



# Homotopic properties of $KA$ -digitizations of $n$ -dimensional Euclidean spaces

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## Abstract

For  $X(\subset \mathbf{R}^n)$ , assume the subspace  $(X, E_X^n)$  induced by the  $n$ -dimensional Euclidean topological space  $(\mathbf{R}^n, E^n)$ . Let  $\mathbf{Z}$  be the set of integers. Khalimsky topology on  $\mathbf{Z}$ , denoted by  $(\mathbf{Z}, \kappa)$ , is generated by the set  $\{\{2m-1, 2m, 2m+1\} \mid m \in \mathbf{Z}\}$  as a subbase. Besides, Khalimsky topology on  $\mathbf{Z}^n, n \in \mathbf{N}$ , denoted by  $(\mathbf{Z}^n, \kappa^n)$ , is a product topology induced by  $(\mathbf{Z}, \kappa)$ . Proceeding with a digitization of  $(X, E_X^n)$  in terms of the Khalimsky ( $K$ -, for short) topology, we obtain a  $K$ -digitized space in  $\mathbf{Z}^n$ , denoted by  $D_K(X)(\subset \mathbf{Z}^n)$ , which is a  $K$ -topological space. Considering further  $D_K(X)$  with  $K$ -adjacency, we obtain a *topological graph related to the  $K$ -topology* (a  $KA$ -space for short) denoted by  $D_{KA}(X)$  (see an algorithm in Section 3). Motivated by an  $A$ -homotopy between  $A$ -maps for  $KA$ -spaces, the present paper establishes a new homotopy, called an  $LA$ -homotopy, which is suitable for studying homotopic properties of both  $(X, E_X^n)$  and  $D_{KA}(X)$  because a homotopy for Euclidean topological spaces has some limitations of digitizing  $(X, E_X^n)$ . The goal of the paper is to study some relationships among an ordinary homotopy equivalence for spaces  $(X, E_X^n)$ , an  $LA$ -homotopy equivalence for spaces  $(X, E_X^n)$ , and an  $A$ -homotopy equivalence for  $KA$ -spaces  $D_{KA}(X)$ . Finally, we classify  $KA$ -spaces (*resp.*  $(X, E_X^n)$ ) via an  $A$ -homotopy equivalence (*resp.* an  $LA$ -homotopy equivalence). This approach can facilitate studies of applied topology, approximation theory and digital geometry.

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**Keywords.** digital topology,  $KA$ -digitization, Khalimsky adjacency,  $A$ -map,  $LA$ -map,  $K$ -topological graph,  $K$ -localized neighborhood,  $LA$ -homotopy equivalence,  $A$ -homotopy equivalence

## 1. Introduction

Let  $\mathbf{Z}$  (*resp.*  $\mathbf{N}$ ) represent the set of integers (*resp.* natural numbers), and  $\mathbf{Z}^n$  the set of points in the Euclidean  $n$ -dimensional space with integer coordinates. Let  $(\mathbf{R}^n, E^n)$  be the  $n$ -dimensional real space with Euclidean topology [33], *i.e.* usual topology on  $\mathbf{R}^n$ . For  $X(\subset \mathbf{R}^n)$ , we consider the subspace  $(X, E_X^n)$  induced by  $(\mathbf{R}^n, E^n)$ . In this paper we denote by  $ETC$  the category of ordinary  $n$ -dimensional Euclidean topological spaces  $(X, E_X^n)$  and Euclidean-topologically continuous maps. To digitize  $(X, E_X^n)$  into a digital space in  $\mathbf{Z}^n$  in

a certain digital topological approach, we have often used graph theory and locally finite topological structures such as Khalimsky ( $K$ -, for short) topology, Alexandroff topology, axiomatic locally finite topology and so forth [1, 11, 19, 21, 25, 27, 29]. To be specific, when digitizing  $(X, E_X^n)$ , it is clear to recognize partitioned shapes such as discs, triangles, rectangles and so forth of the Euclidean space into points in a certain digital topological approach [2, 4, 5, 11, 12, 19, 23, 24, 26–30, 32].

$K$ -topology on  $\mathbf{Z}^n$ , denoted by  $(\mathbf{Z}^n, \kappa^n)$ , was established [20, 21]. Besides, it is well known that whereas a  $K$ -connectedness relation is reflexive and symmetric, a  $K$ -adjacency relation is irreflexive and symmetric, and a  $K$ -topological space is a  $T_0$ -Alexandroff space [22]. Furthermore,  $K$ -connectedness is proved to be equivalent to pathconnectedness of  $K$ -topology [22].

Developing a  $K$ -localized neighborhood of a point  $p \in \mathbf{Z}^n$ , the recent paper [11] proceeded with a  $K$ -digitization of a subspace  $(X, E_X^n)$  (see Definition 3.6 in the present paper and the  $K$ -digitized space in Fig.2) denoted by  $D_K(X) (\subset \mathbf{Z}^n)$ . After considering further  $D_K(X)$  with  $K$ -adjacency, we denote it by  $D_{KA}(X)$  (see Definition 3.9), which can be used in digital geometry because  $D_{KA}(X)$  is a kind of a graph in  $\mathbf{Z}^n$  with digital connectivity originated by Rosenfeld [31].

Since an ordinary continuous map in the category of Euclidean topological spaces has some limitations of mapping of  $(X, E_X^n)$  from the viewpoint of digitization theory (see the map  $g$  of (3.2) in the proof of Theorem 3.14 in the present paper), the recent paper [11] developed an  $LA$ -map (see Definition 3.11 in the present paper) which is not compatible with a Euclidean-topologically continuous map. Furthermore, it established a category, denoted by  $LAC$ , consisting of the two data, the sets of  $(X, E_X^n)$  and  $LA$ -maps. Besides, the paper [15] established another category, denoted by  $KAC$ , consisting of two sets, the set of topological graphs based on  $K$ -topology (for short  $K$ -topological graphs or  $KA$ -spaces) as objects of the category and the set of  $A$ -maps between every ordered pair of  $KA$ -spaces as morphisms of the category. Indeed, this  $LA$ -map can be used to study both  $ETC$  and  $KAC$ .

It turns out that  $KAC$  is broader than the category of  $K$ -topological spaces ( $KTC$  for short) [15], which facilitates studies of both digital topology and digital geometry. Besides, the recent paper [10] proposed a homotopy in  $KAC$ , called an  $A$ -homotopy, which can be suitable for studying homotopic properties of  $D_{KA}(X)$ . To study some homotopic properties of  $(X, E_X^n) \in Ob(ETC)$  and  $D_{KA}(X) \in Ob(KAC)$ , the present paper develops two notions of an  $LA$ -homotopy, and an  $LA$ -homotopy equivalence in  $LAC$  (cf. [6, 7, 9]).

In relation to these homotopies, we may pose the following queries.

Assume two Euclidean topological spaces  $(X, E_X^n)$  and  $(Y, E_Y^n)$  and further, their  $KA$ -digitized spaces (or  $KA$ -spaces)  $D_{KA}(X)$  and  $D_{KA}(Y)$ .

(Q1) If  $F, G : (X, E_X^n) \rightarrow (Y, E_Y^n)$  are homotopic in  $ETC$ , then are  $D_{KA}(F)$  and  $D_{KA}(G)$   $A$ -homotopic?

(Q2) If  $F, G : (X, E_X^n) \rightarrow (Y, E_Y^n)$  are  $LA$ -homotopic in  $LAC$ , then are  $D_{KA}(F)$  and  $D_{KA}(G)$   $A$ -homotopic?

Besides, we have the further queries.

(Q3) Are there some relationships among an ordinary homotopy equivalence in  $ETC$ , an  $LA$ -homotopy equivalence in  $LAC$  and an  $A$ -homotopy equivalence in  $KAC$ ?

(Q4) What are some relationships among the ordinary contractibility in  $ETC$ , the  $LA$ -contractibility in  $LAC$  and the  $A$ -contractibility in  $KAC$ ?

The present paper shall address these issues in Sections 4 and 5. Besides, the dimensions of the spaces  $X$  and  $Y$  of (Q1) and (Q2) need not be equal to each other. Roughly saying, the first question can be answered negatively and the second question can be answered affirmatively. Thus we can observe some advantages of an  $LA$ -homotopy for studying  $LA$ -maps.

The rest of the paper is organized as follows: Section 2 provides some basic notions on digital topology and various notions in  $KAC$ . Section 3 investigates some properties of a  $KA$ -digitization of  $(X, E_X^n)$ . In particular, we prove that an  $LA$ -map is not compatible with a Euclidean-topologically continuous map. Section 4 studies some properties of an  $A$ -homotopy and develops an  $LA$ -homotopy. Section 5 investigates some relationships among a homotopy equivalence in  $ETC$ , an  $LA$ -homotopy equivalence in  $LAC$  and an  $A$ -homotopy equivalence in  $KAC$ . Furthermore, we compare among ordinary contractibility in  $ETC$ ,  $LA$ -contractibility in  $LAC$  and  $A$ -contractibility in  $KAC$ . Section 6 concludes the paper with a remark.

## 2. Preliminaries

The establishment of a topological graph based on a certain digital topological structure plays an important role in topology and applied sciences [4, 15, 22]. Since the  $K$ -topology is one of the important digital topological structures, let us recall some notions related to the  $n$ -dimensional  $K$ -topological space, denoted by  $(\mathbf{Z}^n, \kappa^n)$ , which is an Alexandroff space [1] and a semi- $T_{\frac{1}{2}}$  space [3]. For a set  $X \subset \mathbf{Z}^n$  we consider the subspace  $(X, \kappa_X^n)$  induced by  $(\mathbf{Z}^n, \kappa^n)$ . Under  $(X, \kappa_X^n)$ , for a point  $p \in X$ , in the paper, we will denote by  $SN_K(p)$  the smallest open neighborhood of the given point  $p$ .

For two  $K$ -topological spaces  $(X, \kappa_X^n) := X$  and  $(Y, \kappa_Y^n) := Y$ , if a function  $f : X \rightarrow Y$  satisfies the following property

$$f(SN_K(x)) \subset SN_K(f(x)), \quad (2.1)$$

then we say that the map  $f$  is  $K$ -continuous at a point  $x \in X$  [8]. The property of (2.1) is a representation of the typical  $K$ -continuity of the map  $f : X \rightarrow Y$ . Furthermore, a map  $f : X \rightarrow Y$  is Khalimsky ( $K$ -, for short) continuous if it is  $K$ -continuous at every point  $x \in X$ .

By using  $K$ -continuous maps, we obtain the category of  $K$ -topological spaces, denoted by  $KTC$  [8], consisting of the following data.

- The set of  $(X, \kappa_X^n)$  as objects, denoted by  $Ob(KTC)$ ,
- For every ordered pair of objects  $(X, \kappa_X^n)$  and  $(Y, \kappa_Y^n)$ , the set of all  $K$ -continuous maps  $f : (X, \kappa_X^n) \rightarrow (Y, \kappa_Y^n)$  as morphisms.

For two spaces  $(X, \kappa_X^n) := X$  and  $(Y, \kappa_Y^n) := Y$ , a map  $h : X \rightarrow Y$  is called a  $K$ -homeomorphism if  $h$  is a  $K$ -continuous bijection and further,  $h^{-1} : Y \rightarrow X$  is  $K$ -continuous.

Let us recall the notion of a *digital space*.

**Definition 2.1.** (1) A *digital space* is a pair  $(X, \pi)$ , where  $X$  is a nonempty set and  $\pi$  is a binary symmetric relation on  $X$  such that  $X$  is  $\pi$ -connected, [18].

(2) We say that a *grid space* is a union of some  $\pi$ -connected components with the given relation  $\pi$  instead of just a  $\pi$ -connected component in a digital space, [13].

In Definition 2.1, we say that  $X$  is  $\pi$ -connected if for any two elements  $x$  and  $y$  of  $X$  there is a finite sequence  $(x_i)_{i \in [0, l]_{\mathbf{Z}}}$  of elements in  $X$  such that  $x = x_0$ ,  $y = x_l$  and  $(x_j, x_{j+1}) \in \pi$  for  $j \in [0, l-1]_{\mathbf{Z}}$ .

**Remark 2.2.** In Definition 2.1, depending on the situation, we can consider the relation  $\pi$ . For instance, we may consider the relation  $\pi$  as the  $K$ -adjacency relation of a  $K$ -topological space (see Definition 2.3 below), which it is a symmetric relation.

According to the property (2.1), it turns out that a  $K$ -continuous map is so rigid that it has some limitations of geometric transformations. To be specific, a  $K$ -continuous map does not even include all rotations of a  $K$ -topological space with  $90^\circ$ [15] and cannot support a translation with an odd vector. Thus the recent paper [15] overcame the shortcoming by developing the so-called  $A$ -map (see Definition 2.5 in the present paper) which is a broader than a  $K$ -continuous map. This approach can be substantially helpful to study geometric transformations of  $K$ -topological spaces  $(X, \kappa_X^n)$ .

To guarantee a digital space structure of  $(X, \kappa_X^n)$ , we need to consider a  $K$ -adjacency relation on  $(X, \kappa_X^n)$  as follows:

**Definition 2.3** ([22]). For  $(X, \kappa_X^n)$  we say that two distinct points  $x$  and  $y$  in  $X$  are  $K$ -adjacent if  $y \in SN_K(x)$  or  $x \in SN_K(y)$ .

Considering  $(X, \kappa_X^n)$  with  $K$ -adjacency, we call it a *topological graph based on  $K$ -topology* (for short  *$K$ -topological graph or  $KA$ -space*).

In relation to the establishment of an  $A$ -map, we will use the following  $K$ -adjacency neighborhood of a point  $p \in X$ .

**Definition 2.4** ([15]). For a  $KA$ -space  $(X, T_X^n) := X$  and a point  $p \in X$  we define a  $K$ -adjacency neighborhood of  $p$  to be the set  $A_X(p) \cup \{p\} := AN_X(p)$  which is called an  $A$ -neighborhood of  $p$ , where  $A_X(p) = \{x \in X \mid x \text{ is } K\text{-adjacent to } p\}$ .

Hereafter, for convenience, we will use  $AN(p)$  instead of  $AN_X(p)$  if there is no danger of ambiguity. For a  $KA$ -space  $(X, T_X^n) := X$  and each point  $x \in X$ , since for every  $x \in X$  there is always  $AN(x) \subset X$ , we can develop an  $A$ -map and an  $A$ -isomorphism (see Definitions 2.5 and 2.7).

**Definition 2.5** ([15]). For two  $KA$ -spaces  $(X, T_X^{n_0}) := X$  and  $(Y, T_Y^{n_1}) := Y$ , we say that a function  $f : X \rightarrow Y$  is an  $A$ -map at a point  $x \in X$  if

$$f(AN(x)) \subset AN(f(x)).$$

Furthermore, we say that a map  $f : X \rightarrow Y$  is an  $A$ -map if the map  $f$  is an  $A$ -map at every point  $x \in X$ .

In view of Definition 2.5, we observe that an  $A$ -map  $f : X \rightarrow Y$  implies a map preserving connected subsets of  $X$  into connected ones [15], which can play an important role in studying  $K$ - and  $KA$ -spaces. Let us consider the self-map  $f$  of a simple closed  $K$ -curves with  $l$  elements in  $\mathbf{Z}^n$ , denoted by  $SC_K^{n,l} := (x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$  [15], such that  $f(x_i) = x_{i+1(mod l)}$ ,  $l \geq 4$ , where  $SC_K^{n,l} := (x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$  is a  $K$ -path  $(x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$  such that  $x_i$  and  $x_j$  are  $K$ -adjacent if and only if  $|i - j| = \pm 1(mod l)$ . Then, whereas  $f$  is an  $A$ -map, it is not a  $K$ -continuous map [11], which implies the following:

**Theorem 2.6** (Theorem 4.5 of [15]). *For a map from  $(X, T_X^{n_0}) := X$  to  $(Y, T_Y^{n_1}) := Y$ , a  $K$ -continuous map implies an  $A$ -map. But the converse does not hold.*

Based on the notion of an  $A$ -map, we obtain the following:

**Definition 2.7.** For two  $KA$ -spaces  $(X, T_X^{n_0}) := X$  and  $(Y, T_Y^{n_1}) := Y$ , a map  $h : X \rightarrow Y$  is called an  $A$ -isomorphism if  $h$  is a bijective  $A$ -map (for brevity,  $A$ -bijection) and if  $h^{-1} : Y \rightarrow X$  is an  $A$ -map.

Hereafter, we denote by  $X \approx_K Y$  and  $X \approx_A Y$  a  $K$ -homeomorphism and an  $A$ -isomorphism, respectively.

In view of Theorem 2.6, we obtain the following:

**Corollary 2.8** ([15]). *Let  $f : (X, T_X^{n_0}) := X \rightarrow (Y, T_Y^{n_1}) := Y$  be a map. If  $f$  is a  $K$ -homeomorphism, then it is an  $A$ -isomorphism. But the converse does not hold.*

Using  $A$ -maps, we establish the so called  $KA$ -category [15], denoted by  $KAC$ , consisting of the following data.

- (1) The set of  $KA$ -spaces as objects, denoted by  $Ob(KAC)$ ,
- (2) For every ordered pair of objects  $(X, \kappa_X^n)$  and  $(Y, \kappa_Y^n)$ , the set of all  $A$ -maps  $f : (X, \kappa_X^n) \rightarrow (Y, \kappa_Y^n)$  as morphisms.

**Definition 2.9** ([15]). Let  $(X, \kappa_X^n) := X$  be a  $KA$ -space. Then we define the following:  
 (1) Two distinct points  $x, y \in X$  are called  $KA$ -path connected (or  $KA$ -connected) if there is a sequence (or a path)  $(x_0, x_1, \dots, x_m)$  on  $X$  with  $\{x_0 = x, x_1, \dots, x_m = y\}$  such that  $x_i$  and  $x_{i+1}$  are  $K$ -adjacent,  $i \in [0, m - 1]_{\mathbf{Z}}, m \geq 1$ . This sequence is called an  $KA$ -path. Furthermore, the number  $m$  is called the length of this  $KA$ -path.

(2) A simple  $KA$ -path in  $X$  is the sequence  $(x_i)_{i \in [0, l]_{\mathbf{Z}}}$  such that  $x_i$  and  $x_j$  are  $K$ -adjacent if and only if  $|i - j| = 1$ .  
 In addition, we say that a simple closed  $KA$ -curve with  $l$  elements,  $l \geq 4$ , denoted by  $SC_A^{n, l} := (x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$ , is a  $KA$ -path  $(x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$  such that  $x_i$  and  $x_j$  are  $KA$ -adjacent if and only if  $|i - j| = \pm 1 \pmod{l}$ .

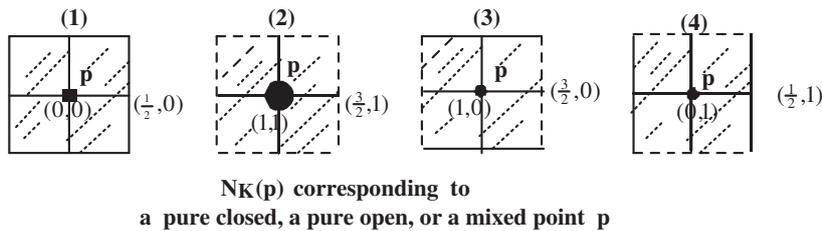
### 3. Some properties of $KA$ -digitizations

This section investigates some properties of a  $KA$ -digitization of  $(X, E_X^n)$ . To do this work, for a point  $p \in \mathbf{Z}^n$ , the papers [11, 19] uses a  $K$ -localized neighborhood of the given point  $p$ , denoted by  $N_K(p)$ , which is substantially related to the  $K$ -topological structure (see Fig.1).

**Definition 3.1** ([11]). In  $\mathbf{R}^n$ , for each point  $p := (p_i)_{i \in [1, n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ , we define the set  $N_K(p) := \{(x_i)_{i \in [1, n]_{\mathbf{Z}}}\}$  which is called the local  $K$ -neighborhood of  $p$  associated with  $(\mathbf{Z}^n, \kappa^n)$ , where

$$\begin{cases} \text{if } p_i = 2m, & \text{then } x_i \in [2m - \frac{1}{2}, 2m + \frac{1}{2}]; \\ \text{if } p_i = 2m + 1, & \text{then } x_i \in (2m + \frac{1}{2}, 2m + \frac{3}{2}). \end{cases}$$

In Fig.1(1)-(4), for a pure closed point, a mixed point and a pure open point  $p$ , we have their corresponding  $K$ -localized neighborhoods  $N_K(p) \subset \mathbf{R}^n$  [11].



**Figure 1.** Configuration of a  $K$ -localized neighborhood depending on the given point  $p \in \mathbf{Z}^2$  [11].

**Definition 3.2** ([11]). For two points  $x, y \in (\mathbf{R}^n, E^n)$ , we say that  $x$  is related to  $y$  if  $x, y \in N_K(p)$  for some point  $p \in \mathbf{Z}^n$ , denoted by  $x \sim_K y$ .

**Lemma 3.3** ([11]). The relation ‘ $\sim_K$ ’ of Definition 3.2 is an equivalence relation on  $\mathbf{R}^n$ .

**Proposition 3.4** ([11]). The set  $\{N_K(p) \mid p \in \mathbf{Z}^n\}$  is a partition of  $\mathbf{R}^n$  associated with the  $K$ -topology.

**Remark 3.5.** In view of Proposition 3.4 and Definition 3.2, for each point  $p \in \mathbf{Z}^n$ ,  $N_K(p)$  can be substantially used to digitize  $(X, E_X^n)$  in  $Ob(ETC)$  into a  $K$ -topological space  $D_K(X)$  in  $Ob(KTC)$ .

Let us recall the  $K$ -digitization of a non-empty space  $(X, E_X^n)$ .

**Definition 3.6** ([11]). For a non-empty space  $(X, E_X^n)$  we define a  $K$ -digitization of  $X$ , denoted by  $D_K(X)$ , to be the space with the  $K$ -topology

$$D_K(X) := \{p \in \mathbf{Z}^n \mid N_K(p) \cap X \neq \emptyset\}.$$

Motivated by Proposition 3.4, we obviously obtain the following:

**Corollary 3.7** ([11]). *Given a non-empty  $n$ -dimensional Euclidean space  $(X, E_X^n)$ , there is a partition of  $\mathbf{R}^n$  associated with the space  $(X, E_X^n)$ :*

$$\{N_K(p), \mathbf{R}^n \setminus \cup_{p \in D_K(X)} N_K(p) \mid p \in D_K(X)\}.$$

**Definition 3.8** ([11]). For a space  $(X, E_X^n)$  and two points  $p, q \in X$ , we say that the point  $p$  is related to  $q$  if there is a point  $x \in D_K(X)$  such that  $p, q \in N_K(x)$ . In this case we use the notation  $(p, q) \in L_X$  and further, the relation set is denoted by  $(X, L_X)$ .

It is clear that the relation  $L_X$  in the set  $(X, L_X)$  of Definition 3.8 is an equivalence relation [11].

After digitizing  $X$  in the  $K$ -topological approach (see Lemma 3.3), we define the following:

**Definition 3.9** ([11]). We say that  $D_{KA}(X)$  is the  $K$ -topological space  $D_K(X)$  with  $K$ -adjacency.

Namely, we see that  $D_K(X) \in Ob(KTC)$  and  $D_{KA}(X) \in Ob(KAC)$ . More precisely, we obtain the following algorithm for proceeding with a  $KA$ -digitization of  $(X, E_X^n)$ .

**Algorithm for establishing  $D_{KA}(X) \in Ob(KAC)$  from  $(X, E_X^n) \in Ob(ETC)$  (or  $Ob(LAC)$  later) [11]**

(A-1) Given  $(X, E_X^n) \in Ob(ETC)$  (or  $Ob(LAC)$  later), take an  $n$ D compact cuboid satisfying  $X \subset D$  such as  $D := \prod_{i=1}^n [m_i - p_i, m_i + p_i] \subset \mathbf{R}^n, m_i \in \mathbf{N}, p_i \geq 2$ .

(A-2) Take all points  $p \in D \cap \mathbf{Z}^n$  such that  $N_K(p) \cap X \neq \emptyset$  and put  $X' := \{p \in D \cap \mathbf{Z}^n \mid N_K(p) \cap X \neq \emptyset\}$ .

(A-3) For each point  $p \in X'$  take  $N_K(p) \subset \mathbf{R}^n$  and further, consider  $N_K(p) \cap X$ .

(A-4) Delete the set  $\mathbf{R}^n \setminus \cup_{p \in X'} N_K(p)$  from  $\mathbf{R}^n$  (see Corollary 3.7).

(A-5) Using Lemma 3.3 and Proposition 3.4, we recognize the set  $N_K(p) \cap X$  to be a singleton  $\{p\} \subset \mathbf{Z}^n$  so that we have  $N_K(p) \cap X := p \in D_K(X)$ .

(A-6) After adopting  $K$ -adjacency to the space  $(D_K(X), \kappa_{D_K(X)}^n)$ , we finally obtain  $D_{KA}(X) \in Ob(KAC)$ .

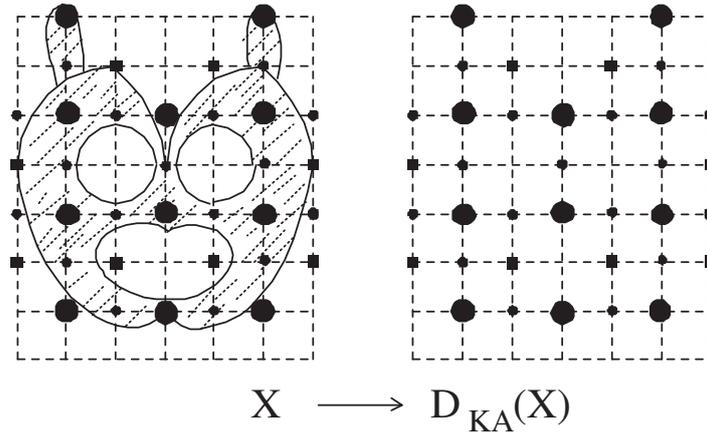
According to the above algorithm, for  $(X, E_X^n) \in Ob(ETC)$  (or  $Ob(LAC)$  later) we obtain its  $KA$ -digitization  $D_{KA}(X)$  (see Fig.2). Then the map

$$D_{KA} : ETC(\text{or } LAC \text{ later}) \rightarrow KAC \text{ given by } D_{KA}((X, E_X^n)) := D_{KA}(X)$$

is called a  $KA$ -digitization of  $(X, E_X^n)$ .

**Remark 3.10.** (1) In Definition 3.9,  $D_{KA}(X)$  is called a topological graph based on  $K$ -topology or a  $KA$ -space. More precisely, we can consider  $D_{KA}(X)$  as a  $K$ -topological graph of which a vertex set is  $D_K(X)$  and an edge between two points in  $D_K(X)$  is defined by using the  $K$ -adjacency of Definition 2.3. Thus we obtain  $D_{KA}(X) \in Ob(KAC)$  as a topological graph related to the  $K$ -topology.

(2) At (Step 1) in the page 169 of the paper [13], the part "satisfying  $X \subset D$ " related to an  $MA$ -digitization of  $X$  should be replaced by "surrounding  $X$ " instead.



**Figure 2.** The process for  $KA$ -digitizing  $(X, E_X^n) := X$ .

Combining a  $K$ -localized neighborhood of Definition 3.1 with an  $A$ -map, we define a map  $G : (X, E_X^{n1}) \rightarrow (Y, E_Y^{n2})$  as a more improved version than the earlier one of [11] which can be used to study both  $(X, E_X^n)$  and  $D_{KA}(X)$  [14], as follows:

**Definition 3.11.** For two spaces  $(X, E_X^{n1}), (Y, E_Y^{n2})$ , take their  $KA$ -digitized spaces  $D_{KA}(X) \subset \mathbf{Z}^{n1}$  and  $D_{KA}(Y) \subset \mathbf{Z}^{n2}$ . Assume an  $A$ -map  $g : D_{KA}(X) \rightarrow D_{KA}(Y)$ . Then, consider a map  $G : (X, E_X^{n1}) \rightarrow (Y, E_Y^{n2})$  induced by the map  $g$  with the property that for each point  $p \in D_{KA}(X)$

$$G(N_K(p) \cap X) \subset N_K(g(p)) \cap Y. \tag{3.1}$$

Then we say that the map  $G$  is a lattice-based  $K$ -adjacency map (an  $LA$ -map, for short). Besides, we use the notation  $D_{KA}(G) := g$  as a  $KA$ -digitization of  $G$ .

**Remark 3.12.** (Improvement of the notion of  $LM$ -map of Definition 11 of [17]) For two spaces  $(X, E_X^2), (Y, E_Y^2)$ , take their  $MA$ -digitized spaces  $D_{MA}(X) \subset \mathbf{Z}^2$  and  $D_{MA}(Y) \subset \mathbf{Z}^2$ . Assume an  $MA$ -map  $g : D_{MA}(X) \rightarrow D_{MA}(Y)$ . Then, consider a map  $G : (X, E_X^2) \rightarrow (Y, E_Y^2)$  induced by the map  $g$  with the property that for each point  $p \in D_{MA}(X)$

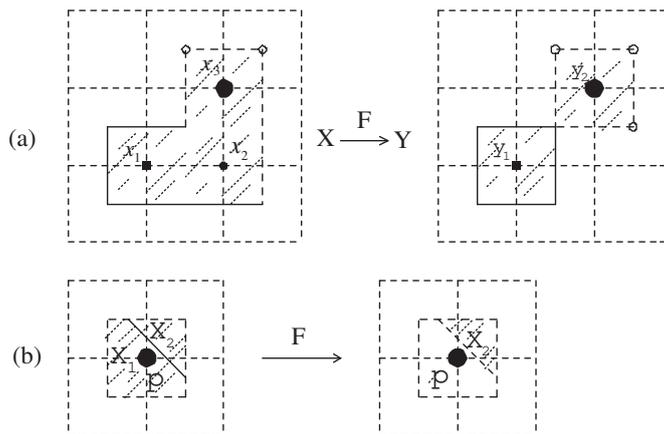
$$G(N_M(p) \cap X) \subset N_M(g(p)) \cap Y.$$

Then we say that the map  $G$  is a lattice-based  $M$ -adjacency map (an  $LM$ -map, for short). Besides, we use the notation  $D_{MA}(G) := g$  as an  $MA$ -digitization of  $G$ .

The paper [15] denotes by  $LAC$  the category consisting of the following data.

- (\* 1) The set of spaces  $(X, E_X^n) := X$  as objects of  $LAC$  denoted by  $Ob(LAC)$ ;
- (\* 2) For every ordered pair of elements in  $Ob(LAC)$ , the set of  $LA$ -maps between them as morphisms of  $LAC$  denoted by  $Mor(LAC)$ .

**Example 3.13.** In Fig.3(a), consider the  $A$ -map  $f : D_{KA}(X) \rightarrow D_{KA}(Y)$  such that  $f(x_1) = y_1, f(\{x_2, x_3\}) = \{y_2\}$ . Then, further consider the map  $F : (X, E_X^2) \rightarrow (Y, E_Y^2)$ , induced by the map  $f$ , given by  $F(N_K(x_1) \cap X) \subset N_K(y_1) \cap Y$  and  $F((N_K(x_2) \cup N_K(x_3)) \cap X) \subset N_K(y_2) \cap Y$ . Then  $F$  is an  $LA$ -map induced by the map  $f$ .



**Figure 3.** (a) Configuration of an  $LA$ -map; (b) Comparison between an  $LA$ -map in  $LAC$  and a continuous map in  $ETC$ .

Let us compare an  $LA$ -map in  $LAC$  and a continuous map in  $ETC$ .

**Theorem 3.14.** *None of an  $LA$ -map in  $LAC$  and a Euclidean-topologically continuous map in  $ETC$  implies the other.*

**Proof.** We prove that an  $LA$ -map in  $LAC$  need not imply a continuous map in  $ETC$ . For convenience, let us assume that  $(X, E_X^n)$  is connected. For some point  $p \in D_{KA}(X)$ , let us consider the subspaces  $X_i \subset N_K(p) \cap X, i \in \{1, 2\}$  such that  $X_1 \cap X_2 = \emptyset$  and  $X_1 \cup X_2 = N_K(p) \cap X$  which is connected (see the spaces  $X_1$  and  $X_2$  in Fig.3(b) as an example, where  $X_2$  is an open set in  $N_K(p) \cap X$ ). Consider an  $A$ -map as the identity map  $1_{D_{KA}(X)}$  with  $1_{D_{KA}(X)}(p) = p$ , where  $D_{KA}(X) = \{p\}$ .

Assume an  $LA$ -map  $G$  induced by the  $A$ -map  $1_{D_{KA}(X)}$  such that  $G(X_i) \subset G(N_K(p) \cap X), i \in \{1, 2\}$  and  $G(X_1) \cup G(X_2)$  is not connected in  $G(N_K(p) \cap X)$  (see Fig.3(b)). In this case the map  $G$  is not a continuous map in  $ETC$ .

To be precise, put  $N_K(p) := X = X_1 \cup X_2$  and  $X_1 \cap X_2 = \emptyset$  (see Fig.3(b)) and consider the map  $G$  on  $N_K(p)$  in Fig.3(b) given by  $G(X_1) = \{p\}$  and  $G(X_2) = 1_{X_2}(X_2)$ , where the point  $p$  is assumed to be a pure open point. Then, it is clear that whereas the map  $G$  is an  $LA$ -map induced by an  $A$ -map  $f : D_{KA}(X) \rightarrow D_{KA}(Y)$ , it is not a Euclidean-topologically continuous map because  $G(X_1 \cup X_2)$  is not connected.

Conversely, let us now prove that a continuous map in  $ETC$  does not imply an  $LA$ -map. Indeed, a continuous map in  $ETC$  need not support the property (3.1) of Definition 3.11. For instance, consider the self-map  $G$  of the unit interval  $([0, 1], E_{[0,1]})$  given by

$$G(t) = 2t, t \in [0, \frac{1}{2}) \text{ and } G(t) = 1, t \in [\frac{1}{2}, 1]. \tag{3.2}$$

While the given map  $G$  is a Euclidean-topologically continuous map, it is not an  $LA$ -map contrary to the property (3.1) of Definition 3.11, which completes the proof.  $\square$

**Remark 3.15.** In view of Theorem 3.14, it turns out that an  $LA$ -map is not compatible with a Euclidean-topologically continuous map. Besides, we see that an ordinary continuous map in  $ETC$  has some limitation of mapping of  $(X, E_X^n)$  from the viewpoint of digitization theory. Furthermore, an  $LA$ -map plays an important role in digitizing  $(X, E_X^n)$  because an  $LA$ -map guarantees to preserve  $N_K(p) \cap X$  into  $N_K(f(p)) \cap Y$ .

Let us now classify spaces  $(X, E_X^n)$  in  $LAC$  by using the  $LA$ -homeomorphism. To do this work, using both an  $A$ -map and a  $K$ -localized neighborhood of Definition 3.1, we need to establish the following notion which is an improved version of the notion of an  $LA$ -homeomorphism in the paper [11].

**Definition 3.16.** Let  $F : (X, E_X^n) := X \rightarrow (Y, E_Y^n) := Y$  be an  $LA$ -map induced by an  $A$ -map  $f : D_{KA}(X) \rightarrow D_{KA}(Y)$ . Then we say that  $F$  is an  $LA$ -homeomorphism if it satisfies the following two properties.

- (1) The map  $f$  is an  $A$ -isomorphism.
- (2) The inverse of  $f$ , denoted by  $g$ , induces an  $LA$ -map  $G : Y \rightarrow X$  such that  $D_{KA}(G) = g$  and further,  $(G \circ F)(X)$  is (Euclidean) homeomorphic to  $X$  and  $(F \circ G)(Y)$  is (Euclidean) homeomorphic to  $Y$ .

In case  $F : (X, E_X^n) \rightarrow (Y, E_Y^n)$  is an  $LA$ -homeomorphism, we say that  $(X, E_X^n)$  is  $LA$ -homeomorphic to  $(Y, E_Y^n)$ . Let us compare an  $LA$ -homeomorphism in  $LAC$  and a homeomorphism in  $ETC$ .

**Corollary 3.17.** *None of the  $LA$ -homeomorphism in  $LAC$  and the homeomorphism in  $ETC$  implies the other.*

**Proof.** Owing to Theorem 3.14, we see that an  $LA$ -homeomorphism need not imply a homeomorphism in  $ETC$ .

Conversely, it is obvious that a homeomorphism in  $ETC$  need not support the property (3.1) of Definition 3.11 (see Theorem 3.14). □

Let us compare an  $LA$ -homeomorphism and an  $A$ -isomorphism.

**Theorem 3.18.** *Consider two spaces  $(X, E_X^n)$  and  $(Y, E_Y^n)$ , and their  $KA$ -digitized spaces  $D_{KA}(X) := X'$  and  $D_{KA}(Y) := Y'$ . If  $(X, E_X^n)$  is  $LA$ -homeomorphic to  $(Y, E_Y^n)$ , then  $X'$  is  $A$ -isomorphic to  $Y'$ . But the converse does not hold.*

**Proof.** Owing to the property of Definition 3.16(1), it is clear that if  $(X, E_X^n)$  is  $LA$ -homeomorphic to  $(Y, E_Y^n)$ , then  $D_{KA}(X) := X'$  is  $A$ -isomorphic to  $D_{KA}(Y) := Y'$ .

Conversely, let us prove that not every  $A$ -isomorphism  $f$  between  $X'$  and  $Y'$  implies an  $LA$ -homeomorphism between  $(X, E_X^2)$  and  $(Y, E_Y^2)$ , where  $D_{KA}(X) = X'$  and  $D_{KA}(Y) = Y'$ . With the hypothesis of an  $A$ -isomorphism of  $f$ , we prove that there are some spaces  $(X, E_X^n)$  and  $(Y, E_Y^n)$  which are not  $LA$ -homeomorphic to each other. To be specific, consider the spaces  $(X, E_X^2)$  consisting of three closed line segments and  $(Y, E_Y^2)$  consisting of two closed line segments and one point  $p_5$  in Fig.4, where

$$X := L_1 \cup L_2 \cup L_3 \text{ and } Y := L_4 \cup \{p_5\} \cup L_5$$

such that  $L_1 \subset N_K(x_2), L_2 \subset N_K(x_1), L_3 \subset N_K(x_3) \cup N_K(x_4)$  and  $L_4 \subset N_K(y_2), p_5 \in N_K(y_1), L_5 \subset N_K(y_3) \cup N_K(y_4)$ .

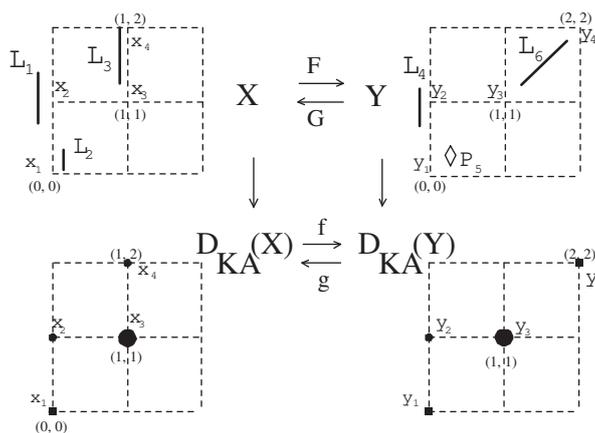
In this case, consider the map  $f : X' \rightarrow Y'$  given by  $f(x_i) = y_i, i \in \{1, 2, 3, 4\}$ . Then it is clear that  $f$  is an  $A$ -isomorphism because

$$|AN(x_i)| = |AN(y_i)|, i \in \{1, 2, 3, 4\},$$

e.g.  $|AN(x_1)| = 3 = |AN(y_1)|, |AN(x_2)| = 3 = |AN(y_2)|, |AN(x_3)| = 4 = |AN(y_3)|, |AN(x_4)| = 2 = |AN(y_4)|$ .

Then consider the map  $F : (X, E_X^2) \rightarrow (Y, E_Y^2)$  induced by  $f$  such that  $F(L_1) \subset L_4, F(L_2) \subset \{p_5\}, F(L_3) \subset L_5$ . Then it is clear that  $F$  is an  $LA$ -map induced by the  $A$ -map  $f$ . Then, for convenience, we denote by  $g$  the inverse of  $f$ . But we do not have an  $LA$ -map  $G : (Y, E_Y^2) \rightarrow (X, E_X^2)$  induced by  $g$  with the property of Definition 3.16(2). More precisely, owing to the point  $p_5 \in Y$ , it is clear that any  $LA$ -map  $G : (Y, E_Y^2) \rightarrow (X, E_X^2)$  satisfying  $D_{KA}(G) := g$  cannot satisfy the following property that  $G \circ F(X)$  is homeomorphic to  $X$  (see the property (3.2)). □

**Remark 3.19.** (1) In view of Lemma 3.3 and Corollary 3.17 and Theorem 3.18, in relation to the  $KA$ -digitization of  $(X, E_X^n)$ , an  $LA$ -homeomorphism in  $LAC$  is helpful to classify the spaces  $(X, E_X^n)$  in  $LAC$ .



**Figure 4.** Comparison among a homeomorphism, an  $LA$ -homeomorphism and an  $A$ -isomorphism

(2) (Improvement of the notion of an  $LM$ -homeomorphism of Definition 11 of the paper [17]) Let  $F : (X, E_X^2) := X \rightarrow (Y, E_Y^2) := Y$  be an  $LM$ -map induced by an  $MA$ -map  $f : D_{MA}(X) \rightarrow D_{MA}(Y)$ . Then we say that  $F$  is an  $LM$ -homeomorphism if it satisfies the following two properties.

- (1) The map  $f$  is an  $MA$ -isomorphism.
- (2) The inverse of  $f$ , denoted by  $g$ , induces an  $LM$ -map  $G : Y \rightarrow X$  such that  $D_{MA}(G) = g$  and further,  $(G \circ F)(X)$  is (Euclidean) homeomorphic to  $X$  and  $(F \circ G)(Y)$  is (Euclidean) homeomorphic to  $Y$ .

#### 4. Homotopic properties in $LAC$ and $KAC$

This section addresses the questions (Q1) and (Q2) posed in Section 1. Given  $(X, E_X^n) \in Ob(LAC)$ , we investigate some relationships between  $D_{KA}(X)$  and  $(X, E_X^n)$  from the viewpoint of homotopy theory. Motivated by Theorem 3.14, we firstly propose a *lattice based  $K$ -adjacency homotopy* ( $LA$ -homotopy for short) in  $LAC$ . Besides, owing to Theorem 3.14 (in particular, the property of (3.2) and the spaces in Fig.7(c)-(d)), although an ordinary homotopy in  $ETC$  does not induce an  $A$ -homotopy in  $KAC$ , we prove that an  $LA$ -homotopy in  $LAC$  induces an  $A$ -homotopy in  $KAC$ , which can play an important role in studying both  $(X, E_X^n)$  and its  $KA$ -digitized space  $D_{KA}(X)$ . To do this work, first of all, we need to recall the notion of an  $A$ -homotopy [10]. For a space  $X \in Ob(KAC)$  let  $B$  be a subset of  $X$ . Then  $(X, B)$  is called a  $KA$ -space pair. Furthermore, if  $B$  is a singleton set  $\{x_0\}$ , then  $(X, x_0)$  is called a pointed space in  $Ob(KAC)$ . To study homotopic properties of  $D_{KA}(X)$ , in this section we use the notions of an  $A$ -homotopy relative to a subset  $B \subset X$  [10],  $A$ -contractibility [10] and an  $A$ -homotopy equivalence [10].

**Definition 4.1** ([10]). Let  $(X, B)$  and  $Y$  be a space pair and a space in  $Ob(KAC)$ , respectively. Let  $f, g : X \rightarrow Y$  be  $A$ -maps. Suppose that there exist  $m \in \mathbf{N}$  and a function  $F : X \times [0, m]_{\mathbf{Z}} \rightarrow Y$  such that

- for all  $x \in X$ ,  $F(x, 0) = f(x)$  and  $F(x, m) = g(x)$ ;
- for all  $x \in X$ , the induced function  $F_x : [0, m]_{\mathbf{Z}} \rightarrow Y$  given by  $F_x(t) = F(x, t)$  for all  $t \in [0, m]_{\mathbf{Z}}$  is an  $A$ -map;
- for all  $t \in [0, m]_{\mathbf{Z}}$ , the induced function  $F_t : X \rightarrow Y$  given by  $F_t(x) = F(x, t)$  for all  $x \in X$  is an  $A$ -map.

Then we say that  $F$  is an  $A$ -homotopy between  $f$  and  $g$ .

- Furthermore, for all  $t \in [0, m]_{\mathbf{Z}}$ , assume that  $F_t(x) = f(x) = g(x)$  for all  $x \in B$  and for all  $t \in [0, m]_{\mathbf{Z}}$ .

Then we call  $F$  an  $A$ -homotopy relative to  $B$  between  $f$  and  $g$ , and we say that  $f$  and  $g$  are  $A$ -homotopic relative to  $B$  in  $Y$ ,  $f \simeq_{Arel.B} g$  in symbol.

In Definition 4.1, if  $B = \{x_0\} \subset X$ , then we say that  $F$  is a pointed  $A$ -homotopy at  $\{x_0\}$ . When  $f$  and  $g$  are pointed  $A$ -homotopic in  $Y$ , we use the notation  $f \simeq_A g$  and  $f \in [g]$  which denotes the  $A$ -homotopy class of  $g$ . If, for some  $x_0 \in X$ ,  $1_X$  is  $A$ -homotopic to the constant map in the singleton  $\{x_0\}$  relative to  $\{x_0\}$ , then we say that  $(X, x_0)$  is *pointed  $A$ -contractible* ( $A$ -contractible if there is no danger of ambiguity) [10].

To study some relations between  $D_{KA}(X)$  and  $(X, E_X^n)$  from the viewpoint of homotopy theory, combining an ordinary homotopy in  $ETC$  and an  $A$ -homotopy in  $KAC$ , we develop the following  $LA$ -homotopy.

**Definition 4.2.** Consider  $(X, E_X^{n_0}) := X$ ,  $(Y, E_Y^{n_1}) := Y$  and  $(B, E_B^{n_0}) := B$  with  $B \subsetneq X$ . Let  $f, g : X \rightarrow Y$  be  $LA$ -maps induced by some  $A$ -maps  $f', g' : D_{KA}(X) \rightarrow D_{KA}(Y)$ , respectively. Suppose that there exist  $m \in \mathbf{N}$  and a function  $F : X \times [0, m]_{\mathbf{Z}} \rightarrow Y$  such that

- for all  $x \in X$ ,  $F(x, 0) = f(x)$  and  $F(x, m) = g(x)$ ;
- for all  $x \in X$ , the induced function  $F_x : [0, m]_{\mathbf{Z}} \rightarrow Y$  given by  $F_x(t) = F(x, t)$  for all  $t \in [0, m]_{\mathbf{Z}}$  is an  $LA$ -map induced by an  $A$ -map  $D_{KA}(F_x)$ ;
- for all  $t \in [0, m]_{\mathbf{Z}}$ , the induced function  $F_t : X \rightarrow Y$  given by  $F_t(x) = F(x, t)$  for all  $x \in X$  is an  $LA$ -map induced by an  $A$ -map  $D_{KA}(F_t)$ .

Then we say that  $F$  is an  $LA$ -homotopy between  $f$  and  $g$ .

- Furthermore, for all  $t \in [0, m]_{\mathbf{Z}}$ , assume that  $F_t(x) = f(x) = g(x)$  for all  $x \in B$  and for all  $t \in [0, m]_{\mathbf{Z}}$ .

Then we call  $F$  an  $LA$ -homotopy relative to  $B$  between  $f$  and  $g$ , and we say that  $f$  and  $g$  are  $LA$ -homotopic relative to  $B$  in  $Y$ ,  $f \simeq_{LArel.B} g$  in symbol.

If, for some  $x_0 \in X$ ,  $1_X$  is  $LA$ -homotopic to the constant map in the singleton  $\{x_0\}$  relative to  $\{x_0\}$ , then we say that  $(X, x_0)$  is *pointed  $LA$ -contractible* ( $LA$ -contractible if there is no danger of ambiguity).

Let us investigate some properties of an  $LA$ -homotopy and an  $LA$ -contractibility.

**Example 4.3.** (1) Let us consider two closed curves  $(Y_i, E_{Y_i}^2), i \in \{1, 2\}$  in Fig.5(a). Then, by using  $D_{KA}(Y_i)$  in Fig.5(a), we observe that  $1_{D_{KA}(Y_1)}$  is  $A$ -homotopic to  $1_{D_{KA}(Y_2)}$ . Namely, we concluded that there is an  $LA$ -homotopy between  $1_{(Y_1, E_{Y_1}^2)}$  and  $1_{(Y_2, E_{Y_2}^2)}$ .

(2) Consider the space  $(Z, E_Z^2)$  in Fig.5(b), where  $Z = Z_1 \cup Z_2$ ,  $Z_1 = \{(x, y) \mid |x-1| + |y| = 1\}$  and  $Z_2 := \{(x, y) \mid (x-p_1)^2 + (y-p_2)^2 = \frac{1}{r}, r \geq 4, (p_1, p_2) \in \{(0, 0), (1, 1), (1, -1), (2, 0)\}\}$ . Then  $(Z, E_Z^2)$  is  $LA$ -contractible, which is a quite distinctive feature compared with the contractibility in  $ETC$ . More precisely, we see that  $D_{KA}(Z)$  is  $SC_A^{2,4}$  (see Fig.5(b)). Considering any self-map  $f$  of  $(Z, E_Z^2)$  as a constant map such that

$$f(Z) = \{t\}, t \in \{(0, 0), (1, 1), (1, -1), (2, 0)\}.$$

Then the map  $f$  is  $LA$ -homotopic to  $1_Z$  relative to the singleton  $\{t\}$  by using the method suggested in Fig.5(b), which implies that  $(Z, E_Z^2)$  is  $LA$ -contractible.

**Remark 4.4.** In view of Example 4.3, we observe some difference between the contractibility in  $ETC$  and the  $LA$ -contractibility in  $LAC$ .

Let us now investigate some relationships between an  $LA$ -homotopy and an  $A$ -homotopy. To do this work, we recall some notions related to a  $KA$ -digitization map. The paper [19] studies the connectedness preserving ( $CP$ -, for short) property of a  $K$ -digitization, as follows:

**Lemma 4.5** ([19]). *If  $(X, E_X^n)$  is connected, then  $D_K(X)$  is  $K$ -connected.*

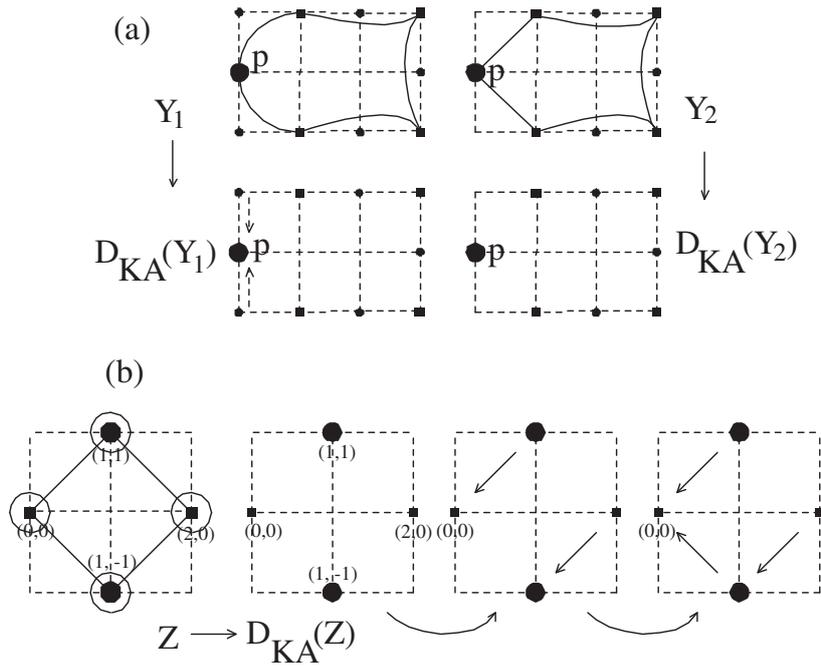


Figure 5. Explanation of an LA-homotopy and the LA-contractibility

Let us prove that an LA-homotopy induces an A-homotopy, as follows:

**Theorem 4.6.** Consider two LA-maps  $f, g : (X, E_X^n) \rightarrow (Y, E_Y^n)$  induced by their KA-digitized maps  $D_{KA}(f), D_{KA}(g) : D_{KA}(X) \rightarrow D_{KA}(Y)$ , respectively. If there is an LA-homotopy between  $f$  and  $g$ , then we obtain an A-homotopy between  $D_{KA}(f)$  and  $D_{KA}(g)$  derived from the given LA-homotopy.

**Proof.** (Case 1) Assume that  $(X, E_X^n)$  and  $(Y, E_Y^n)$  are connected. Firstly, by Lemma 4.5, it is clear that if  $(X, E_X^n)$  is connected, then  $D_{KA}(X)$  is KA-connected because for two distinct points  $p$  and  $q$  in  $(\mathbf{Z}^n, \kappa^n)$  such that  $p \in N_{3^n-1}(q)$ , the  $K$ -connectedness of these points is equivalent to their  $K$ -adjacency [22], where  $N_{3^n-1}(p) := \{q \mid q \text{ is } (3^n - 1)\text{-adjacent to } p\}$  [31]. Concretely, a KA-digitization map has the *connectedness preserving* (CP-, for brevity) property, i.e. if  $(X, E_X^n)$  is connected, then by Lemma 4.5 we obtain that  $D_{KA}(X)$  is KA-connected.

Secondly, assume an LA-homotopy  $H$  in LAC between two LA-maps  $f, g : (X, E_X^n) \rightarrow (Y, E_Y^n)$  induced by  $D_{KA}(f), D_{KA}(g)$ , respectively. Namely, we have

$$H : X \times [0, m]_{\mathbf{Z}} \rightarrow Y \text{ such that } H(x, 0) = f(x) \text{ and } H(x, m) = g(x)$$

satisfying the property of Definition 4.2. By Proposition 3.4, Corollary 3.7 and Lemma 4.5, we obtain

$$D_{KA}(H) : D_{KA}(X) \times [0, m]_{\mathbf{Z}} \rightarrow D_{KA}(Y) \text{ such that}$$

- for all  $x' \in D_{KA}(X)$

$$\left\{ \begin{array}{l} D_{KA}(H)(x', 0) = D_{KA}(f)(x') \text{ and} \\ D_{KA}(H)(x', m) = D_{KA}(g)(x'). \end{array} \right\}$$

- for all  $x' \in D_{KA}(X)$ , the induced function  $D_{KA}(H)_{x'} : [0, m]_{\mathbf{Z}} \rightarrow D_{KA}(Y)$  given by  $D_{KA}(H)_{x'}(t) = D_{KA}(H)(x', t)$  for all  $t \in [0, m]_{\mathbf{Z}}$  is an A-map;

- for all  $t \in [0, m]_{\mathbf{Z}}$ , the induced function  $D_{KA}(H)_t : D_{KA}(X) \rightarrow D_{KA}(Y)$  given by  $(D_{KA}(H)_t)(x') = D_{KA}(H)(x', t)$  for all  $x' \in D_{KA}(X)$  is an  $A$ -map, which implies that  $D_{KA}(H)$  is an  $A$ -homotopy between the above  $A$ -maps  $D_{KA}(f)$  and  $D_{KA}(g)$ .

(Case 2) Assume that  $(X, E_X^n)$  and  $(Y, E_Y^n)$  are not connected. Owing to Proposition 3.4 and Corollary 3.7, even for this case, by the method similar to the proof of Case 1, we complete the proof.  $\square$

### 5. A comparison among an ordinary homotopy equivalence, an $LA$ -homotopy equivalence and an $A$ -homotopy equivalence

This section addresses the issues (Q3) and (Q4) posed in Section 1. To do these works, we start with an ordinary homotopy equivalence (*resp.* contractibility) of  $(X, E_X^n)$  and an  $A$ -homotopy equivalence (*resp.*  $A$ -contractibility) of  $D_{KA}(X)$ . Besides, we develop the notions of an  $LA$ -homotopy equivalence (*resp.*  $LA$ -contractibility) of  $(X, E_X^n)$  and an  $A$ -homotopy equivalence (*resp.*  $A$ -contractibility) of  $D_{KA}(X)$ . Furthermore, we investigate some relationships among these kinds of homotopy equivalences (*resp.* contractibilities). Given  $(X, E_X^n) \in Ob(LAC)$ , by using these homotopic tools, we study homotopic properties of both  $D_{KA}(X)$  and  $(X, E_X^n)$ . Let us now recall an  $A$ -homotopy equivalence in  $KAC$ .

**Definition 5.1** ([10]). In  $KAC$ , for two spaces  $X$  and  $Y$ , if there are  $A$ -maps  $h : X \rightarrow Y$  and  $l : Y \rightarrow X$  such that  $l \circ h$  is  $A$ -homotopic to  $1_X$  and  $h \circ l$  is  $A$ -homotopic to  $1_Y$ , then the map  $h : X \rightarrow Y$  is called an  $A$ -homotopy equivalence. Then we use the notation  $X \simeq_{A.h.e} Y$ .

**Theorem 5.2** ([10]). *The composite preserves an  $A$ -homotopy equivalence in  $KAC$ . Namely, if  $X \simeq_{A.h.e} Y$  and  $Y \simeq_{A.h.e} Z$ , then  $X \simeq_{A.h.e} Z$ .*

As referred to in Example 4.3, we obtain the following:

**Lemma 5.3** ([10]). *In  $KAC$ ,  $SC_A^{2,4}$  is  $A$ -contractible.*

Motivated by several types of digital versions of homotopy equivalences [6–9], let us propose the notion of an  $LA$ -homotopy equivalence in  $LAC$ .

**Definition 5.4.** In  $LAC$ , for two spaces  $(X, E_X^n) := X$  and  $(Y, E_Y^n) := Y$ , if there are  $LA$ -maps  $h : X \rightarrow Y$  and  $l : Y \rightarrow X$  such that  $l \circ h$  is  $LA$ -homotopic to  $1_X$  and  $h \circ l$  is  $LA$ -homotopic to  $1_Y$ , then the map  $h : X \rightarrow Y$  is called an  $LA$ -homotopy equivalence. Then we use the notation  $X \simeq_{LA.h.e} Y$ .

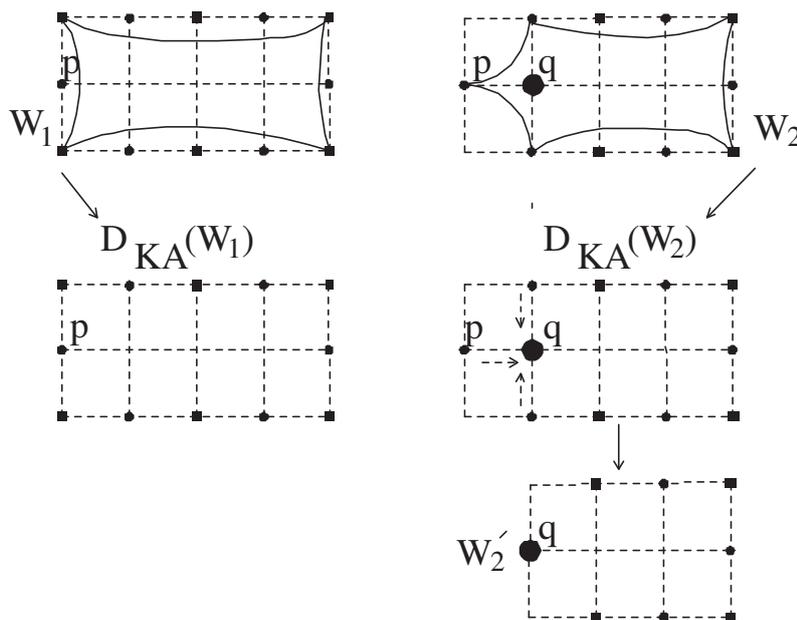
**Example 5.5.** Consider the two closed curves  $(W_1, E_{W_1}^2)$  and  $(W_2, E_{W_2}^2)$  in Fig.6. Since  $D_{KA}(W_1)$  is  $SC_A^{2,12}$  and  $D_{KA}(W_2)$  is  $SC_A^{2,10}$  (see Fig.6), any  $LA$ -map from  $(W_2, E_{W_2}^2)$  to  $(W_1, E_{W_1}^2)$  is not an  $LA$ -homotopy equivalence between them because  $D_{KA}(W_1)$  is not  $A$ -homotopy equivalent to  $D_{KA}(W_2)$  (see Theorem 4.6).

Comparing an  $LA$ -homotopy equivalence and an ordinary homeomorphism in [33], we can observe that an  $LA$ -homotopy equivalence has some advantages in classifying spaces in  $Ob(LAC)$ .

**Theorem 5.6.** *The composite preserves an  $LA$ -homotopy equivalence in  $LAC$ . Namely, if  $X \simeq_{LA.h.e} Y$  and  $Y \simeq_{LA.h.e} Z$ , then  $X \simeq_{LA.h.e} Z$ .*

Let us now compare among an ordinary homotopy equivalence in  $ETC$ , an  $LA$ -homotopy equivalence in  $LAC$  and an  $A$ -homotopy equivalence in  $KAC$ .

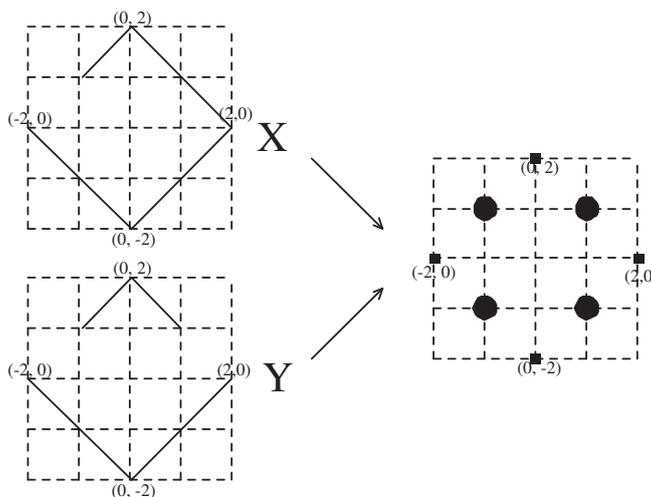
**Theorem 5.7.** *Consider two Euclidean topological spaces  $(X, E_X^n)$  and  $(Y, E_Y^n)$  and their  $KA$ -digitized spaces  $D_{KA}(X)$  and  $D_{KA}(Y)$ . None of a homotopy equivalence in  $ETC$  and an  $A$ -homotopy equivalence in  $KAC$  implies the other.*



**Figure 6.** Comparison among a homotopy equivalence in  $ETC$ , an  $LA$ - and an  $A$ -homotopy equivalence.

**Proof.** Since the notion of an  $A$ -homotopy equivalence is stronger than that an  $LA$ -homotopy equivalence, by Theorem 3.14, we prove that a homotopy equivalence between  $(X, E_X^n)$  and  $(Y, E_Y^n)$  in  $ETC$  does not imply an  $A$ -homotopy equivalence between  $D_{KA}(X)$  and  $D_{KA}(Y)$  in  $KAC$ .

Conversely, consider the spaces  $(X, E_X^2)$  and  $(Y, E_Y^2)$  in Fig.7. While their  $KA$ -spaces  $D_{KA}(X)$  and  $D_{KA}(Y)$  in Fig.7(a) are equal to each other so that  $D_{KA}(X)$  and  $D_{KA}(Y)$  are  $A$ -homotopy equivalent to each other, However, by Theorem 3.8, it is clear that the space  $(X, E_X^2)$  is not homotopy equivalent to  $(Y, E_Y^2)$  in  $ETC$ .  $\square$



**Figure 7.** Comparison among a homotopy equivalence in  $ETC$ , an  $LA$ - and an  $A$ -homotopy equivalence.

By using the method similar to the proof of Theorem 5.7, we obtain the following:

**Theorem 5.8.** *None of a homotopy equivalence in ETC and an LA-homotopy equivalence in LAC implies the other.*

**Proof.** By using the method similar to the proof of Theorem 5.7, it is clear that a homotopy equivalence in ETC does not imply an LA-homotopy equivalence in LAC. Conversely, let us prove that an LA-homotopy equivalence in LAC does not imply a homotopy equivalence in ETC.

Even in the case some spaces  $(Y, E_Y^2)$  and  $(W, E_W^2)$  are LA-homotopy equivalent to each other, by Theorem 5.6, they need not be homotopy equivalent to each other in ETC.  $\square$

Let us compare between an LA-homotopy equivalence in LAC and an A-homotopy equivalence in KAC.

**Theorem 5.9.** *An LA-homotopy equivalence between  $(X, E_X^n)$  and  $(Y, E_Y^n)$  in LAC implies an A-homotopy equivalence between  $D_{KA}(X)$  and  $D_{KA}(Y)$  in KAC.*

**Proof.** Consider two topological spaces  $(X, E_X^n)$  and  $(Y, E_Y^n)$  in  $Ob(LAC)$  and their KA-digitized spaces  $D_{KA}(X)$  and  $D_{KA}(Y)$ . By Theorem 4.6, we conclude that an LA-homotopy equivalence between  $(X, E_X^n)$  and  $(Y, E_Y^n)$  in LAC implies an A-homotopy equivalence between  $D_{KA}(X)$  and  $D_{KA}(Y)$  in LAC.  $\square$

H.E. in ETC --> A-H.E	H.E. in ETC --> LA-H.E	LA-H.E --> H.E in ETC	LA-H.E --> A-H.E	A-H.E. --> H.E. in ETC
NO	NO	NO	YES	NO

**Figure 8.** Comparison among a homotopy equivalence in ETC, an LA- and an A-homotopy equivalence.

**Remark 5.10.** In view of Theorems 5.7 and 5.9 the notion of an LA-homotopy equivalence in LAC can be used to study both  $(X, E_X^n)$  and its KA-digitized space  $D_{KA}(X)$  from the viewpoint of homotopy theory.

Let us now compare the contractibility of  $(X, E_X^n)$ , the LA-contractibility of  $(X, E_X^n)$  the A-contractibility of  $D_{KA}(X)$ .

**Theorem 5.11.** *None of the contractibility of  $(X, E_X^n)$  and the A-contractibility of  $D_{KA}(X)$  implies the other.*

**Proof.** Let us consider the case that the contractibility of  $(X, E_X^n)$  does not imply the A-contractibility of  $D_{KA}(X)$ . Consider the space  $(X, E_X^2)$  in Fig.9(a). It is clear that whereas  $(X, E_X^2)$  is a kind of arc which is contractible in ETC and  $D_{KA}(X)$  is not A-contractible (see Fig.9(a)). More precisely, since  $D_{KA}(X)$  is a kind of  $SC_A^8$  (see Fig.9(a)), it cannot be A-contractible.

Conversely, let us prove that the A-contractibility of  $D_{KA}(X)$  does not imply the contractibility of  $(X, E_X^n)$ . Consider the space  $(Z, E_Z^2)$  in Fig.9(c) which is not contractible in ETC. But it is clear that  $D_{KA}(Z)$  is A-contractible.  $\square$

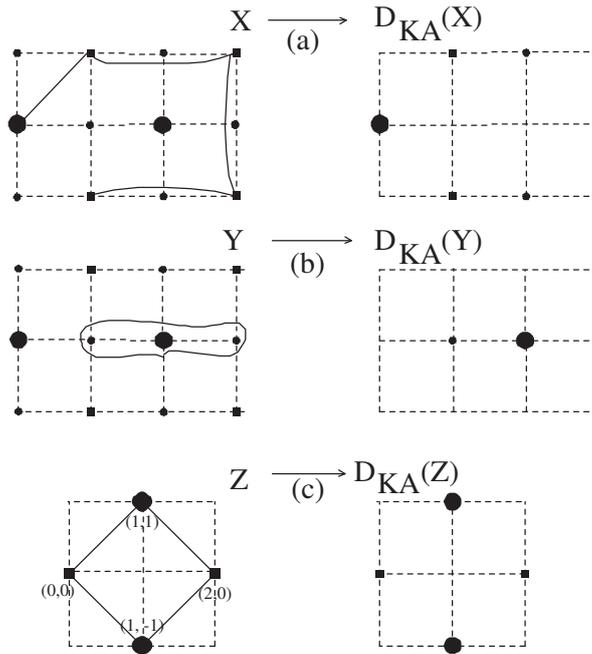
**Remark 5.12.** Consider the space in  $(Y, E_Y^2)$  in Fig.9(b). Although it is not contractible, we see that  $D_{KA}(Y)$  is A-contractible.

By using Theorem 5.9, we obtain the following:

**Proposition 5.13.** *The  $LA$ -contractibility of  $(X, E_X^n)$  implies the  $A$ -contractibility of  $D_{KA}(X)$ .*

**Remark 5.14.** For an efficient process for examining the  $LA$ -contractibility, we can write an algorithm, as follows:

- (Step 1) Proceed with a  $KA$ -digitization of  $(X, E_X^n)$ .
- (Step 2) Take  $D_{KA}(X)$ .
- (Step 3) Examine if the  $A$ -contractibility of  $D_{KA}(X)$  holds.



**Figure 9.** Comparison among ordinary contractibility in  $ETC$ ,  $LA$ -contractibility, and  $A$ -contractibility.

## 6. Summary

We have studied various properties of an  $LA$ -homotopy, an  $LA$ -homotopy equivalence and  $LA$ -contractibility. Besides, comparing a Euclidean-topologically continuous map with an  $LA$ -map, we observed that an  $LA$ -map has strong advantages of digitizing  $(X, E_X^n)$ . Furthermore, comparing a Euclidean homotopy with an  $LA$ -homotopy, we concluded that an  $LA$ -homotopy is a suitable homotopy for digitizing  $(X, E_X^n)$  in  $ETC$  and further, obtaining a  $K$ -topological graph in  $KAC$ . Besides, the paper investigated some relationships between subspaces  $(X, E_X^n)$  and their  $KA$ -spaces  $D_{KA}(X)$  by using an  $LA$ -homotopy equivalence and an  $A$ -homotopy equivalence. Finally, we investigate some relationships among Euclidean contractibility,  $A$ -contractibility and  $LA$ -contractibility. Compared with several kinds of homotopies for digital spaces [13, 16], the current homotopies have their own utilities.

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