6974
2^{-24}	2.04E-02	6.45E-03	2.13E-03	6.93E-04	2.19E-04
	1.6654	1.595	1.6226	1.6635	1.6974
2^{-28}	2.04E-02	6.45E-03	2.13E-03	6.93E-04	2.19E-04
	1.6654	1.595	1.6226	1.6635	1.6974
2^{-32}	2.04E-02	6.45E-03	2.13E-03	6.93E-04	2.19E-04
	1.6654	1.595	1.6226	1.6635	1.6974
E^N	2.06E-02	7.26E-03	3.10E-03	1.41E-03	6.68E-04
r^N	1.5082	1.2290	1.1305	1.0825	1.0586

Table 2. Maximum point-wise error and rate of convergence for Example 6.1 with $\varepsilon = 2^{-5}$ and $\delta = 0.5 \times \varepsilon$.

(λ_1, λ_2)	N				
	32	64	128	256	512
$(1/6, 1/3)$	2.60E-02	8.60E-03	2.91E-03	9.71E-04	3.20E-04
	1.5978	1.5615	1.5852	1.5999	1.6129
$(1/12, 5/12)$	1.91E-02	6.61E-03	2.26E-03	7.77E-04	2.66E-04
	1.5294	1.5482	1.5430	1.5468	1.3627
$(1/18, 4/9)$	2.34E-02	7.80E-03	2.65E-03	8.90E-04	2.95E-04
	1.5851	1.5557	1.5771	1.5914	1.6038
$(1/14, 3/7)$	2.38E-02	7.92E-03	2.69E-03	9.01E-04	2.99E-04
	1.587	1.5565	1.5784	1.5927	1.6052
$(1/30, 14/30)$	2.29E-02	7.64E-03	2.60E-03	8.73E-04	2.90E-04
	1.5824	1.5544	1.5753	1.5896	1.6018
$(1/24, 11/24)$	2.31E-02	7.70E-03	2.62E-03	8.80E-04	2.92E-04
	1.5834	1.5549	1.576	1.5903	1.6026

Example 6.2. (Constant Coefficient)

$$\begin{cases} \varepsilon u''(x) + u'(x-\delta) - u(x) = 0, \\ u(x) = 1,\ -\delta \leqslant x \leqslant 0,\quad u(1) = 1. \end{cases}$$

Table 3 presents the maximum point-wise error, rate of convergence and uniform error for different choices of ε and N. Table 4 shows the comparison between the proposed method and the method in [11]. It has been observed that the proposed method gives more accurate result than the method in [11]. Table 5 indicates maximum point-wise error and rate of convergence for different choices of λ_1 and λ_2. It is observed from the table, the

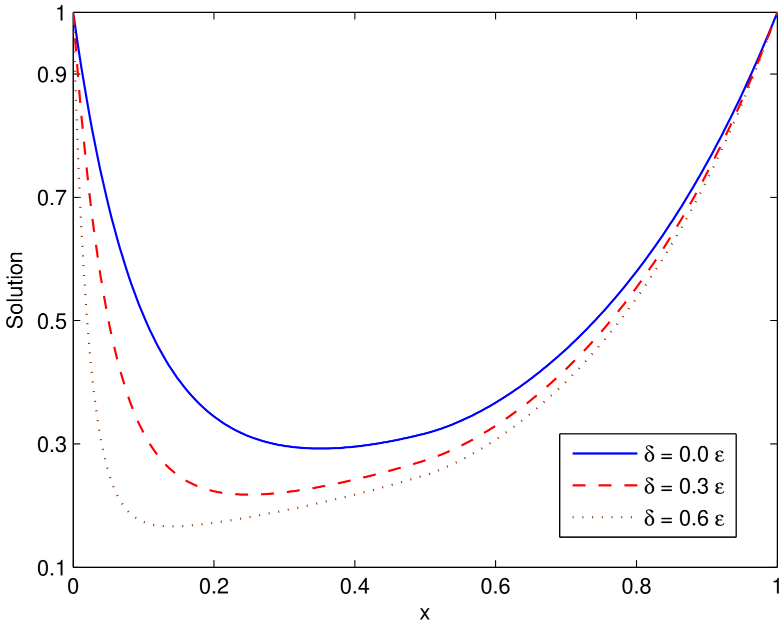

Figure 1. Numerical solution with $\varepsilon = 10^{-1}$ for Example 6.1.

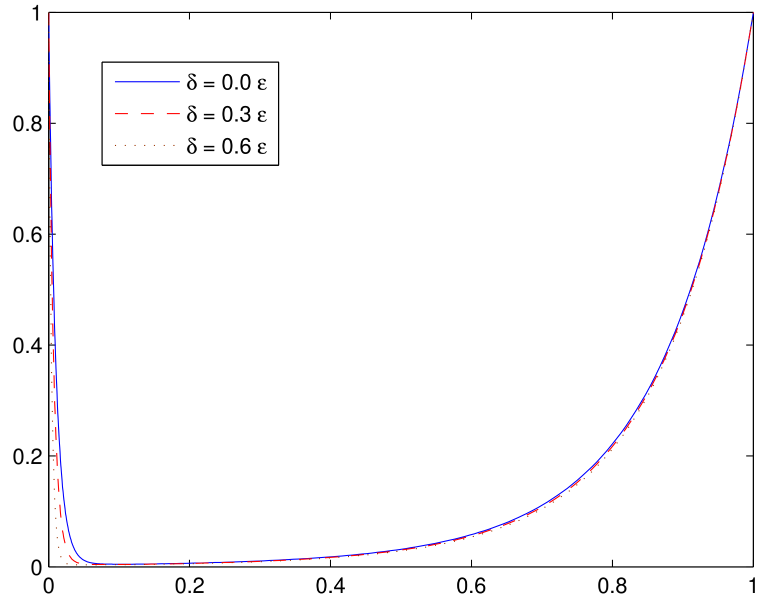

Figure 2. Numerical solution with $\varepsilon = 10^{-2}$ for Example 6.1.

maximum point-wise error decreases as the mesh size decreases and the proposed scheme is ε-uniformly convergent. Figures 4-5 indicate the numerical solution for different values of delay parameter and is observed that the thickness of the boundary layer decreases as delay parameter increases. Figure 6 represents the loglog plot of maximum point-wise error.

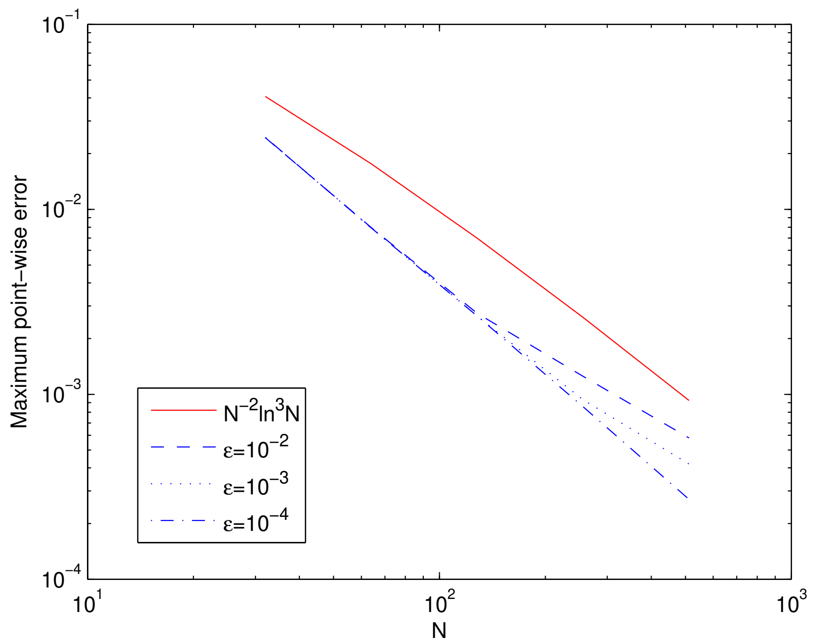

Figure 3. Loglog plots of Maximum point-wise error for Example 6.1.

Table 3. Maximum point-wise error (E_ε^N), rate of convergence (r_ε^N), uniform error (E^N) for Example 6.2 with $\lambda_1 = 1/12,\ \lambda_2 = 5/12,\ \delta = 0.5 \times \varepsilon$.

	Number of mesh points (N)				
ε	32	64	128	256	512
2^{-4}	1.52E-02	4.63E-03	1.16E-03	3.05E-04	8.50E-05
	1.7099	1.9955	1.928	1.8452	1.7182
2^{-8}	1.67E-02	5.59E-03	1.85E-03	6.01E-04	1.90E-04
	1.5753	1.597	1.6205	1.6595	1.6871
2^{-12}	1.67E-02	5.61E-03	1.85E-03	6.02E-04	1.90E-04
	1.5755	1.5982	1.6236	1.6646	1.6975
2^{-16}	1.67E-02	5.61E-03	1.85E-03	6.02E-04	1.90E-04
	1.5755	1.5983	1.6238	1.6649	1.6982
2^{-20}	1.67E-02	5.61E-03	1.85E-03	6.02E-04	1.90E-04
	1.5755	1.5983	1.6238	1.665	1.6982
2^{-24}	1.67E-02	5.61E-03	1.85E-03	6.02E-04	1.90E-04
	1.5755	1.5983	1.6238	1.665	1.6982
2^{-28}	1.67E-02	5.61E-03	1.85E-03	6.02E-04	1.90E-04
	1.5755	1.5983	1.6238	1.665	1.6982
2^{-32}	1.67E-02	5.61E-03	1.85E-03	6.02E-04	1.90E-04
	1.5755	1.5983	1.6238	1.665	1.6982
E^N	1.67E-02	5.61E-03	1.85E-03	6.02E-04	1.90E-04
r^N	1.5755	1.5983	1.6238	1.6605	1.6871

Example 6.3. (Nonlinear Problem)

$$\begin{cases} \varepsilon u''(x) + u(x)u'(x-\delta) - u(x) = 0, \\ u(x) = 1,\ -\delta \leqslant x \leqslant 0, \quad u(1) = 1. \end{cases}$$

Table 4. Comparison of ε-uniform error (E^N) and rate of convergence (r^N) for Example 6.2.

		Number of mesh points (N)			
		64	128	256	512
Fitted mesh method [11]	E^N	3.20E-02	2.06E-02	1.38E-02	1.00E-02
	r^N	0.6354	0.5779	0.4646	-
Proposed method	E^N	9.63E-02	3.10E-02	9.00E-03	2.80E-03
	r^N	1.6338	1.7854	1.6867	-

Table 5. Maximum point-wise error and rate of convergence for Example 6.2 with $\varepsilon = 2^{-5}$ and $\delta = 0.5 \times \varepsilon$.

(λ_1, λ_2)	N				
	32	64	128	256	512
$(1/6, 1/3)$	2.07E-02	6.63E-03	2.21E-03	7.11E-04	2.25E-04
	1.639	1.5873	1.6337	1.6616	1.6855
$(1/12, 5/12)$	1.61E-02	5.38E-03	1.78E-03	5.80E-04	1.85E-04
	1.5810	1.5975	1.6159	1.6496	1.6666
$(1/18, 4/9)$	2.02E-02	6.50E-03	2.16E-03	6.97E-04	2.20E-04
	1.637	1.5868	1.6333	1.6611	1.6849
$(1/14, 3/7)$	2.03E-02	6.52E-03	2.17E-03	6.99E-04	2.21E-04
	1.6373	1.5868	1.6334	1.6612	1.685
$(1/30, 14/30)$	2.01E-02	6.47E-03	2.15E-03	6.94E-04	2.20E-04
	1.6366	1.5866	1.6332	1.661	1.6848
$(1/24, 11/24)$	2.02E-02	6.48E-03	2.16E-03	6.95E-04	2.20E-04
	1.6368	1.5867	1.6332	1.661	1.6848

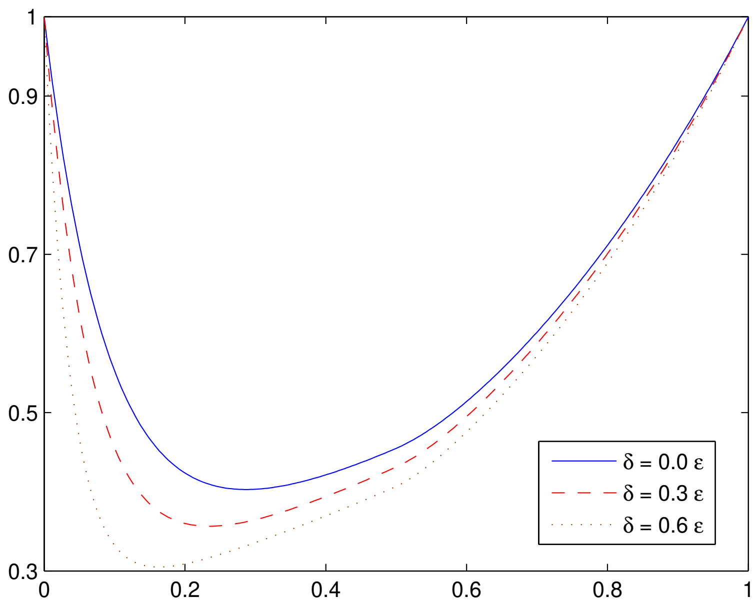

Figure 4. Numerical solution with $\varepsilon = 10^{-1}$ for Example 6.2.

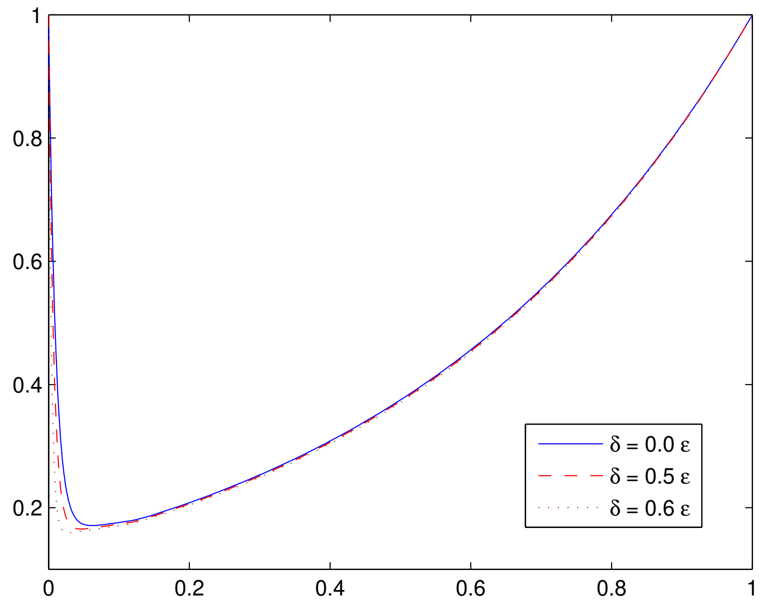

Figure 5. Numerical solution with $\varepsilon = 10^{-2}$ for Example 6.2.

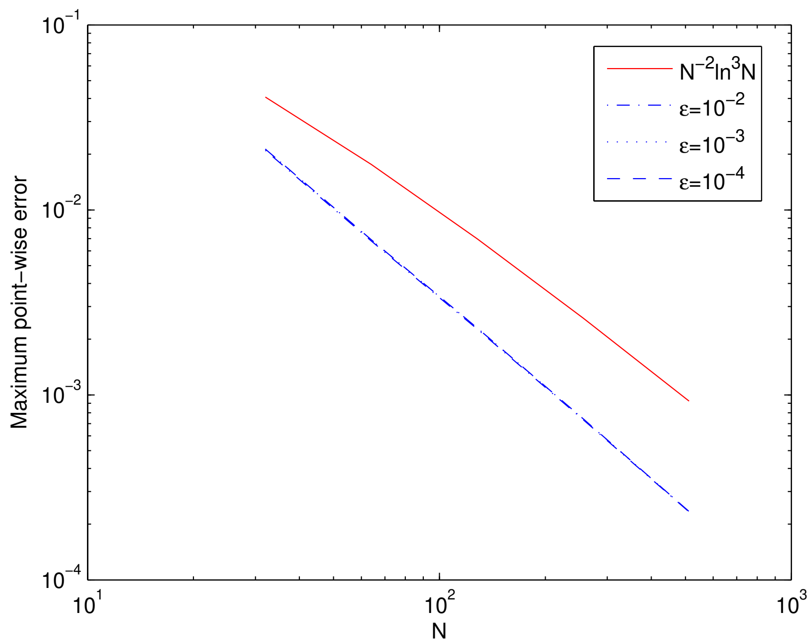

Figure 6. Loglog plots of Maximum point-wise error for Example 6.2.

The maximum point-wise errors for different values of ε and N with $\lambda_1 = 1/12,\ \lambda_2 = 5/12$ and $\delta = 0.5\varepsilon$ are shown in Table 6. It is observed from the table that the proposed scheme is ε-uniform convergent and also the maximum point-wise error decreases as the number of mesh points increases.

Table 6. Maximum point-wise error for Example 6.3 with $\delta = 0.5 \times \varepsilon$

	Number of mesh points (N)					
ε	16	32	64	128	256	512
2^{-2}	5.95E-03	2.17E-03	1.01E-03	6.90E-04	1.68E-04	5.47E-05
2^{-4}	1.01E-02	3.34E-03	1.12E-03	5.15E-04	1.63E-04	6.44E-05
2^{-8}	2.01E-02	6.34E-03	3.12E-03	9.15E-04	4.63E-04	2.44E-04
2^{-12}	2.00E-02	1.09E-02	5.71E-03	3.21E-03	1.72E-03	9.95E-04
2^{-16}	2.00E-02	1.01E-02	4.56E-03	2.86E-03	1.57E-03	8.59E-04
2^{-18}	2.00E-02	1.01E-02	4.27E-03	2.28E-03	1.49E-03	7.92E-04
2^{-20}	2.00E-02	1.01E-02	4.20E-03	2.13E-03	1.43E-03	6.33E-04
2^{-24}	2.00E-02	1.01E-02	4.18E-03	2.08E-03	1.40E-03	5.79E-04
2^{-28}	2.00E-02	1.01E-02	4.18E-03	2.08E-03	1.40E-03	5.79E-04
2^{-32}	2.00E-02	1.01E-02	4.18E-03	2.08E-03	1.40E-03	5.79E-04

7. Conclusion

In this article, we have proposed a hybrid scheme for a class of singularly perturbed convection delay problems on piece-wise uniform mesh. The solution of the tension spline method gives the oscillations in the outer region for smaller values of ε. In order to retain its stability, we use the mid point approximation for the convective term in the outer region. Error estimates for the method is analyzed and is shown that the method is convergent of order two. We also proved that the proposed method is ε-uniformly convergent.

Acknowledgment. The authors would like to express their sincere thanks and gratitude to the reviewers for their insightful comments and suggestions for the improvement of this paper.

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