EÜFBED - Fen Bilimleri Enstitüsü Dergisi Cilt-Sayı: 4-2 Yıl: 2011 135-144

SOME CONVEX FAMILIES OF MULTIVALENTLY ANALYTIC FUNCTIONS DEFINED BY USING CERTAIN OPERATORS OF FRACTIONAL DERIVATIVES

KESİRSEL TÜREV OPERATÖRÜ KULLANILARAK TANIMLANAN ÇOKDEĞERLİ ANALİTİK FONKSİYONLARIN BAZI KONVEKS SINIFLARI

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Geliş Tarihi: 22 Haziran 2010 **Kabul Tarihi:** 8 Aralık 2011

ABSTRACT:

Making use of fractional derivative operator, we introduce a general class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$ of functions which are analytic and *p*-valent in the open unit disk *U*, and obtain a necessary and sufficient condition for a function to be in the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$, distortion bounds, inclusion property and the radii of *p*-valently close-to-convexity, *p*-valently starlikeness, *p*-valently convexity for this generalized class of p-valent function.

2000 Mathematics Subject Classification. Primary 30C45, 26A33, Secondary 33C05

Key Words and Phrases: *p*-valent, fractional integral, fractional derivative, distortion theorems, maximum modulus theorem, radii of *p*-valently close-to-convex, starlikeness and convexity.

ÖZET:

Kesirsel türev operatörü kullanılarak *U* birim diskinde analitik ve p-değerli fonksiyonların genel bir $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$ sınıfı tanımlandı ve bir fonksiyonun $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$ sınıfında olması için gerek ve yeter şart, büyüme ve bükülme

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sınırları, kapsama özelliği ve p-değerli fonksiyonların bu genelleştirilmiş sınıfı için yıldızılık, konvekslik, konvekse yakınlık yarıçapları elde edildi.

Anahtar kelimeler: *P*-değerli, kesirsel integral, kesirsel türev, büyüme teoremleri, maksimum modül teoremi, p-değerli konvekse yakınlık, yıldızıllık ve konvekslik yarıçapları.

1. INTRODUCTION

Let $T(n,p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \qquad \left(a_k \geq 0; n, p \in N = \{1, 2, 3, \ldots\}\right), \quad (1.1)$$

which are *analytic* and *p-valent* in the *open unit* disk

$$U = \{z; z \in C \; and \; |z| < 1\}$$

A function $f(z) \in T(n, p)$ is said to be *p-valently starlike of order* α if it satisfies the inequality

$$\Re\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha, \; \left(0 \leq \alpha < p; \, p \in N; \, z \in U\right). \quad (1.2)$$

A function $f(z) \in T(n, p)$ is said to be *p-valently convex of order* α if it satisfies the inequality

$$\Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha, \; \left(0 \leq \alpha < p; \, p \in N; \, z \in U\right). \quad (1.3)$$

Furthermore, a function $f(z) \in T(n, p)$ is said to be *p-valently close-to-convex of order* α if it satisfies the inequality

$$\Re\left\{z^{1-p} f'(z)\right\} > \alpha, \; \left(0 \leq \alpha < p; \, p \in N; \, z \in U\right). \quad (1.4)$$

It is easily seen that a function $f(z)$ is *p-valently convex of order* $\alpha\left(0 \leq \alpha < p; \, p \in N\right)$ *if and only if* $\frac{zf'(z)}{p}$ is *p-valently starlike of order* $\alpha\left(0 \leq \alpha < p; \, p \in N\right)$ (see, [7], [8], [13]).

The following definitions of fractional integral and fractional derivative by Owa (1978) (also Srivastava and Owa(1987), Owa and Srivastava(1992)) will be required in our investigation.

Definition 1 (Fractional Integral Operator). The fractional integral of order δ is defined, for a function $f(z)$, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi, \quad (\delta > 0), \tag{1.5}$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring $\log(z-\xi)$ to be real $(z-\xi) > 0$.

Definition 2 (Fractional Derivative Operator). The fractional derivative of order $q+\delta$ is defined, for a function, by

$$D_z^{q+\delta} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\delta)} \dfrac{d}{dz} \displaystyle\int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi, & (0 \le \delta < 1;\ q = 0) \\ \dfrac{d^q}{dz^q} D_z^{\delta} f(z), & (0 \le \delta < 1;\ q \in N), \end{cases} \tag{1.6}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\xi)^{\delta}$ is removed, as in Definition 1.

Using $D_z^{q+\delta} f$, we denote by $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$ the subclass of function $f(z)$ in the class $T(n, p)$ which also satisfy the inequality

$$\mathrm{Re}\left\{\Gamma(p-q-\delta+1) z^{q-p+\delta}\left[\beta D_z^{q+\delta} f(z) + \lambda z D_z^{q+\delta+1} f(z)\right]\right\} > \alpha, \quad (z \in U), \tag{1.7}$$

Where $p>q; 0\le\delta<1; \lambda\ge 0; \beta>\lambda\cdot\delta; 0<\alpha\le p![\beta+\lambda(p-q-\delta)]; n,p\in N; q\in N_0 = N\cup\{0\}$.

In view of the inequality (1.7), it is easily verified that the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$ can be identified with the class of

(i) p-valently close-to-convex functions of order $\alpha\ (0 < \alpha \le p;\quad p \in N)$ when

$$\lambda = \delta = 0, \quad q = 1, \text{ and } \beta = \frac{1}{(p-1)!};$$

(ii) close-to-convex functions of order $\alpha\ (0 < \alpha \le 1)$ when

$$\lambda = \delta = 0, \quad p = q = 1, \text{ and } \beta = 1.$$

We note that by giving specific values to *p, q, n,* δ, β and λ, we obtain the following important subclasses studied by various authors in earlier papers:

(i) $K_{p,n}^{q,\delta}(0,1,\alpha) \subseteq K_{p,n}^{q,\delta}(1,0,\alpha)$;

(ii) $K_{p,n}^{1,0}\left(\frac{\lambda}{(p-1)!}, \frac{1}{(p-1)!}, \alpha\right) \equiv C_n(p,\alpha,\lambda), \ (\lambda \ge 0; 0 < \alpha \le p(p\lambda - \lambda + 1); n, p \in N)$;

(iii) $K_{1,n}^{1,\delta}(\lambda, 1-\lambda, \alpha) \equiv F_\delta(n,\lambda,\alpha), \ (0 \le \delta < 1; 0 \le \lambda \le 1; 0 < \alpha \le 1; \delta + \alpha < 1; n \in N)$;

(iv) $K_{1,n}^{1,0}(\lambda, 1-\lambda, \alpha) \equiv F_\lambda(1,\lambda,\alpha), \ (0 \le \delta < 1; 0 \le \lambda \le 1)$.

The class $C_n(p,\alpha,\lambda)$ was studied by Chen *et al.*, [5], the class $F_\delta(n,\lambda,\alpha)$ was studied by Altıntaş *et al,*.[1], and the class $F_\lambda(\alpha)$ was studied by Bhoosnurmath and Swamy [3] and see, for example Chen *et al.*, [4], Chen *et al.*, [6], Irmak [9]; Irmak *et al.*, [10] and Altıntaş *et al.*, [2]).

2. Basic Properties of the Class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$

A necessary and sufficient condition for a function $f(z) \in T(n,p)$ to be in the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$ is provided by following theorem.

Theorem 1. Let a function *f* (*z*) is defined by (1.1) be in the class *T*(*n*,*p*). Then, the function *f* (*z*) belongs to the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$ *if and only if*

$$\sum_{k=n+p}^{\infty} \frac{k!\left[\beta + \lambda(k-q-\delta)\right]}{\Gamma(k-q-\delta+1)} a_k \le \frac{p!\left[\beta + \lambda(p-q-\delta)\right] - \alpha}{\Gamma(p-q-\delta+1)}, \tag{2.1}$$

where $(p > q; 0 \le \delta < 1; \lambda \ge 0; \beta > \lambda\delta; 0 < \alpha \le p![\beta + \lambda(p-q-\delta)]; p \in N; q \in N_0 = N \cup \{0\})$.

The result is sharp for the function $f(z)$ given by
$$f(z)=z^p-\frac{\Gamma(n+p-q-\delta+1)\{p![\beta+\lambda(p-q-\delta)]-\alpha\}}{(n+p)!\Gamma(p-q-\delta+1)[\beta+\lambda(n+p-q-\delta)]}z^{n+p},$$
(2.2)

where $p>q; n,p\in N; q\in N_0$.

Proof. Suppose that the function $f(z)$, given by (1.1), is in the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$. It is easily seen from the definition of fractional derivatives that

$$D_z^{q+\delta}f(z)=\frac{p!}{\Gamma(p-q-\delta+1)}z^{p-q-\delta}-\sum_{k=n+p}^{\infty}\frac{k!}{\Gamma(k-q-\delta+1)}z^{k-q-\delta}a_k, \qquad (2.3)$$

where $p>p; 0\le\delta<1; n,p\in N; q\in N_0$

Making use of (1.7) and (2.3), choosing values of z on the real axis, and then $z\to 1^-$ through real values, we arrive at the assertion (2.1) of Theorem 1.

Conversely, we suppose that the inequality (2.1) holds true and let

$$z\in\partial U=\{z;\ z\in C\ and\ |z|=1\}.$$

Then, we find from (1.1), (1.7) and (2.3) that

$$\left|\Gamma(p-q-\delta+1)z^{q-p+\delta}\{\beta D_z^{q+\delta}f(z)+\lambda z D_z^{q+\delta+1}f(z)\}-p!\,[\beta+\lambda(p-q-\delta)]\right|$$
$$\le\sum_{k=n+p}^{\infty}\frac{k![\beta+\lambda(k-q-\delta)]}{\Gamma(k-q-\delta+1)}a_k\le\frac{p![\beta+\lambda(p-q-\delta)]-\alpha}{\Gamma(p-q-\delta+1)} \qquad (2.4)$$

where $z\in\partial U; p>q, 0\le\delta<1; \lambda\ge 0, \beta>\lambda\cdot\delta; 0<\alpha\le p![\beta+\lambda(p-q-\delta)]; n,p\in N; q\in N_0$,

Thus, by the maximum modulus theorem, we conclude from (2.4) that $f(z)\in K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$.

Finally, by observing the function $f(z)$ given by (2.2) is indeed an extremal function for assertion (2.1), we complete the proof of Theorem 1.

Next, by appealing to the assertion (2.1) of Theorem 1, it is not difficult to prove the following inclusion property for the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$. The details may be omitted.

Theorem 2. Let a function $f(z)$ is defined by (1.1) and the function *g(z)* is defined by

$$g(z)=z^p-\sum_{k=n+p}^{\infty}b_k z^k, \qquad (b_k\geq 0;\, n,p\in N) \tag{2.5}$$

be in the same class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$. Then the function *h(z)* defined by

$$h(z)=(1-\mu)f(z)+\mu\, g(z)=z^p-\sum_{k=n+p}^{\infty}c_k z^k, \tag{2.6}$$

where $C_K=(1-\mu)a_k+\mu b_k\geq 0;\, 0\leq\mu\leq 1;\, n,p\in N$ is also in the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$.

Theorem 3. If $f(z)\in K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$, then

$$\left(p!-\frac{p![\beta+\lambda(p-q-\delta)]-\alpha}{\beta+\lambda(n+p-q-\delta)}|z|^n\right)\frac{|z|^{p-q-\delta}}{\Gamma(p-q-\delta+1)}\leq\left|D_z^{q+\delta}f(z)\right|$$
$$\leq\left(p!+\frac{p![\beta+\lambda(p-q-\delta)]-\alpha}{\beta+\lambda(n+p-q-\delta)}|z|^n\right)\frac{|z|^{p-q-\delta}}{\Gamma(p-q-\delta+1)}, \tag{2.7}$$

where
$z\in U; p>q; 0\leq\delta<1; \lambda\geq 0; \beta>\lambda\cdot\delta; 0<\alpha\leq p![\beta+\lambda(p-q-\delta)]; n,p\in N; q\in N_0$, and the fractional operator $D_z^{q+\delta}f(z)$ given by (2.3). The result is sharp for the function $f(z)$ given by (2.2).

Proof. Under the hypothesis of Theorem 3, we find from the assertion (2.1) of Theorem 1, that

$$[\beta+\lambda(n+p-q-\delta)]\sum_{k=n+p}^{\infty}\frac{k!}{\Gamma(k-q-\delta+1)}a_k\leq\sum_{k=n+p}^{\infty}\frac{k![\beta+\lambda(k-q-\delta)]}{\Gamma(p-q-\delta+1)}a_k,$$

which readily yields

$$\sum_{k=n+p}^{\infty}\frac{k!}{\Gamma(k-q-\delta+1)}a_k \leq \frac{p![\beta+\lambda(k-q-\delta)]-\alpha}{\Gamma(p-q-\delta+1)},\ (p>q;n,p\in N;q\in N_0) \quad (2.8)$$

Now the inequality (2.7) would readily when we make use of (2.8) in conjunction with the series expansion for $D_z^{q+\delta}f(z)\quad (p>q;\, n,p\in N;\, q\in N_0)$ given by (2.3).

Theorem 4. If $f(z)\in K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$, then

$$\left[\frac{|z|^{p-q-\delta}}{\Gamma(p-q-\delta+1)}-\frac{\Gamma(n+p-q-\delta+1)[\beta+\lambda(p-q-\delta)]-\alpha}{\Gamma(p-q-\delta+1)\Gamma(n+p-q-\delta+1)[\beta+\lambda(n+p-q-\delta)]}|z|^n\right]\cdot p!|z|^{p-q+\delta}$$

$$\leq\left|D_z^{q+\delta}f(z)\right| \quad (2.9)$$

$$\leq p!|z|^{p-q+\delta}\left[\frac{|z|^{p-q-\delta}}{\Gamma(p-q-\delta+1)}+\frac{\Gamma(n+p-q-\delta+1)[\beta+\lambda(p-q-\delta)]-\alpha}{\Gamma(p-q-\delta+1)\Gamma(n+p-q-\delta+1)[\beta+\lambda(n+p-q-\delta)]}|z|^n\right]$$

where $z\in U;\, p>p;\, n,p\in N;\, q\in N_0$ and the fractional integral operator $D_z^{q-\delta}f(z)$ for a function $f(z)$ is given by (1.1) defined by

$$D_z^{q-\delta}f(z)=\frac{p!}{\Gamma(p-q+\delta+1)}z^{p-q+\delta}-\sum_{k=n+p}^{\infty}\frac{k!}{\Gamma(k-q+\delta+1)}a_k z^{k-q+\delta}, \quad (2.10)$$

Each of these results is sharp for the function $f(z)$ given by (2.2).

Proof. Under the hypothesis of Theorem 4, we find from the assertion of Theorem 1, that

$$\sum_{k=n+p}^{\infty}k!a_k \leq \frac{p!\Gamma(n+p-q-\delta+1)\{[\beta+\lambda(p-q-\delta)]-\alpha\}}{\Gamma(p-q-\delta+1)[\beta+\lambda(n+p-q-\delta)]}, \quad (2.11)$$

where $p>p;\, 0\leq\delta<1;\, n,p\in N;\, q\in N_0$.

Now the inequalities (2.9) follows readily when we use (2.11) in conjunction with the series expansion for $D_z^{q-\delta}f(z)\quad (\delta>q;\, q\in N_0)$ given by (2.10).

Finally, we determine the radii of *p*-valently starlikenes, *p*-valently convexity, and *p*-valently close-to-convexity, for function in the class $K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$.

Theorem 5. If $f(z)\in K_{p,n}^{q,\delta}(\lambda,\beta,\alpha)$, then $f(z)$ is *p*-valently starlike of order $\xi\ (0\le\xi<p;\ p\in N)\ in\ |z|<r_1$, *p*-valently convex of order $\xi\ (0\le\xi<p;\ p\in N)\ in\ |z|<r_2$, and *p*-valently close-to-convex of order $\xi\ (0\le\xi<p;\ p\in N)\ in\ |z|<r_3$, where

$$r_1\le\inf_k\left[\frac{(p-\xi)k!\,\Gamma(p-q-\delta+1)[\beta+\lambda(k-q-\delta)]}{(k-\xi)\Gamma(k-q-\delta+1)\{p![\beta+\lambda(p-q-\delta)]\}}\right]^{\frac{1}{k-p}},\quad(2.12)$$

$$r_2\le\inf_k\left[\frac{p(p-\xi)(k-1)!\,\Gamma(p-q-\delta+1)[\beta+\lambda(k-q-\delta)]}{(k-\xi)\Gamma(k-q-\delta+1)\{p![\beta+\lambda(p-q-\delta)]\}}\right]^{\frac{1}{k-p}}\quad(2.13)$$

and

$$r_3\le\inf_k\left[\frac{(p-\xi)(k-1)!\,\Gamma(p-q-\delta+1)[\beta+\lambda(k-q-\delta)]}{\Gamma(k-q-\delta+1)\{p![\beta+\lambda(p-q-\delta)]\}}\right]^{\frac{1}{k-p}}\quad(2.14)$$

for $k\ge n+p;\ n,p\in N;0<\alpha\le p;\ 0\le\xi<p.$

Each of these result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{\Gamma(n+p-q-\delta+1)\{p![\beta+\lambda(p-q-\delta)]-\alpha\}}{(n+p)!\,\Gamma(p-q-\delta+1)[\beta+\lambda(n+p-q-\delta)]}z^{n+p},\quad(2.15)$$

where $k\ge n+p;\ n,p\in N;\ q\in N_0.$

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-p\right|<p-\xi,\quad(|z|<r_1;\ 0\le\xi<p;\ z\in U;\ p\in N),\quad(2.16)$$

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|<p-\xi,\quad(|z|<r_2;\ 0\le\xi<p;\ z\in U;\ p\in N)\ (2.17)$$

and

$$\left|z^{1-p} f'(z) - p\right| < p - \xi, \quad \left(|z| < r_3;\, 0 \le \xi < p;\, z \in U;\, p \in N\right), \qquad (2.18)$$

for a function $f(z) \in K_{p,n}^{q,\delta}(\lambda, \beta, \alpha)$, where $r_1, r_2, and\ r_3$ are defined by (2.12), (2.13), and (2.14) respectively. The details involved are fairly straightforward and may be omitted.

REFERENCES

Altıntaş, O., Irmak H. and Srivastava, H. M. 1995. A Subclass of Analytic Functions Defined by Using Certain Operators of Fractional Calculus, *Comput. Math. Appl*. 30, 1-9.

Altıntaş, O., Irmak H. and Srivastava, H. M. 1995. fractional Calculus and Certain Starlike Functions With Negative Coefficients, *Comput Math. Appl.* 30 (2), 9-15.

Bhoosnurmath, S. S. and Swamy, S. R. 1985. Certain Classes of Analytic Functions With Negative Coefficients, *Indian J. Math.* 27, 89-98.

Chen, M. -P., Irmak H. and Srivastava, H. M. 1996. Some Multivalently Functions With Negative Coefficients Defined by Using a Differantial Operator, *Pan Amer. Math. J.*6 (2), 55-64.

Chen, M. -P., Irmak H. and Srivastava, H. M. 1997. Some Families of Multivalently Analytic Functions With Negative Coefficients, *J. Math. Anal. Appl.* 214 , 674-690.

Chen, M. -P., Irmak, H. and Srivastava, H. M. 1998. A Certain Subclass of Analytic Functions Involving Operators of Fractional Calculus, *Comput. Math. Appl.* 35 (5) , 83-91.

Duren, P. L. 1983. *Univalent Functions*, Grunlehren der Mathematischen Wissenschaften, Bd. 259, Springer-Verlag, New York, Berlin, Heiderberg, and Tokyo.

Goodman, A. W. 1983. *Univalent Functions,* Vols. I and II, Polygonal Publishing Conpany Washington, New Jersey.

Irmak, H. 1995. Certain Subclasses of P-Valently Starlike Functions With Negative Coefficients, *Bull. Calcutta Math. Soc.* 87, 589-598.

Irmak, H., Lee, S. H. and Cho, N. E. 1997. Some Multivalently Analytic Functions With Negative Coefficients and Their Subclasses Defined by Using a Differantial Operators, *Kyungpook Math. J.* 37 (1), 43-51.

Owa, S. 1978. On the Distortion Theorems. I, *Kyungpook Maht. J.*, 18,53-59.

Owa, S. and Srivastava, H. M. 1987. Univalent and Starlike Generalized Hypergeometric Function, *Canad. J. Math.*, 39, 1057-1077.

Srivastava, H. M. and Owa, S. 1992. (Editors), *Current Topics in Analitic Function Theory*, Word Scientific Publishing Company, Singapure, New Jersey, London, and HongKong.