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CYCLIC PRESENTATIONS AND TORUS KNOTS K(d, 2)

DEVİRLİ TEMSİLLER VE K(d, 2) TOR DÜĞÜMLERİ

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ABSTRACT

In this paper, we have shown that the polynomials associated with the cyclically presented groups obtained from the word w generated with Dunwoody parameters $(1,k,0,2),(1,k,0,k),(\frac{k+1}{2},1,0,\frac{k+1}{2}),(\frac{k+1}{2},1,0,\frac{k+3}{2})$, where k is an odd positive integer and $d=k+2$, coincide (up to sign) with the Alexander polynomial of the torus knot $K(d,2)$.

Key words: Alexander polynomial, cyclic presentation, Dunwoody parameters, Torus knots.

ÖZET

Bu çalışmada, k pozitif tek tamsayı ve $d=k+2$ olmak üzere $(1,k,0,2),(1,k,0,k),(\frac{k+1}{2},1,0,\frac{k+1}{2}),(\frac{k+1}{2},1,0,\frac{k+3}{2})$ Dunwoody parametrelerine karşılık gelen w kelimesinden elde edilen devirli temsillenen gruplarla eşlenen polinomların $K(d,2)$ tor düğümünün Alexander polinomu ile çakıştığı gösterilmiştir.

Anahtar kelimeler: Alexander polinomu, Devirli temsil, Dunwoody parametreleri, Torus düğümleri.

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1. INTRODUCTION

In order to investigate the relations between cyclic branced covering of knots in S^3 and manifolds admitting cyclically presented fundemental groups, M. J. Dunwoody introduced in (Dunwoody, 1995) a class of 3-manifolds depending on six integer parameters. An interesting problem is to find the Dunwoody parameters of the cyclic

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branced coverings of important classes of (1,1)-knots, in particular when the knots lies in S^3. This type of result has been obtained in (Grasselli and Mulazzani, 2001) for all 2-bridge knots. Aydin *et al.* (2003) obtained the Dunwoody parameters for all cyclic branced coverings of torus knots of type $K(p, mp \pm 1)$, with $m > 0$ and $p > 1$.

In this paper we show the polynomials associated with the cyclically presented groups obtained from the word w generated with Dunwoody parameters are equal to the Alexander polynomial of torus knot $K(d, 2)$.

2. MATERIALS AND METHODS

Let F_n be the free group on free generators $x_0, x_1, x_2, \ldots, x_{n-1}$. Let $\theta : F_n \to F_n$ be the automorphism such that

$$\theta(x_i) = x_{i+1}, \quad i = 0,1,\ldots,n-2, \quad \theta(x_{n-1}) = x_0.$$

For $w \in F_n$, $G_n(w)$ is defined as $G_n(w) = F_n / R$ where R is the normal closure in F_n of the set $\{w, \theta(w), \theta^2(w), \ldots, \theta^{n-1}(w)\}$ (Johnson, 1990). For a reduced word $w \in F_n$, the cyclically presented group $G_n(w)$ is given by $G_n(w) = \langle x_0, x_1, \ldots, x_{n-1} \mid w, \theta(w), \ldots, \theta^{n-1}(w) \rangle$ (Grasselli and Mulazzani, 2001).

Definition 2.1: A group G is said to have a cyclic presentation if $G \cong G_n(w)$ for some n and w (Cavicchioli, et al. 2001).

The polynomial associated with the cyclically presented group $G = G_n(w)$ is given by $f(t) = \sum_{i=0}^{n-1} a_i t^i$

where a_i is the exponent sum of x_i in w, $1 \le i \le n$ (Dunwoody, 1995).

Let a, b, c, n be integers such that $n > 0$, $a, b, c \ge 0$ and $a + b + c > 0$. Let $\overline{\tau}(a, b, c)$ be the graph shown in Figure 1. This is an infinite graph with an automorphism θ such that $\theta(u_n) = u_{n+1}$ and

$\theta(v_n) = v_{n+1}$. The labels indicate the number of edges joining a pair of vertices. Thus, there are a edges joining u_1 and u_2. We see that the $\bar{\tau}(a,b,c)$ is d-regular where $d = 2a+b+c$. Let $\tau_n = \tau_n(a,b,c)$ denote the graph obtained from $\bar{\tau}(a,b,c)$ by identifying all edges and vertices in each orbit of θ^n. Thus τ_n has $2n$ vertices (Dunwoody, 1995).

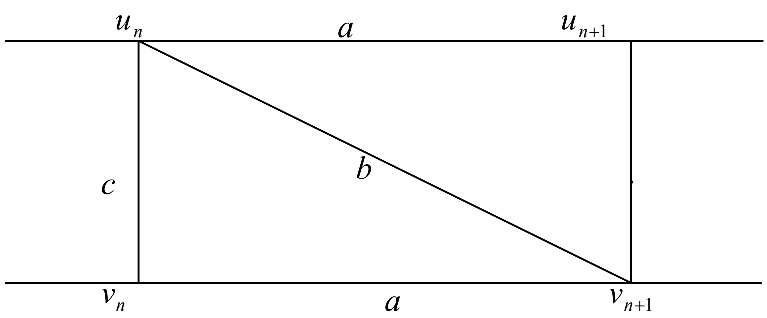

Figure 1.

We say that the 6-tuple (a,b,c,r,s,n) has property M if it corresponds to the Heegaard diagram of a 3-manifold. An algorithm determining which 6-tuples have property M is now described. Put $d = 2a+b+c$ and let

$X = \{-d, \ -d+1, \ \ldots, \ -1, \ 1, \ 2, \ldots, \ d\}$.

Let α, β be the permutations of X defined as follows:

$$\alpha = (1,d)(2,d-1)\ldots(a, \ d-a+1)(a+1,-a-c-1)(a+2, \ -a-c-2)\ldots$$
$$(a+b,-a-c-b)(a+b+1,-a-1)(a+b+2,-a-2)\ldots(a+b+c,-a-c)(-1,-d)$$

and

$$\beta(j) = \begin{cases} -(j+r) & , \text{ if } \ j>0 \text{ and } j+r \le d \text{ or } j<0 \text{ and } j+r<0 \\ -(j+r-d) & , \text{ if } \ j+r \ge 0 \end{cases}$$

The following theorem characterizes the 6-tuples (a,b,c,r,s,n) that have property M. Detail and the proof of this theorem can be found in (Dunwoody, 1995).

Theorem 2.1: Let $d = 2a+b+c$ be odd. The 6-tuple (a,b,c,r,s,n) has property M if and only if the following two conditions hold simultaneously:

(i). $\alpha\beta$ has two cycles of length d

(ii). $ps+q \equiv 0 \pmod n$

where p is the difference between the number of arrows pointing down the page and the number of arrows pointing up, whereas q is the number of arrows pointing from left to right minus the number of arrows pointing from right to left in the oriented path determined by $\alpha\beta$. The entries in the first cycle of $\alpha\beta$ contain one vertex from each line segment of the diagram. There exists an integer s such that $ps+q \equiv 0 \pmod n$. The first cycle of $\alpha\beta$ and the value of s can also be used to calculate the word w of the corresponding cyclic presentation.

Recall that $K(p,q) = K(p',q')$ if and only if (p',q') is equal to one of the following pairs: $(p,q),(q,p),(-p,-q),(-q,-p)$ and that $K(-p,-q) = -K(p,q)$ (Burde and Zieschang, 1985).

The Alexander polynomial of the torus knot $K(p,q)$, $p > q \geq 2$ is

$$\Delta_{p,q}(t) = \frac{(1-t^{pq})(1-t)}{(1-t^p)(1-t^q)} = 1 - t + t^q - \ldots - t^{pq-p-q} + t^{pq+1-p-q}$$

(Cavicchioli *et al.* 1999).

3. RESULTS AND DISCUSSIONS

We can now state our theorems:

Theorem 3.1 Theorem 3.1 (Ankaralioglu and Aydin, 2008): The cyclically presented groups obtained from the word w generated with Dunwoody parameters $(1,b,0,2)$ are isomorphic to the groups

$S\left((d+1)/2,\, d\right)$ when b is an odd positive integer and $d=2a+b+c$.

Theorem 3.2: The polynomial associated with the cyclically presented group obtained from the word w generated with Dunwoody parameters $(1,k,0,2)$, where k is an odd positive integer and $d=k+2$, coincides with the Alexander polynomial of the torus knot $K(d,2)$.

Proof: As stated in the proof of theorem 3.1, the defining word w corresponding to Dunwoody parameters $(1,k,0,2)$ has the following form

$$x_{k+1}^{-1}x_{k-1}^{-1}x_{k-3}^{-1}\ldots x_2^{-1}x_0^{-1}x_1x_3x_5\ldots x_k, \qquad (1)$$

where $d=k+2$.

The corresponding polynomial with (1) is

$$f(t)=-1+t-t^2+\ldots+t^k-t^{k+1} \qquad (2)$$

or more generally

$$f(t)=\sum_{j=0}^{k+1}(-1)^{j+1}t^j \quad , \; j\equiv 0(\operatorname{mod} d).$$

According to the values $p=d$ and $q=2$, the Alexander polynomial of the torus knot $K(d,2)$ is

$$\Delta(t)=-\frac{(1-t^{2d})(1-t)}{(1-t^d)(1-t^2)}=-\frac{1+t^d}{1+t}=-\frac{1+t^{k+2}}{1+t}=-1+t-t^2+\ldots+t^k-t^{k+1}. \qquad (3)$$

Note that (2) and (3) are equivalent. This completes the proof.

Theorem 3.3 (Ankaralioglu and Aydin, 2008): The cyclically presented group obtained from the word w generated with Dunwoody parameters $(1,b,0,d-2)$ has the cyclic presentation

$$< x_1, x_2, \ldots, x_d \mid x_{i+d-1}x_{i+d-3}x_{i+d-5}\ldots x_{i+2}x_i = x_{i+b}x_{i+b-2}\ldots x_{i+5}x_{i+3}x_{i+1} >,$$

when b is an odd positive integer and $d=2a+b+c$.

Theorem 3.4: The polynomial associated with the cyclically presented group obtained from the word w generated with Dunwoody parameters $(1,k,0,k)$ where k is an odd positive integer and $d=k+2$, coincides with the Alexander polynomial of the torus knot $K(d,2)$.

Proof: As stated in the proof of theorem 3.3, the defining word w corresponding to Dunwoody parameters $(1,k,0,k)$ has the following form

$$x_1^{-1}x_3^{-1}x_5^{-1}\dots x_k^{-1}x_{k+1}x_{k-1}x_{k-3}\dots x_2x_0, \qquad (4)$$

where $d=k+2$.

The corresponding polynomial with (4) is

$$f(t)=1-t+t^2-...-t^k+t^{k+1} \qquad (5)$$

or more generally

$$f(t)=\sum_{j=0}^{k+1}(-1)^j t^j \quad , \; j\equiv 0(\mathrm{mod}\, d).$$

According to the values $p=d$ and $q=2$, the Alexander polynomial of the torus knot $K(d,2)$ is

$$\Delta(t)=\frac{(1-t^{2d})(1-t)}{(1-t^d)(1-t^2)}=\frac{1+t^d}{1+t}=\frac{1+t^{k+2}}{1+t}= 1-t+t^2-...-t^k+t^{k+1}. \qquad (6)$$

Note that (5) and (6) are equivalent. The proof is complete.

Lemma 3.1 (Cattabriga and Mulazzani, 2005)**:**

a) $K(a,b,c,r)$ and $K(a,c,b,-r)$ are equivalent;

b) $K(a,0,c,r)$ and $K(a,c,0,r)$ are equivalent.

Observe that not every 4-tuple of non-negatif integers (a,b,c,r) determines a torus knot.

It can be easily seen that the polynomials associated with the cyclically presented groups obtained from the word w generated with Dunwoody parameters $(1,k,0,2)$ and $(1,0,k,2)$, and $(1,k,0,k)$ and $(1,0,k,k)$, where k is an odd positive integer, are equivalent and coincide with the Alexander polynomial of the torus knot $K(d,2)$.

Theorem 3.5 (Ankaralioglu and Aydin, 2008) : The cyclically presented group obtained from the word w generated with Dunwoody parameters $(a,1,0,a)$ has the cyclic presentation $< x_1, x_2, ..., x_d \mid x_{i+d-1}x_i = x_{i+d-2}x_{i+d-3}^{-1}x_{i+d-4}...x_{i+3}x_{i+2}^{-1}x_{i+1} >$,

when a is a positive integer and $d = 2a+b+c$.

Theorem 3.6: The polynomial associated with the cyclically presented group obtained from the word w generated with Dunwoody parameters $(\frac{k+1}{2},1,0,\frac{k+1}{2})$, where k is an odd positive integer and $d = k+2$, coincides with the Alexander polynomial of the torus knot $K(d,2)$.

Proof: As stated in the proof of theorem 3.5, the defining word w corresponding to Dunwoody parameters $(\frac{k+1}{2},1,0,\frac{k+1}{2})$ has the following form

$$x_1^{-1}x_2x_3^{-1}\dots x_{k-2}^{-1}x_{k-1}x_k^{-1}x_{k+1}x_0, \qquad (7)$$

where $d = k+2$.

The corresponding polynomial with (7) is

$$f(t) = 1 - t + t^2 - ... - t^k + t^{k+1} \qquad (8)$$

or more generally

$$f(t) = \sum_{j=0}^{k+1}(-1)^j t^j \quad , \quad j \equiv 0(\mathrm{mod}\, d).$$

According to the values $p = d$ and $q = 2$, the Alexander polynomial of the torus knot $K(d,2)$ is

$$\Delta(t) = \frac{(1-t^{2d})(1-t)}{(1-t^d)(1-t^2)} = \frac{1+t^d}{1+t} = \frac{1+t^{k+2}}{1+t} = 1 - t + t^2 - ... - t^k + t^{k+1}. \qquad (9)$$

Note that (8) and (9) are equivalent. We are done.

Theorem 3.7 (Ankaralioglu and Aydin, 2008) : The cyclically presented group obtained from the word w generated with Dunwoody parameters $(a,1,0,a+1)$ has the cyclic presentation

$$< x_1, x_2, \ldots, x_d \mid x_{i+1}x_{i+2}^{-1}x_{i+3}\ldots x_{i+d-4}x_{i+d-3}^{-1}x_{i+d-2} = x_i x_{i+d-1} >,$$

when a is a positive integer and $d = 2a+b+c$.

Theorem 3.8: The polynomial associated with the cyclically presented group obtained from the word w generated with Dunwoody parameters $(\frac{k+1}{2},1,0,\frac{k+3}{2})$, where k is an odd positive integer and $d = k+2$, coincides with the Alexander polynomial of the torus knot $K(d,2)$.

Proof: As stated in the proof of theorem 3.7, the defining word w corresponding to Dunwoody parameters $(\frac{k+1}{2},1,0,\frac{k+3}{2})$ has the following form

$$x_{k+1}^{-1}x_0^{-1}x_1x_2^{-1}x_3\ldots x_{k-2}x_{k-1}^{-1}x_k \,, \tag{10}$$

where $d = k+2$.

The corresponding polynomial with (10) is

$$f(t) = -1 + t - t^2 + \ldots + t^k - t^{k+1} \tag{11}$$

or more generally

$$f(t) = \sum_{j=0}^{k+1} (-1)^{j+1} t^j \quad , \; j \equiv 0 (\mathrm{mod}\, d).$$

According to the values $p = d$ and $q = 2$, the Alexander polynomial of the torus knot $K(d,2)$ is

$$\Delta(t) = -\frac{(1-t^{2d})(1-t)}{(1-t^d)(1-t^2)} = -\frac{1+t^d}{1+t} = -\frac{1+t^{k+2}}{1+t} = -1+t-t^2+\ldots+t^k-t^{k+1}. \tag{12}$$

Note that (11) and (12) are equivalent. This completes the proof.

It can be easily seen that the polynomials associated with the cyclically presented groups obtained from the word w generated with Dunwoody parameters $(\frac{k+1}{2},1,0,\frac{k+1}{2})$ and $(\frac{k+1}{2},0,1,\frac{k+1}{2})$, and

$(\frac{k+1}{2},1,0,\frac{k+3}{2})$ and $(\frac{k+1}{2},0,1,\frac{k+3}{2})$, where k is an odd positive integer, are equivalent and coincide with the Alexander polynomial of the torus knot $K(d,2)$.

Corollary 3.1: The polynomials associated with the cyclically presented groups obtained from the word w generated with Dunwoody parameters $(1,k,0,2),(1,0,k,2),(1,k,0,k),(1,0,k,k),(\frac{k+1}{2},1,0,\frac{k+1}{2}),(\frac{k+1}{2},0,1,\frac{k+1}{2}),(\frac{k+1}{2},1,0,\frac{k+3}{2}),(\frac{k+1}{2},0,1,\frac{k+3}{2})$, where k is an odd positive integer and $d=k+2$, coincide (up to sign) with the Alexander polynomial of the torus knot $K(d,2)$.

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