Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (2) (2019), 472 – 478
DOI : 10.15672/HJMS.2017.531

Research Article

On generalized autocommutativity degree of finite groups

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Abstract

Let H be a subgroup of a finite group G and $\operatorname{Aut}(G)$ be the automorphism group of G. In this paper we introduce and study the probability that the autocommutator of a randomly chosen pair of elements, one from H and the other from $\operatorname{Aut}(G)$, is equal to a given element of G.

Mathematics Subject Classification (2010). 20D60, 20P05, 20F28

Keywords. automorphism group, autocommutativity degree, autoisoclinism

1. Introduction

Throughout the paper H denotes a subgroup of a finite group G and $\operatorname{Aut}(G)$ denotes automorphism group of G. The autocommutativity degree of G, denoted by $\Pr(G, \operatorname{Aut}(G))$, is the probability that an automorphism fixes an element of G. In other words,

$$\Pr(G, \operatorname{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \operatorname{Aut}(G) : [x, \alpha] = 1\}|}{|G||\operatorname{Aut}(G)|}$$

where $[x, \alpha] = x^{-1}\alpha(x)$ is the autocommutator of x and α. The study of autocommutativity degree of finite groups was initiated by Sherman [10] in 1975. Many results on $\Pr(G, \operatorname{Aut}(G))$, including some characterizations of G in terms of $\Pr(G, \operatorname{Aut}(G))$, can be found in [1, 3]. In the year 2015, Rismanchian and Sepehrizadeh [9] generalized the concept of autocommutativity degree and studied relative autocommutativity degree of H, that is the probability that an automorphism of G fixes an element of H. However in the year 2011, Moghaddam et al. [8] also studied this notion. We write $\Pr(H, \operatorname{Aut}(G))$ to denote the relative autocommutativity degree of H. Recently, we have obtained several new results on $\Pr(H, \operatorname{Aut}(G))$ in [2]. In this paper, we introduce a new probability concept called the generalized relative autocommutativity degree of H given by the following ratio

$$\Pr_g(H, \operatorname{Aut}(G)) = \frac{|\{(x, \alpha) \in H \times \operatorname{Aut}(G) : [x, \alpha] = g\}|}{|H||\operatorname{Aut}(G)|} \tag{1.1}$$

where g is an element of G. In other words $\Pr_g(H, \operatorname{Aut}(G))$ is the probability that the autocommutator of a randomly chosen pair of elements, one from H and the other from $\operatorname{Aut}(G)$, is equal to a given element $g \in G$. Clearly, if $g = 1$ (the identity element of G) then $\Pr_g(H, \operatorname{Aut}(G)) = \Pr(H, \operatorname{Aut}(G))$. In the forthcoming sections, we obtain some computing

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Received: 02.06.2017; Accepted: 08.11.2017

formulae and bounds for $\Pr_g(H, \operatorname{Aut}(G))$. We also obtain some characterizations of groups through $\Pr_g(H, \operatorname{Aut}(G))$.

Let $S(H, \operatorname{Aut}(G)) = \{[x, \alpha] : x \in H \text{ and } \alpha \in \operatorname{Aut}(G)\}$ and $[H, \operatorname{Aut}(G)]$ be the subgroup generated by $S(H, \operatorname{Aut}(G))$. Let $L(H, \operatorname{Aut}(G)) = \{x \in H : [x, \alpha] = 1 \text{ for all } \alpha \in \operatorname{Aut}(G)\}$ and $L(G) = L(G, \operatorname{Aut}(G))$, the absolute center of G defined in [5]. Clearly, $L(H, \operatorname{Aut}(G))$ is a normal subgroup of H contained in $H \cap Z(G)$. Let $C_{\operatorname{Aut}(G)}(x) = \{\alpha \in \operatorname{Aut}(G) : \alpha(x) = x\}$ for $x \in G$ and $C_{\operatorname{Aut}(G)}(H) = \{\alpha \in \operatorname{Aut}(G) : \alpha(x) = x \text{ for all } x \in H\}$. Then $C_{\operatorname{Aut}(G)}(x)$ is a subgroup of $\operatorname{Aut}(G)$ and $C_{\operatorname{Aut}(G)}(H) = \bigcap_{x \in H} C_{\operatorname{Aut}(G)}(x)$. Note that if $g \notin S(H, \operatorname{Aut}(G))$ then $\Pr_g(H, \operatorname{Aut}(G)) = 0$, therefore throughout the paper we consider $g \in S(H, \operatorname{Aut}(G))$.

2. Some computing formulae

We begin with the following results.

Proposition 2.1. *Let H be a subgroup of G. If $g \in G$ then*

$$\Pr_{g^{-1}}(H, \operatorname{Aut}(G)) = \Pr_g(H, \operatorname{Aut}(G)).$$

Proof. Let $A = \{(x, \alpha) \in H \times \operatorname{Aut}(G) : [x, \alpha] = g\}$ and $B = \{(y, \beta) \in H \times \operatorname{Aut}(G) : [y, \beta] = g^{-1}\}$. Then $(x, \alpha) \mapsto (\alpha(x), \alpha^{-1})$ gives a bijection between A and B. Therefore, $|A| = |B|$ and hence the result follows from (1.1). □

Proposition 2.2. *Let G_1 and G_2 be two finite groups such that $\gcd(|G_1|, |G_2|) = 1$. Let H_1 and H_2 be subgroups of G_1 and G_2 respectively. If $(g_1, g_2) \in G_1 \times G_2$ then*

$$\Pr_{(g_1,g_2)}(H_1 \times H_2, \operatorname{Aut}(G_1 \times G_2)) = \Pr_{g_1}(H_1, \operatorname{Aut}(G_1))\Pr_{g_2}(H_2, \operatorname{Aut}(G_2)).$$

Proof. Let

$$\begin{aligned}
\mathcal{X} &= \{((x, y), \alpha_{G_1 \times G_2}) \in (H_1 \times H_2) \times \operatorname{Aut}(G_1 \times G_2) : \\
&\qquad\qquad [(x, y), \alpha_{G_1 \times G_2}] = (g_1, g_2)\},\\
\mathcal{Y} &= \{(x, \alpha_{G_1}) \in H_1 \times \operatorname{Aut}(G_1) : [x, \alpha_{G_1}] = g_1\} \text{ and}\\
\mathcal{Z} &= \{(y, \alpha_{G_2}) \in H_2 \times \operatorname{Aut}(G_2) : [y, \alpha_{G_2}] = g_2\}.
\end{aligned}$$

Since $\gcd(|G_1|, |G_2|) = 1$, by [6, Lemma 2.1], we have $\operatorname{Aut}(G_1 \times G_2) = \operatorname{Aut}(G_1) \times \operatorname{Aut}(G_2)$. Therefore, for every $\alpha_{G_1 \times G_2} \in \operatorname{Aut}(G_1 \times G_2)$ there exist unique $\alpha_{G_1} \in \operatorname{Aut}(G_1)$ and $\alpha_{G_2} \in \operatorname{Aut}(G_2)$ such that $\alpha_{G_1 \times G_2} = \alpha_{G_1} \times \alpha_{G_2}$, where $\alpha_{G_1} \times \alpha_{G_2}((x, y)) = (\alpha_{G_1}(x), \alpha_{G_2}(y))$ for all $(x, y) \in H_1 \times H_2$. Also, for all $(x, y) \in H_1 \times H_2$, we have $[(x, y), \alpha_{G_1 \times G_2}] = (g_1, g_2)$ if and only if $[x, \alpha_{G_1}] = g_1$ and $[y, \alpha_{G_2}] = g_2$. These lead to show that $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$. Therefore

$$\frac{|\mathcal{X}|}{|H_1 \times H_2||\operatorname{Aut}(G_1 \times G_2)|} = \frac{|\mathcal{Y}|}{|H_1||\operatorname{Aut}(G_1)|} \cdot \frac{|\mathcal{Z}|}{|H_2||\operatorname{Aut}(G_2)|}.$$

Hence, the result follows from (1.1). □

In the year 1940, Hall [4] introduced the concept of isoclinism between two groups. Following Hall, Moghaddam et al. [7] have defined autoisoclinism between two groups, in the year 2013. Recently in [2], we generalize the notion of autoisoclinism between two groups. Let H_1 and H_2 be subgroups of the groups G_1 and G_2 respectively. The pairs (H_1, G_1) and (H_2, G_2) are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{H_1}{L(H_1, \operatorname{Aut} G_1)} \to \frac{H_2}{L(H_2, \operatorname{Aut}(G_2))}$, $\beta : [H_1, \operatorname{Aut}(G_1)] \to [H_2, \operatorname{Aut}(G_2)]$ and $\gamma : \operatorname{Aut}(G_1) \to \operatorname{Aut}(G_2)$ such that the following diagram commutes

$$\begin{array}{ccc}
\frac{H_1}{L(H_1, \operatorname{Aut}(G_1))} \times \operatorname{Aut}(G_1) & \xrightarrow{\psi \times \gamma} & \frac{H_2}{L(H_2, \operatorname{Aut}(G_2))} \times \operatorname{Aut}(G_2) \\
\Big\downarrow a_{(H_1, \operatorname{Aut}(G_1))} & & \Big\downarrow a_{(H_2, \operatorname{Aut}(G_2))} \\
[H_1, \operatorname{Aut}(G_1)] & \xrightarrow{\beta} & [H_2, \operatorname{Aut}(G_2)]
\end{array}$$

where the maps $a_{(H_i,\mathrm{Aut}(G_i))} : \frac{H_i}{L(H_i,\mathrm{Aut}(G_i))} \times \mathrm{Aut}(G_i) \to [H_i, \mathrm{Aut}(G_i)]$, for $i = 1, 2$, are given by

$$a_{(H_i,\mathrm{Aut}(G_i))}(x_i L(H_i, \mathrm{Aut}(G_i)), \alpha_i) = [x_i, \alpha_i].$$

Such a pair $(\psi\times\gamma, \beta)$ is said to be an autoisoclinism between the pairs of groups (H_1, G_1) and (H_2, G_2). We have the following generalization of [3, Theorem 5.1] and [9, Lemma 2.5].

Theorem 2.3. *Let G_1 and G_2 be two finite groups with subgroups H_1 and H_2 respectively. If $(\psi\times\gamma, \beta)$ is an autoisoclinism between the pairs (H_1, G_1) and (H_2, G_2) then, for $g \in G_1$,*

$$\mathrm{Pr}_{\beta(g)}(H_2, \mathrm{Aut}(G_2)) = \mathrm{Pr}_g(H_1, \mathrm{Aut}(G_1)).$$

Proof. Let $\mathcal{S}_g = \{(x_1 L(H_1, \mathrm{Aut}(G_1)), \alpha_1) \in \frac{H_1}{L(H_1,\mathrm{Aut}(G_1))} \times \mathrm{Aut}(G_1) : [x_1, \alpha_1] = g\}$ and $\mathcal{T}_{\beta(g)} = \{(x_2 L(H_2, \mathrm{Aut}(G_2)), \alpha_2) \in \frac{H_2}{L(H_2,\mathrm{Aut}(G_2))} \times \mathrm{Aut}(G_2) : [x_2, \alpha_2] = \beta(g)\}$. Since (H_1, G_1) is autoisoclinic to (H_2, G_2) we have $|\mathcal{S}_g| = |\mathcal{T}_{\beta(g)}|$. Again, it is clear that

$$|\{(x_1, \alpha_1) \in H_1 \times \mathrm{Aut}(G_1) : [x_1, \alpha_1] = g\}| = |L(H_1, \mathrm{Aut}(G_1))||\mathcal{S}_g| \tag{2.1}$$

and

$$|\{(x_2, \alpha_2) \in H_2 \times \mathrm{Aut}(G_2) : [x_2, \alpha_2] = \beta(g)\}| = |L(H_2, \mathrm{Aut}(G_2))||\mathcal{T}_{\beta(g)}|. \tag{2.2}$$

Hence, the result follows from (1.1), (2.1) and (2.2). □

Note that $\mathrm{Aut}(G)$ acts on G by the action $(\alpha, x) \mapsto \alpha(x)$ where $\alpha \in \mathrm{Aut}(G)$ and $x \in G$. Let $\mathrm{orb}(x) = \{\alpha(x) : \alpha \in \mathrm{Aut}(G)\}$ be the orbit of $x \in G$. Then by orbit-stabilizer theorem, we have

$$|\mathrm{orb}(x)| = \frac{|\mathrm{Aut}(G)|}{|C_{\mathrm{Aut}(G)}(x)|}.$$

Now we obtain the following computing formula for $\mathrm{Pr}_g(H, \mathrm{Aut}(G))$ in terms of the order of $C_{\mathrm{Aut}(G)}(x)$ and $\mathrm{orb}(x)$.

Theorem 2.4. *Let H be a subgroup of G. If $g \in G$ then*

$$\mathrm{Pr}_g(H, \mathrm{Aut}(G)) = \frac{1}{|H||\mathrm{Aut}(G)|} \sum_{\substack{x\in H\\ xg\in\mathrm{orb}(x)}} |C_{\mathrm{Aut}(G)}(x)| = \frac{1}{|H|} \sum_{\substack{x\in H\\ xg\in\mathrm{orb}(x)}} \frac{1}{|\mathrm{orb}(x)|}.$$

Proof. Let $T_{x,g}(H, G) = \{\alpha \in \mathrm{Aut}(G) : [x, \alpha] = g\}$ for any $x \in H$. Then $T_{x,g}(H, G) \neq \emptyset$ if and only if $xg \in \mathrm{orb}(x)$. We also have

$$\{(x, \alpha) \in H \times \mathrm{Aut}(G) : [x, \alpha] = g\} = \sqcup_{x\in H}(\{x\} \times T_{x,g}(H, G)),$$

where $\sqcup$ represents the union of disjoint sets. Therefore, by (1.1), we have

$$|H||\mathrm{Aut}(G)|\mathrm{Pr}_g(H, \mathrm{Aut}(G)) = |\sqcup_{x\in H}(\{x\} \times T_{x,g}(H, G))| = \sum_{x\in H} |T_{x,g}(H, G)|. \tag{2.3}$$

Let $\sigma \in T_{x,g}(H, G)$ and $\beta \in \sigma C_{\mathrm{Aut}(G)}(x)$. Then $\beta = \sigma\alpha$ for some $\alpha \in C_{\mathrm{Aut}(G)}(x)$. We have

$$[x, \beta] = [x, \sigma\alpha] = x^{-1}\sigma(\alpha(x)) = [x, \sigma] = g.$$

Therefore, $\beta \in T_{x,g}(H, G)$ and so $\sigma C_{\mathrm{Aut}(G)}(x) \subseteq T_{x,g}(H, G)$. Again, let $\gamma \in T_{x,g}(H, G)$ then $\gamma(x) = xg$. We have $\sigma^{-1}\gamma(x) = \sigma^{-1}(xg) = x$ and so $\sigma^{-1}\gamma \in C_{\mathrm{Aut}(G)}(x)$. Therefore, $\gamma \in \sigma C_{\mathrm{Aut}(G)}(x)$ which gives $T_{x,g}(H, G) \subseteq \sigma C_{\mathrm{Aut}(G)}(x)$. Thus, $\sigma C_{\mathrm{Aut}(G)}(x) = T_{x,g}(H, G)$ and hence

$$|T_{x,g}(H, G)| = |C_{\mathrm{Aut}(G)}(x)| = \frac{|\mathrm{Aut}(G)|}{|\mathrm{orb}(x)|}. \tag{2.4}$$

Therefore, the result follows from (2.3) and (2.4). □

Putting $g = 1$ in Theorem 2.4 we get the following corollary.

Corollary 2.5. *Let H be a subgroup of G. Then*

$$\Pr(H, \operatorname{Aut}(G)) = \frac{1}{|H||\operatorname{Aut}(G)|}\sum_{x\in H}|C_{\operatorname{Aut}(G)}(x)| = \frac{|\operatorname{orb}(H)|}{|H|}$$

where $\operatorname{orb}(H) = \{\operatorname{orb}(x) : x \in H\}$.

As an application of Theorem 2.4 we have the following result.

Proposition 2.6. *Let H be a subgroup of G. If $\operatorname{orb}(x) = x[H, \operatorname{Aut}(G)]$ for all $x \in H \setminus L(H, \operatorname{Aut}(G))$ then*

$$\Pr_g(H, \operatorname{Aut}(G)) = \begin{cases} \dfrac{1}{|[H, \operatorname{Aut}(G)]|}\left(1 + \dfrac{|[H, \operatorname{Aut}(G)]| - 1}{|H : L(H, \operatorname{Aut}(G))|}\right), & \text{if } g = 1 \\ \dfrac{1}{|[H, \operatorname{Aut}(G)]|}\left(1 - \dfrac{1}{|H : L(H, \operatorname{Aut}(G))|}\right), & \text{if } g \neq 1. \end{cases}$$

Proof. If $g = 1$ then the result follows from [2, Proposition 3.4]. If $g \neq 1$, we have $xg \notin \operatorname{orb}(x)$ for all $x \in L(H, \operatorname{Aut}(G))$. Again, since $g \in S(H, \operatorname{Aut}(G)) \subseteq [H, \operatorname{Aut}(G)]$ therefore $xg \in x[H, \operatorname{Aut}(G)] = \operatorname{orb}(x)$ for all $x \in H \setminus L(H, \operatorname{Aut}(G))$. Now from Theorem 2.4 we have

$$\begin{aligned} \Pr_g(H, \operatorname{Aut}(G)) &= \frac{1}{|H|}\sum_{\substack{x\in H\setminus L(H,\operatorname{Aut}(G))\\ xg\in\operatorname{orb}(x)}}\frac{1}{|\operatorname{orb}(x)|} \\ &= \frac{1}{|H|}\sum_{\substack{x\in H\setminus L(H,\operatorname{Aut}(G))\\ xg\in\operatorname{orb}(x)}}\frac{1}{[H, \operatorname{Aut}(G)]} \\ &= \frac{1}{|[H, \operatorname{Aut}(G)]|}\left(1 - \frac{1}{|H : L(H, \operatorname{Aut}(G))|}\right). \end{aligned}$$

□

3. Various bounds

In this section, we obtain various bounds for $\Pr_g(H, \operatorname{Aut}(G))$. We begin with the following lower bounds.

Proposition 3.1. *Let H be a subgroup of G. Then, for $g \in G$, we have*

$$\Pr_g(H, \operatorname{Aut}(G)) \geq \begin{cases} \dfrac{|L(H, \operatorname{Aut}(G))|}{|H|} + \dfrac{|C_{\operatorname{Aut}(G)}(H)|(|H| - |L(H, \operatorname{Aut}(G))|)}{|H||\operatorname{Aut}(G)|}, & \text{if } g = 1 \\ \dfrac{|L(H, \operatorname{Aut}(G))||C_{\operatorname{Aut}(G)}(H)|}{|H||\operatorname{Aut}(G)|}, & \text{if } g \neq 1. \end{cases}$$

Proof. Let $\mathcal{C}$ denotes the set $\{(x, \alpha) \in H \times \operatorname{Aut}(G) : [x, \alpha] = g\}$.

If $g = 1$ then $(L(H, \operatorname{Aut}(G)) \times \operatorname{Aut}(G)) \cup (H \times C_{\operatorname{Aut}(G)}(H))$ is a subset of $\mathcal{C}$ and $|(L(H, \operatorname{Aut}(G)) \times \operatorname{Aut}(G)) \cup (H \times C_{\operatorname{Aut}(G)}(H))| = |L(H, \operatorname{Aut}(G))||\operatorname{Aut}(G)| + |C_{\operatorname{Aut}(G)}(H)||H| - |L(H, \operatorname{Aut}(G))||C_{\operatorname{Aut}(G)}(H)|$. Hence, the result follows from (1.1).

If $g \neq 1$ then $\mathcal{C}$ is non-empty since $g \in S(H, \operatorname{Aut}(G))$. Let $(y, \beta) \in \mathcal{C}$ then $(y, \beta) \notin L(H, \operatorname{Aut}(G)) \times C_{\operatorname{Aut}(G)}(H)$ otherwise $[y, \beta] = 1$. It is easy to see that the coset $(y, \beta)(L(H, \operatorname{Aut}(G)) \times C_{\operatorname{Aut}(G)}(H))$ having order $|L(H, \operatorname{Aut}(G))||C_{\operatorname{Aut}(G)}(H)|$ is a subset of $\mathcal{C}$. Hence, the result follows from (1.1). □

Proposition 3.2. *Let H be a subgroup of G. If $g \in G$ then*

$$\Pr_g(H, \operatorname{Aut}(G)) \leq \Pr(H, \operatorname{Aut}(G)).$$

The equality holds if and only if $g = 1$.

Proof. By Theorem 2.4, we have

$$\begin{aligned}\Pr_g(H, \operatorname{Aut}(G)) &= \frac{1}{|H||\operatorname{Aut}(G)|} \sum_{\substack{x\in H\\ xg\in \operatorname{orb}(x)}} |C_{\operatorname{Aut}(G)}(x)| \\ &\leq \frac{1}{|H||\operatorname{Aut}(G)|}\sum_{x\in H}|C_{\operatorname{Aut}(G)}(x)| = \Pr(H, \operatorname{Aut}(G)).\end{aligned}$$

Clearly the equality holds if and only if $g = 1$. □

Proposition 3.3. *Let H be a subgroup of G. Let $g \in G$ and p the smallest prime dividing $|\operatorname{Aut}(G)|$. If $g \neq 1$ then*

$$\Pr_g(H, \operatorname{Aut}(G)) \leq \frac{|H| - |L(H, \operatorname{Aut}(G))|}{p|H|} < \frac{1}{p}.$$

Proof. By Theorem 2.4, we have

$$\Pr_g(H, \operatorname{Aut}(G)) = \frac{1}{|H|} \sum_{\substack{x\in H\setminus L(H,\operatorname{Aut}(G))\\ xg\in \operatorname{orb}(x)}} \frac{1}{|\operatorname{orb}(x)|} \tag{3.1}$$

noting that for $x \in L(H, \operatorname{Aut}(G))$ we have $xg \notin \operatorname{orb}(x)$. Also, for $x \in H \setminus L(H, \operatorname{Aut}(G))$ and $xg \in \operatorname{orb}(x)$ we have $|\operatorname{orb}(x)| > 1$. Since $|\operatorname{orb}(x)|$ is a divisor of $|\operatorname{Aut}(G)|$ we have $|\operatorname{orb}(x)| \geq p$. Hence, the result follows from (3.1). □

Proposition 3.4. *Let H_1 and H_2 be two subgroups of G such that $H_1 \subseteq H_2$. Then*

$$\Pr_g(H_1, \operatorname{Aut}(G)) \leq |H_2 : H_1|\Pr_g(H_2, \operatorname{Aut}(G)).$$

The equality holds if and only if $xg \notin \operatorname{orb}(x)$ for all $x \in H_2 \setminus H_1$.

Proof. By Theorem 2.4, we have

$$\begin{aligned}|H_1||\operatorname{Aut}(G)|\Pr_g(H_1, \operatorname{Aut}(G)) &= \sum_{\substack{x\in H_1\\ xg\in \operatorname{orb}(x)}} |C_{\operatorname{Aut}(G)}(x)| \\ &\leq \sum_{\substack{x\in H_2\\ xg\in \operatorname{orb}(x)}} |C_{\operatorname{Aut}(G)}(x)| \\ &= |H_2||\operatorname{Aut}(G)|\Pr_g(H_2, \operatorname{Aut}(G)).\end{aligned}$$

Hence, the result follows. □

We conclude this section with the following result.

Proposition 3.5. *Let H be a subgroup of G. If $g \in G$ then*

$$\Pr_g(H, \operatorname{Aut}(G)) \leq |G : H| \Pr(G, \operatorname{Aut}(G))$$

with equality if and only if $g = 1$ and $H = G$.

Proof. By Proposition 3.2, we have

$$\begin{aligned}\Pr_g(H, \operatorname{Aut}(G)) &\leq \Pr(H, \operatorname{Aut}(G)) \\ &= \frac{1}{|H||\operatorname{Aut}(G)|}\sum_{x\in H}|C_{\operatorname{Aut}(G)}(x)| \\ &\leq \frac{1}{|H||\operatorname{Aut}(G)|}\sum_{x\in G}|C_{\operatorname{Aut}(G)}(x)| \\ &= |G : H| \Pr(G, \operatorname{Aut}(G)).\end{aligned}$$

Hence, the result follows from Corollary 2.5. □

4. Characterizations through $\Pr_g(H, \text{Aut}(G))$

In this section, we obtain some characterizations of groups through $\Pr_g(H, \text{Aut}(G))$. The following lemma is useful in this regard.

Lemma 4.1. *Let H be a subgroup of G. If p is the smallest prime divisor of $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]| = p$ then $\text{orb}(x) = x[H, \text{Aut}(G)]$ for all $x \in H \setminus L(H, \text{Aut}(G))$.*

Proof. We have $\text{orb}(x) \subseteq x[H, \text{Aut}(G)]$ for all $x \in H$. Also, $|\text{orb}(x)|$ is a divisor of $|\text{Aut}(G)|$ for all $x \in H$. Therefore, $|\text{orb}(x)| \geq p$ for all $x \in H \setminus L(H, \text{Aut}(G))$. Hence, $|\text{orb}(x)| = |x[H, \text{Aut}(G)]| = p$ for all $x \in H \setminus L(H, \text{Aut}(G))$ and the result follows. □

Now we derive the following characterizations.

Theorem 4.2. *Let H be a subgroup of a finite group G and $g \in G$. Let p be the smallest prime dividing $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]| = p$. If $g \neq 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n-1}{np}$ or $g = 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n+p-1}{np}$ (where n is a positive integer) then $\frac{H}{L(H,\text{Aut}(G))}$ is isomorphic to a group of order n. In particular,*

1. *if $n = q$ or q^2 for some prime q then $\frac{H}{L(H,\text{Aut}(G))} \cong \mathbb{Z}_q, \mathbb{Z}_{q^2}$ or $\mathbb{Z}_q \times \mathbb{Z}_q$.*
2. *if H is abelian and $n = q_1^{k_1} q_2^{k_2} \dots q_m^{k_m}$, where q_i's are primes not necessarily distinct, then $\frac{H}{L(H,\text{Aut}(G))} \cong \mathbb{Z}_{q_1^{k_1}} \times \mathbb{Z}_{q_2^{k_2}} \times \cdots \times \mathbb{Z}_{q_m^{k_m}}$.*

Proof. If $g \neq 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n-1}{np}$ then, by Lemma 4.1 and Proposition 2.6, we have

$$\frac{n-1}{np} = \frac{1}{p}\left(1 - \frac{1}{|H : L(H, \text{Aut}(G))|}\right)$$

which gives $|H : L(H, \text{Aut}(G))| = n$.

If $g = 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n+p-1}{np}$ then, by Lemma 4.1 and Proposition 2.6, we have

$$\frac{n+p-1}{np} = \frac{1}{p}\left(1 + \frac{p-1}{|H : L(H, \text{Aut}(G))|}\right)$$

which also gives $|H : L(H, \text{Aut}(G))| = n$.

Hence, $\frac{H}{L(H,\text{Aut}(G))}$ is isomorphic to a group of order n.

(1) If $n = q$ or q^2 for some prime q then $|H : L(H, \text{Aut}(G))| = q$ or q^2. Therefore $\frac{H}{L(H,\text{Aut}(G))}$ is abelian. Hence, the result follows from fundamental theorem of finite abelian groups.

(2) If H is abelian and $n = q_1^{k_1} q_2^{k_2} \dots q_m^{k_m}$, where q_i's are primes not necessarily distinct then $\frac{H}{L(H,\text{Aut}(G))}$ is an abelian group of order $q_1^{k_1} q_2^{k_2} \dots q_m^{k_m}$. Hence, the result follows from fundamental theorem of finite abelian groups. □

Putting $H = G$, in Theorem 4.2, we have the following corollary.

Corollary 4.3. *Let G be a finite group and $g \in G$. Let p be the smallest prime dividing $|\text{Aut}(G)|$ and $|[G, \text{Aut}(G)]| = p$. If $g \neq 1$ and $\Pr_g(G, \text{Aut}(G)) = \frac{n-1}{np}$ or $g = 1$ and $\Pr_g(G, \text{Aut}(G)) = \frac{n+p-1}{np}$ (where n is a positive integer) then $\frac{G}{L(G)}$ is isomorphic to a group of order n. In particular,*

1. *if $n = q$ or q^2 for some prime q then $\frac{G}{L(G)} \cong \mathbb{Z}_q, \mathbb{Z}_{q^2}$ or $\mathbb{Z}_q \times \mathbb{Z}_q$.*
2. *if G is abelian and $n = q_1^{k_1} q_2^{k_2} \dots q_m^{k_m}$, where q_i's are primes not necessarily distinct, then $\frac{G}{L(G))} \cong \mathbb{Z}_{q_1^{k_1}} \times \mathbb{Z}_{q_2^{k_2}} \times \cdots \times \mathbb{Z}_{q_m^{k_m}}$.*

We conclude the paper with the following result which gives converse of Theorem 4.2.

Theorem 4.4. *Let H be a subgroup of a finite group G and $g \in G$. Let p be the smallest prime dividing $|\operatorname{Aut}(G)|$ and $|[H, \operatorname{Aut}(G)]| = p$. If $\frac{H}{L(H,\operatorname{Aut}(G))}$ is isomorphic to a group of order n then*

$$\Pr_g(H, \operatorname{Aut}(G)) = \begin{cases} \dfrac{n-1}{np}, & \text{if } g \neq 1 \\ \dfrac{n+p-1}{np}, & \text{if } g = 1. \end{cases}$$

Proof. If p is the smallest prime dividing $|\operatorname{Aut}(G)|$ and $|[H, \operatorname{Aut}(G)]| = p$ then, by Lemma 4.1, we have $\operatorname{orb}(x) = x[H, \operatorname{Aut}(G)]$ for all $x \in H \setminus L(H, \operatorname{Aut}(G))$. Therefore, by Proposition 2.6, we have

$$\Pr_g(H, \operatorname{Aut}(G)) = \begin{cases} \dfrac{1}{p}\left(1 + \dfrac{p-1}{|H : L(H, \operatorname{Aut}(G))|}\right), & \text{if } g = 1 \\ \dfrac{1}{p}\left(1 - \dfrac{1}{|H : L(H, \operatorname{Aut}(G))|}\right), & \text{if } g \neq 1. \end{cases}$$

If $\frac{H}{L(H,\operatorname{Aut}(G))}$ is isomorphic to a group of order n then $|H : L(H, \operatorname{Aut}(G))| = n$ and hence the result follows. □

Note that putting $H = G$ in Theorem 4.4, we get the converse of Corollary 4.3.

Acknowledgment. The authors would like to thank the referee for his/her valuable comments and suggestions. The first author would like to thank DST for the INSPIRE Fellowship.

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