ST_2, ΔT_2, ST_3, ΔT_3, Tychonoff, compact and ∂-connected objects in the category of proximity spaces

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Abstract

In this paper, an explicit characterization of the separation properties ST_2, ΔT_2, ST_3, ΔT_3 and Tychonoff objects are given in the topological category of proximity space. Furthermore, the (strongly) compact object and ∂-connected object are also characterized in the category of proximity space. Moreover, we investigate the relationships among ST_2, ΔT_2, ST_3, ΔT_3, the separation properties at a point p, the generalized separation properties T_i, $i = 0, 1, 2$, $\mathbf{T_0}$, $\mathbf{T_1}$, $\mathbf{T_2}$ and Tychonoff objects in this category. Finally, we investigate the relationships between ∂-connected object and (strongly) connected object in the topological category of proximity space.

1. Introduction

The notion of proximity on a set X was introduced in 1950 by Efremovich [18]. He characterized the proximity relation "A is close to B" as a binary relation on subsets of a set X. In the meanwhile, in 1941, a study was made by Wallace [39,40] regarding "separation of sets". This study can be considered as the primordial version of the proximity concept. A large part of the early work in proximity spaces was done by Smirnov [37] and [38].

All our preliminary information on proximity spaces and more information can be found in [32].

In later years, some authors such as Leader [28], Lodato [29] and Pervin [33] have worked with weaker axioms than Efremovich's proximity axioms.

Various generalizations of the usual separation properties of topology and for an arbitrary topological category over sets separation properties at a point p are given in [2]. Baran [2] defined separation properties first at a point p, i.e., locally (see [3,5,6,10,13,24, 25]), then they are generalized this to point free definitions by using the generic element, [22, p. 39], method of topos theory.

One of the uses of local separation properties is to define the notions of closedness and strong closedness on arbitrary topological categories in set based topological categories.

These notions are introduced by Baran [2,4,9] and they are used in [2,7,11,15,24] to generalize each of the notions of compactness, connectedness, Hausdorffness, and perfectness to arbitrary set based topological categories. Also, it is shown in [10,11,13] that closedness and strong closedness form an appropriate closure operator in the sense of Dikranjan and Giuli [17] in some well-known topological categories. Moreover, the notions of each of (strongly) closed morphisms and (strongly) compact objects in a topological category $\mathcal{E}$ over SET are introduced in [7].

The main goal of this paper is

(1) to give the characterization of the separation properties ST_2, ΔT_2, ST_3, ΔT_3 and Tychonoff objects in the topological category of proximity space,

(2) to characterize the (strongly) compact object and ∂-connected object in the topological category of proximity space,

(3) to show that the relationships among ST_2, ΔT_2, ST_3, ΔT_3 and the separation properties at a point p, the generalized separation properties T_i, $i = 0, 1, 2$, $\mathbf{T_0}$, $\mathbf{T_1}$, $\mathbf{T_2}$ and Tychonoff objects in this category, and between ∂-connected object and (strongly) connected object in the topological category of proximity space.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful and amnestic), has small (i.e., sets) fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure, [1,5,12,30,34].

Recall in [1] or [34], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \mathrm{Ob}(\mathcal{E})$), a topological category, is discrete iff every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

Definition 2.1. ([32]). An (Efremovich) proximity space is a pair (X, δ), where X is a set and δ is a binary relation on the powerset of X such that

(P1) $A\delta B$ iff $B\delta A$;

(P2) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;

(P3) $A\delta B$ implies $A, B \neq \emptyset$;

(P4) $A \cap B \neq \emptyset$ implies $A\delta B$;

(P5) $A\bar{\delta}B$ implies there is an $E \subseteq X$ such that $A\bar{\delta}E$ and $(X - E)\bar{\delta}B$;

where $A\bar{\delta}B$ means it is not true that $A\delta B$.

A function $f : (X, \delta) \to (Y, \delta')$ between two proximity spaces is called a *proximity mapping (or a p-map)* iff $f(A)\delta'f(B)$ whenever $A\delta B$. It can easily be shown that f is a p-map iff $f^{-1}(C)\bar{\delta}f^{-1}(D)$ whenever $C\bar{\delta'}D$.

In a (quasi-)proximity space (X, δ), we write $A \ll B$ if and only if $A \bar{\delta} (X - B)$. The relation $\ll$ is called *p*-neighborhood relation or the strong inclusion. When $A \ll B$, we say that B is a *p*-neighborhood of A or A is strongly contained in B [20] or [32].

We denote the category of proximity spaces and proximity mappings by **Prox**. Hunsaker and Sharma [21] showed that the functor $\mathcal{U} : \mathbf{Prox} \longrightarrow \mathbf{Set}$ is topological.

Definition 2.2. ([35]). Let X be a nonempty set. A proximity-base on X is a binary relation $\mathfrak{B}$ on $P(X)$ satisfying the axioms $(B1)$ through $(B5)$ given below:

(B1) $(\emptyset, X) \notin \mathfrak{B}$;

(B2) If $A \cap B \neq \emptyset$ implies $(A, B) \in \mathfrak{B}$;

(B3) $(A, B) \in \mathfrak{B}$ iff $(B, A) \in \mathfrak{B}$;

(B4) If $(A, B) \in \mathfrak{B}$ and $A \subseteq A^*$, $B \subseteq B^*$ then $(A^*, B^*) \in \mathfrak{B}$;

(B5) If $(A, B) \notin \mathfrak{B}$ then there exists a set $E \subseteq X$ such that $(A, E) \notin \mathfrak{B}$ and $(X - E, B) \notin \mathfrak{B}$.

2.3 Let $\mathfrak{B}$ be a proximity-base on a set X and let a binary relation δ on $P(X)$ be defined as follows: $(A, B) \in \delta$ if, given any finite covers $\{A_i : 1 \leq i \leq n\}$ and $\{B_j : 1 \leq j \leq m\}$ of A and B respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$. δ is a proximity on X finer than the relation $\mathfrak{B}$ [21] or [35].

2.4 Let X be a non-empty set, for each $i \in I$, (X_i, δ_i) be a proximity space and $f_i : X \to (X_i, \delta_i)$ be a source in **Prox**. Define a binary relation $\mathfrak{B}$ on $P(X)$ as follows: for $A, B \in P(X)$, $A\mathfrak{B}B$ iff $f_i(A)\delta_i f_i(B)$, for all $i \in I$. $\mathfrak{B}$ is a proximity-base on X [35, Theorem 3.8]. The initial proximity structure δ on X generated by the proximity base $\mathfrak{B}$ is given by for $A, B \in P(X)$, $A\delta B$ iff for any finite covers $\{A_i : 1 \leq i \leq n\}$ and $\{B_j : 1 \leq j \leq m\}$ of A and B respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$ [35].

2.5 Let (X, δ) be a proximity space, Y a non-empty set and f a function from a proximity space (X, δ) onto a set Y. The strong inclusion $\ll^*$ induced by the finest proximity δ^* on Y making f proximally continuous is given by: for every $A, B \subset Y$, $A \ll^* B$ if and only if, for each binary rational s in $[0, 1]$, there is some $C_s \subset Y$ such that $C_0 = A$, $C_1 = B$ and $s < t$ implies $f^{-1}(C_s) \ll_\delta f^{-1}(C_t)$ [20] or [41, p. 276], where $\ll_\delta$ represents the strong inclusion induced by the proximity δ on X. In addition, if $f : (X, \delta) \to (X, \delta^*)$ be a one-to-one p map, then $A \delta^* B$ if and only if $f^{-1}(A) \delta f^{-1}(B)$ [20, p. 591].

2.6 We write Δ for the diagonal in X^2, where $X \in$ **Prox**. For $X \in$ **Prox** we define the wedge $X^2 \vee_\Delta X^2$, as the final structure, with respect to the map $X^2 \coprod X^2 \longrightarrow X^2 \vee_\Delta X^2$, that is the identification of the two copies of X^2 along the diagonal Δ. An epi sink $\{i_1, i_2 : (X^2, \delta) \longrightarrow (X^2 \vee_\Delta X^2, \delta')\}$, where i_1, i_2 are the canonical injections, in **Prox** is a final lift if and only if the following statement holds. For each pair A, B in the different component of $X^2 \vee_\Delta X^2$, $A\delta'B$ iff there exist sets C, D in X^2 such that $C\delta\{(x, y)\}$ and $\{(x, y)\}\delta D$ with $i_k^{-1}(A) = C$ and $i_j^{-1}(B) = D$ for $k, j = 1, 2$ and $k \neq j$. If A and B are in the same component of wedge, then $A\delta'B$ iff there exist sets C, D in X^2 such that $C\delta D$ and $i_k^{-1}(A) = C$ and $i_k^{-1}(B) = D$ for some $k = 1, 2$. Specially, if $i_k(E) = A$ and $i_k(F) = B$, then $(i_k(E), i_k(F)) \in \delta'$ iff $(i_k^{-1}(i_k(E)), i_k^{-1}(i_k(F))) = (E, F) \in \delta$. This is a special case of 2.5.

2.7 Let X be a non-empty set. The discrete proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ iff $A \cap B \neq \emptyset$ [32, p. 9].

2.8 Let X be a non-empty set. The indiscrete proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$ [19, p. 5].

3. ST_2, ΔT_2, ST_3 and ΔT_3 objects in proximity spaces

In this section, the characterization of ST_2, ΔT_2 ST_3 and ΔT_3 objects in this category are given. Furthermore, we investigate the relationships among ST_2, ΔT_2, ST_3, ΔT_3, the separation properties at a point p, the generalized separation properties T_i, $i = 0, 1, 2$, **T₀**, **T₁** and **T₂** in the topological category of (Efremovich) proximity spaces.

Let B be set and $p \in B$. The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times \ldots$ be the countable cartesian product of B. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, \ldots, p, x, p, \ldots)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, \ldots, p, x, p, \ldots)$ (infinite principal p-axis map), and $\nabla_p^\infty : \vee_p^\infty B \longrightarrow B$ by $\nabla_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map), [2, 4].

Note, also, that A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, id, is in the j-th place [11].

Definition 3.1. (cf. [2,4]). Let $\mathcal{U} : \mathcal{E} \longrightarrow \boldsymbol{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifying F with a point $\{*\}$.

Let p be a point in B.

(1) p is closed iff the initial lift of the $\mathcal{U}$-source $\{A_p^\infty : \vee_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.

(2) $F \subset X$ is closed iff $\{*\}$, the image of F, is closed in X/F or $F = \emptyset$.

(3) $F \subset X$ is strongly closed iff X/F is T_1 at $\{*\}$ or $F = \emptyset$.

(4) If $B = F = \emptyset$, then we define F to be both closed and strongly closed.

(5) X is ST_2 iff Δ, the diagonal, is strongly closed in X^2, [4].

(6) X is ΔT_2 iff Δ, the diagonal, is closed in X^2, [4].

(7) X is ΔT_3 iff X is T_1 and X/F is ΔT_2 if it is T_1, where $F \neq \emptyset$ in $U(X)$, [8].

(8) X is ST_3 iff X is T_1 and X/F is ST_2 if it is T_1, where $F \neq \emptyset$ in $U(X)$, [8].

Recall that a prebornological space is a pair $(B, \mathfrak{F})$, where $\mathfrak{F}$ is a family of subsets of B that is closed under nonempty finite union and contains all finite nonempty subsets of B. A morphism $(B, \mathfrak{F}) \to (B_1, \mathfrak{F}_1)$ of such spaces is a function $f : B \to B_1$ such that $f(C) \in \mathfrak{F}_1$ if $C \in \mathfrak{F}$. We denote by $\boldsymbol{PBorn}$, the category thus obtained. This category is topological category over $\boldsymbol{Set}$, [9].

The category $\boldsymbol{Prord}$ of preordered spaces has as objects the pairs (B, R), where B is a set and R is a reflexive and transitive relation on B and has as morphism $(B, R) \to (B_1, R_1)$ those functions $f : B \to B_1$ such that if aRb, then $f(a)R_1 f(b)$ for all $a, b \in B$. This category is topological category over $\boldsymbol{Set}$, [13].

Lemma 3.2. ([13, Theorem 3.6]). *Let (B, R) be a preordered set (i.e., R is a reflexive and transitive relation on B), and $\emptyset \neq F \subset B$. Then,*

(i) *F is a closed subset of B iff for any $x \in B$, if there exists $a, b \in F$ such that xRa and bRx, then $x \in F$.*

(ii) *F is a strongly closed subset of B iff for each $x \in B$, if there exists $a \in F$ such that xRa or aRx, then $x \in F$.*

Lemma 3.3. ([4, Theorem 3.9 and 3.10]). *Let $(B, \mathfrak{F})$ be a prebornological space. Then,*

(i) *A subset $F \subset B$ is closed iff $B = F$ or $F = \emptyset$.*

(ii) *All subsets of B are strongly closed.*

Remark 3.4.   (1) In $\boldsymbol{Top}$, the notion of closedness coincides with the usual one [2] and F is strongly closed iff F is closed and for each $x \notin F$ there exists a neighbourhood of F missing x, [2]. If a topological space is T_1, then the notions of closedness and strong closedness coincide, [2].

(2) In general, for an arbitrary topological category, the notions of closedness and strong closedness are independent of each other. To see this, let $B = \{-1, 1\}$, $R = \{(-1, 1), (-1, -1), (1, 1)\}$ and $F = \{1\}$. Then (B, R) is a preordered set and by 3.2, F is closed, but F is not strongly closed. On the other hand, let $B = \mathbb{R}$, the set of real numbers, and $\mathfrak{F} = P(\mathbb{R}) - \{\emptyset\}$, the set of all nonempty subsets of $\mathbb{R}$. Note, [9] Remark 3.2, that $(B, \mathfrak{F})$ is a prebornological space and by 3.3, Q, the set of rational numbers, is strongly closed, but Q is not closed.

Theorem 3.5. (cf. [26]). *Let (X, δ) be a (Efremovich) proximity space and $p \in X$.*

(1) *$\{p\}$ is closed in X iff for any $B \subset X$, if $\{p\}\delta B$, then $p \in B$.*

(2) $\emptyset \neq F \subset X$ *is closed iff* $x \in F$ *whenever* $\{x\}\delta F$ *for all* $x \in X$.

(3) $\emptyset \neq F \subset X$ *is strongly closed iff* $x \in F$ *whenever* $\{x\}\delta F$ *for all* $x \in X$.

Definition 3.6. Let $\mathcal{E}$ be a topological category over ***Set***, X an object in $\mathcal{E}$ and F be a nonempty subset of X.

(1) $F \subset X$ is open iff F^c, the complement of F, is closed in X.

(2) $F \subset X$ is strongly open iff F^c, the complement of F, is strongly closed in X, [15].

Note that in ***Top*** the notion of openness coincides with the usual one, [15]. If a topological space is T_1, then the notions of openness and strong openness coincide, [15].

Theorem 3.7. ([26]). *Let* (X, δ) *be a (Efremovich) proximity space.* $\emptyset \neq F \subset X$ *is (strongly) open iff* $x \in F^c$ *whenever* $\{x\}\delta F^c$ *for all* $x \in X$.

Definition 3.8. ([41, p. 268]). Let (X, δ) be a (Efremovich) proximity space and $A \subset X$. Define $\bar{A} = \{x | x\delta A\}$ and if $\bar{A} = A$, then A is said to be closed.

Remark 3.9.   (1) Let (X, δ) be a (Efremovich) proximity space and $A \subset X$. A is closed (in the usual above sense) iff for each $x \in X$, if $x\delta A$, then $x \in A$.

(2) Let (X, δ) be a (Efremovich) proximity space. It follows from 3.5 and Definition 3.8 that the notions of closedness (in our sense) and strong closedness coincide with the notion of closedness in the usual sense, [26].

Theorem 3.10. *Let* (X, δ) *be a (Efremovich) proximity space. Then* (X, δ) *is* ST_2 *or* ΔT_2 *iff* δ *is separated (Hausdorff) (Efremovich) proximity i.e., if* $\{x\}\delta\{y\}$, *then* $x = y$.

Proof. (X, δ) is ST_2 or ΔT_2 iff by Definition 3.1 (5) ((6)) Δ is strongly closed (closed) iff by Theorem 3.5 (3) (Theorem 3.5 (2)), letting $F = \Delta$ for each $(x, y) \in X^2$ if there exists (a, a) in Δ such that $\{(x, y)\}\delta^2\{(a, a)\}$ (δ^2 is the product proximity structure on X^2), then $(x, y) \in \Delta$ i.e., $x = y$. We will show that if (X, δ) is ST_2 or ΔT_2, then δ is separated (Hausdorff) proximity. If $\{x\}\delta\{y\}$, then we have clearly $\{(x, y)\}\delta^2\{(y, y)\}$ or $\{(x, x)\}\delta^2\{(x, y)\}$ and consequently $(x, y) \in \Delta$ i.e., $x = y$ since (X, δ) is ST_2 or ΔT_2. Hence δ is separated (Hausdorff) (Efremovich) proximity.

Conversely if δ is separated (Hausdorff) (Efremovich) proximity, then clearly Δ is strongly closed (closed) i.e., (X, δ) is ST_2 or ΔT_2. $\square$

Example 3.11. Let $X = \{a, b\}$ and $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X)\}$. Then (X, δ) is ST_2 or ΔT_2 since $\{a\}\delta\{b\}$, then $a = b$.

Theorem 3.12. *Let* (X, δ) *be a (Efremovich) proximity space. If* (X, δ) *is* ST_2 *or* ΔT_2, *then* $(X/F, \delta^*)$ *is* ST_2 *or* ΔT_2.

Proof. Suppose (X, δ) is ST_2 or ΔT_2. Let x and y be any distinct pair of points in X/F. By Theorem 3.10, we only need to show that $(\{x\}, \{y\}) \notin \delta^*$, where δ^* is the structure on X/F induced by q.

Suppose that $x \neq *$. By definition of q map, there exist $x \in X$ and $F \subset X$ such that $q(x) = x$ and $q(z) = *$ for any $z \in F$. Since $x \neq z$ for any $z \in F$ ($x \notin F$) and (X, δ) is ST_2 or ΔT_2, then $\{x\} \bar{\delta} \{z\}$. By the condition (P2) of Definition 2.1 we obtain $\{x\} \bar{\delta} F$. Then we have $\{x\} \bar{\delta} F = q^{-1}(\{x\}) \bar{\delta} q^{-1}(\{*\})$. It follows that by p-neighborhood relation definition and 2.5, for each binary rational s in $[0, 1]$ there is some $C_s \subset X/F$ such that $C_0 = \{x\}$, $C_1 = \{*\}^z$ and $s < t$ implies $q^{-1}(C_s) \ll_\delta q^{-1}(C_t) = q^{-1}(\{x\}) \ll_\delta (q^{-1}(\{*\}))^z = q^{-1}(\{x\}) \ll_\delta q^{-1}(\{*\}^z)$ if and only if $\{x\} \ll^* \{*\}^z$. Hence $\{x\} \bar{\delta^*} \{*\}$, i.e., $(\{x\}, \{*\}) \notin \delta^*$.

Let $x \neq y \neq *$. By definition of q map, there exists a pair $x, y \in X$ such that $q(x) = x$ and $q(y) = y$. In this case q map can be considered as one-to-one map. Suppose that $\{x\} \delta^* \{y\}$. By definition of q map and 2.5, we have $\{x\} \delta^* \{y\}$ if and only if $q^{-1}(\{x\}) \delta q^{-1}(\{y\}) = \{x\} \delta \{y\}$. But $\{x\} \bar{\delta} \{y\}$ since (X, δ) is ST_2 or ΔT_2. Hence $\{x\} \bar{\delta^*} \{y\}$ i.e., $(\{x\}, \{y\}) \notin \delta^*$.

Consequently for each distinct points x and y in X/F, we have $(\{x\}, \{y\}) \notin \delta^*$. Hence by Theorem 3.10, $(X/F, \delta^*)$ is ST_2 or ΔT_2. $\square$

Theorem 3.13. (cf. [26, 27]). *Let (X, δ) be a (Efremovich) proximity space and $p \in X$.*
(1) *(X, δ) is T_1 at p iff for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$.*
(2) *(X, δ) is $\overline{T_0}$ at p iff for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$.*
(3) *All (Efremovich) proximity spaces are T_0' at p.*
(4) *(X, δ) is $\overline{T_0}$ if and only if, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*
(5) *An Efremovich proximity space is T_0'.*
(6) *(X, δ) is T_1 if and only if, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*
(7) *An Efremovich proximity space is $Pre\overline{T_2}$, ([12]).*
(8) *(X, δ) is $PreT_2'$ ([12]) if and only if, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*
(9) *(X, δ) is $\overline{T_2}$ ([11]) iff, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*
(10) *(X, δ) is T_2' ([11]) iff, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*

Definition 3.14. (cf. [32, 36]). An Efremovich proximity space (X, δ) is said to be a
- **T₀-space** if $x \neq y$ for $x, y \in X$ implies that $x \bar{\delta} y$.
- **T₁-space** if $x \neq y$ for $x, y \in X$ implies that $x \bar{\delta} y$.
- **T₂-space (Hausdorff)** if $x \, \delta \, y$ for $x, y \in X$ implies that $x = y$.

Theorem 3.15. *An (Efremovich) proximity space (X, δ) is ST_3 or ΔT_3 if and only if, δ is separated (Hausdorff) (Efremovich) proximity i.e., if $\{x\}\delta\{y\}$, then $x = y$.*

Proof. It follows from Definition 3.1 (7), (8) and Theorems 3.12, 3.13 (6). $\square$

We give explicit relationships among the generalized separation properties ST_2, ΔT_2, ST_3, ΔT_3, the separation properties at a point p, the generalized separation properties T_i, $i = 0, 1, 2$, **T₀**, **T₁** and **T₂** in the topological category of (Efremovich) proximity spaces.

Remark 3.16. Let (X, δ) be a (Efremovich) proximity space and $p \in A$.
(i) By Theorems 3.10, 3.13 and 3.15, then the followings are equivalent:
    (1) (X, δ) is $\overline{T_0}$ at p for all $p \in A$.
    (2) (X, δ) is T_1 at p for all $p \in A$.
    (3) (X, δ) is ST_i, $i = 2, 3$.
    (4) (X, δ) is ΔT_i, $i = 2, 3$.
(ii) By Theorems 3.10, 3.13 and 3.15, if (X, δ) is ST_i or ΔT_i, $i = 2, 3$, then (X, δ) is T_0' at p for all $p \in A$. But the reverse of implication is not true, in general. For example, let $X = \{a, b\}$ and $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$. Then (X, δ) is T_0' at a but it is not ST_i or ΔT_i, $i = 2, 3$, at a since $(\{a\}, \{b\}) \in \delta$ but $a \neq b$.
(iii) By Theorems 3.10, 3.13, 3.15, and Definition 3.14, then the followings are equivalent:
    (1) (X, δ) is $\overline{T_0}$.
    (2) (X, δ) is **T_i**, **i** $= 0, 1, 2$.
    (3) (X, δ) is T_1.
    (4) (X, δ) is $PreT_2'$.
    (5) (X, δ) is $\overline{T_2}$.
    (6) (X, δ) is T_2'.
    (7) (X, δ) is ST_i, $i = 2, 3$.
    (8) (X, δ) is ΔT_i, $i = 2, 3$.
    (9) For any distinct pair of points a and b in X, $(\{a\}, \{b\}) \notin \delta$.
(iv) By Theorems 3.10, 3.13 and 3.15, if (X, δ) is ST_i or ΔT_i, $i = 2, 3$, then (X, δ) is T_0' or $Pre\overline{T_2}$. But the reverse of implication is not true, in general. For example, let $X =$

$\{a,b\}$ and $\delta = \{(X,X),(\{a\},\{a\}),(\{b\},\{b\}),(X,\{a\}),(\{a\},X),(X,\{b\}),(\{b\},X),$ $(\{a\},\{b\}),(\{b\},\{a\})\}$. Then, (X,δ) is T_0' and $Pre\overline{T}_2$ but it is not ST_i or ΔT_i, $i = 2,3$, since $(\{a\},\{b\}) \in \delta$ but $a \neq b$.

4. Connectedness and compactness

In this section, the characterization of the notion of the ∂-connected object and (strongly) Compact object in this category are given. We investigate the relationships between ∂-connected object and (strongly) connected object in this category.

Recall that the notions of each of (strongly) closed morphisms and (strongly) compact objects in a topological category $\mathcal{E}$ over SET are introduced in [7].

Definition 4.1. (cf. [7]) Let $\mathcal{U} : \mathcal{E} \longrightarrow \boldsymbol{Set}$ be a topological functor, X and Y be objects in $\mathcal{E}$, and $f : X \to Y$ a morphism in $\mathcal{E}$. Then,

    (1) f is said to be closed iff the image of each closed subobject of X is a closed subobject of Y.

    (2) f is said to be strongly closed iff the image of each strongly closed subobject of X is a strongly closed subobject of Y.

    (3) X is compact iff the projection $\pi_2 : X \times Y \to Y$ is closed for each object Y in $\mathcal{E}$.

    (4) X is strongly compact iff the projection $\pi_2 : X \times Y \to Y$ is strongly closed for each object Y in $\mathcal{E}$.

Note that for the category $\boldsymbol{Top}$ of topological spaces, the notions of closed morphism and compactness reduce to the usual ones ([16, p.97 and p.103]).

Lemma 4.2. *(1). Let $f : (X,\delta) \to (Y,\delta')$ be a p-map in* **Prox**. *If $D \subset Y$ is (strongly) closed, so also is $f^{-1}(D)$.*

(2) Let (Y,δ') be a (Efremovich) proximity space. If $N \subset Y$ is (strongly) closed and $M \subset N$ is (strongly) closed, so also is $M \subset Y$.

Proof. (1) Suppose $D \subset Y$ is (strongly) closed and $x \in f^{-1}(D)$. By 3.5 (2) (3.5 (3)), $y \in D$ whenever $\{y\}\delta'D$ for all $y \in Y$. We need to show that, $x \in f^{-1}(D)$ whenever $\{x\}\delta f^{-1}(D)$ for all $x \in X$. Note that $f(x) \in f(f^{-1}(D)) \subset D$ and $\{f(x)\}\delta'D$ since f is p-map and $D \subset B$ is closed. Thus, $f^{-1}(D)$ is closed.

The proof for strongly closedness is similar.

(2) Suppose $N \subset Y$ and $M \subset N$ are strongly closed, $y \in Y$ and there exists $a \in M$ such that $y\delta'a$. By 3.5 (3), we need to show that $y \in M$. Since $N \subset Y$ is strongly closed and $M \subset N$, by 3.5 (3), $y \in N$. It follows that $y \in M$ since $M \subset N$ is strongly closed.

The proof for closedness is similar. $\qquad\square$

Lemma 4.3. *All objects in* **Prox** *are (strongly) compact.*

Proof. Let (B,δ) be a (Efremovich) proximity space. By Definition 4.1 (3) (4.1 (4)), we need to show that for all proximity spaces (A,δ'), $\pi_2 : (B,\delta) \times (A,\delta') \to (A,\delta')$ is (strongly) closed. Suppose $M \subset B \times A$ is (strongly) closed. To show that π_2M is (strongly) closed, we assume the contrary and apply Theorem 3.5 (2) (3.5 (3)). Thus for some point $a \in A$ with $a \notin \pi_2M$ whenever $\{a\}\delta'\pi_2M$. Since $M \subset B \times A$ is (strongly) closed, $(b,a) \in M$ whenever $\{(b,a)\}\delta''M$ for all $(b,a) \in B \times A$, where δ'' is the product proximity structure on $B \times A$. Hence $\pi_2\{(b,a)\}\delta'\pi_2M = \{a\}\delta'\pi_2M$, by definition of product proximity structure. Since $(b,a) \in M$, $\pi_2(b,a) = a \in \pi_2M$. This is a contradiction since M is (strongly) closed, by Theorem 3.5 (2) (3.5 (3)). Hence, by Theorem 3.5 (2) (3.5 (3)), π_2M must be (strongly) closed and consequently, by Definition 4.1 (3) (4.1 (4)), (B,δ) is (strongly) compact. $\quad\square$

Theorem 4.4. *Let $f : X \to Y$ be a p-map in* **Prox**. *If (X,δ) is (strongly) compact, then $(f(X),\delta')$ is (strongly) compact.*

Proof. It follows from Lemma 4.3. $\qquad\square$

We now give the characterization of ∂-connected object in the category of (Efremovich) proximity spaces and investigate the relationships between ∂-connected object and (strongly) connected object in this category.

Definition 4.5. Let $\mathcal{E}$ be a topological category over ***Set*** and X be an object in $\mathcal{E}$.

   (1) X is connected iff the only subsets of X both strongly open and strongly closed are X and $\emptyset$, [15].

   (2) X is strongly connected iff the only subsets of X both open and closed are X and $\emptyset$, [15].

   (3) X is ∂-connected iff the boundary of any non-empty proper subsets of X is non-empty set, i.e., $\partial F \setminus \mathring{F} \neq \emptyset$, [23].

   (4) X is D-connected iff any morphism from X to any discrete object is constant, (cf. [15, 34]).

Note that for the category ***Top*** of topological spaces, the notion of strongly connectedness, ∂-connected and D-connectedness coincides with the usual notion of connectedness. If a topological space X is T_1, then, by 4.5, the notions of connectedness, strong connectedness and ∂-connected coincide, [15].

Theorem 4.6. *A (Efremovich) proximity space (X, δ) is ∂-connected iff for any non-empty proper subset F of X, either the condition* (1) *or* (2) *holds.*

   (1) $x \notin F$ *whenever* $\{x\}\delta F$ *for some* $x \in X$.

   (2) $x \notin F^c$ *whenever* $\{x\}\delta F^c$ *for some* $x \in X$.

Proof. Suppose that (X, δ) is ∂-connected but conditions (1) and (2) do not hold for some non-empty proper subset F of X. Since the condition (1) does not hold, we get $x \in F$ whenever $\{x\}\delta F$ for all $x \in X$ which means that subset F is (strongly) closed by 3.5 (2) or 3.5 (3). Since the condition (2) does not hold, we get $x \in F^c$ for all $x \in X$, whenever $\{x\}\delta F^c$. This means that F^c is (strongly) closed. So F is (strongly) open by 3.7. Hence F is (strongly) open and (strongly) closed, i.e., $\partial F \setminus \mathring{F} = F \setminus F = \emptyset$. But this is a contradiction since (X, δ) is ∂-connected.

Conversely, suppose that the condition (1) holds. Then $x \notin F$ whenever $\{x\}\delta F$ for some $x \in X$ and F is not (strongly) closed 3.5 (2) or 3.5 (3). Suppose that the condition (2) holds. Then for some $x \in X$, $x \notin F^c$ whenever $\{x\}\delta F^c$. This means that F^c is not (strongly) closed. So F is not (strongly) open by 3.7. Hence the only subsets of X both (strongly) open and (strongly) closed are X and $\emptyset$. Hence $\partial F \setminus \mathring{F} \neq \emptyset$. From here (X, δ) is ∂-connected. $\qquad\square$

Example 4.7. Let $X = \{a, b\}$ and $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}),$ $(\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$. Then (X, δ) is ∂-connected since non-empty proper subset $F = \{a\}$ of X, $b \notin F$ whenever $\{b\}\delta F$ for some $b \in X$. The case $F = \{b\}$ of X can be handled similarly.

Theorem 4.8. *A (Efremovich) proximity space (X, δ) is (strongly) connected iff for any non-empty proper subset F of X, either the condition* (1) *or* (2) *holds.*

   (1) $x \notin F$ *whenever* $\{x\}\delta F$ *for some* $x \in X$.

   (2) $x \notin F^c$ *whenever* $\{x\}\delta F^c$ *for some* $x \in X$, [26].

Remark 4.9. Let (X, δ) be in **Prox**. By Theorem 4.6 and Theorem 4.8, (X, δ) is (strongly) connected iff (X, δ) is ∂-connected.

Lemma 4.10. *Let $f : (X, \delta) \to (Y, \delta')$ be a p-map in* **Prox**. *If (X, δ) is (strongly) connected, ∂-connected or D-connected, then $f(X)$ is (strongly) connected, ∂-connected or D-connected, respectively.*

Proof. Let (X, δ), (Y, δ') be in **Prox** and M is any non-empty proper subset of $f(X)$. Since $f^{-1}(M) \subset X$ and (X, δ) is (strongly) connected, either conditions (I) or (II) in Theorem 4.8 holds. Suppose condition (I) in Theorem 4.8 holds. Then, $x \notin f^{-1}(M)$ whenever $\{x\}\delta f^{-1}(M)$ for some $x \in X$. Hence, $f(x) \notin f(f^{-1}(M)) \subset M \Rightarrow f(x) \notin M$ whenever $\{f(x)\}\delta' f(f^{-1}(M))$ for some $f(x) \in f(X)$. Similarly, if the condition (II) of Theorem 4.8 holds, $f(X)$ is strongly connected.

The proof for ∂-connected and D-connectedness is similar. □

5. Tychonoff objects

In this section, the characterization of Tychonoff objects in this category is given. Furthermore, we investigate the relationships between Tychonoff objects and ST_2, ΔT_2, ST_3, ΔT_3, generalized separation properties and separation properties at a point p in this category.

Definition 5.1. (cf. [7, 8, 14]). Let $\mathcal{U} : \mathcal{E} \longrightarrow \boldsymbol{Set}$ be a topological functor and X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.

(1) X is $C\Delta T_{3\frac{1}{2}}$ iff X is a subspace of a compact ΔT_2.
(2) X is $CST_{3\frac{1}{2}}$ iff X is a subspace of a compact ST_2.
(3) X is $LT_{3\frac{1}{2}}$ iff X is a subspace of a compact T_2'.
(4) X is $S\Delta T_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact ΔT_2.
(5) X is $SST_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact ST_2.
(6) X is $SLT_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact T_2'.

Remark 5.2. For the category $\boldsymbol{Top}$ of topological spaces, all six of the properties defined in Definition 5.1 are equivalent and reduce to the usual $T_{3\frac{1}{2}}=$ Tychonoff, i.e, completely regular T_1, spaces ([31, Remark 5.2, and Remark 6.2]).

Theorem 5.3. *Let (X, δ) be a (Efremovich) proximity space. Then the followings are equivalent:*

(1) (X, δ) *is* $C\Delta T_{3\frac{1}{2}}$.
(2) (X, δ) *is* $CST_{3\frac{1}{2}}$.
(3) (X, δ) *is* $LT_{3\frac{1}{2}}$.
(4) (X, δ) *is* $S\Delta T_{3\frac{1}{2}}$.
(5) (X, δ) *is* $SST_{3\frac{1}{2}}$.
(6) (X, δ) *is* $SLT_{3\frac{1}{2}}$.
(7) (X, δ) *is separated (Hausdorff) (Efremovich) proximity i.e., if $\{a\}\delta\{b\}$, then $a = b$.*

Proof. It follows from Theorem 3.10, Lemma 4.3 and Definition 5.1. □

Example 5.4. Let $X = \{a, b\}$, $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X),$ $(X, \{b\}), (\{b\}, X)\}$ and $\delta_1 = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}),$ $(\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$. Then (X, δ) is $C\Delta T_{3\frac{1}{2}}$, but (X, δ_1) is not $C\Delta T_{3\frac{1}{2}}$, since $(\{a\}, \{b\}) \in \delta$ with $a \neq b$.

By Remark 3.16 and Theorem 5.3, we need only to give explicit relationships among $C\Delta T_{3\frac{1}{2}}$, $\overline{T}_0$ at p, T_0' at p, $\overline{T}_0$, $\mathbf{T_0}$, T_0' and $Pre\overline{T}_2$ in the topological category of (Efremovich) proximity spaces.

Remark 5.5. Let (X, δ) be a (Efremovich) proximity space and $p \in A$.

1. By Remark 3.16 and Theorem 5.3, then the followings are equivalent:

(i) (X, δ) is $\overline{T}_0$ at p for all $p \in A$.

(ii) (X, δ) is $C\Delta T_{3\frac{1}{2}}$.

(iii) For each $a \neq p$, $(\{a\}, \{p\}) \notin \delta$.

2. By Theorem 3.13 (3) and Theorem 5.3, if (X, δ) is $C\Delta T_{3\frac{1}{2}}$, then (X, δ) is T_0' at p for all $p \in A$. But the converse of implication is not true. For example, take (X, δ) to be the proximity space in Remark 3.16 (ii). Then (X, δ) is T_0' at a but it is not $C\Delta T_{3\frac{1}{2}}$ at a.

3. By Remark 3.16, Theorem 5.3 and Definition 3.14, then the followings are equivalent:

(i) (X, δ) is $\overline{T}_0$.

(ii) (X, δ) is $\mathbf{T_0}$.

(iii) (X, δ) is $C\Delta T_{3\frac{1}{2}}$.

(iv) For each distinct pair of points a and b in X, $(\{a\}, \{b\}) \notin \delta$.

4. By Theorem 3.13 (5), Theorem 3.13 (7) and Theorem 5.3, if (X, δ) is $C\Delta T_{3\frac{1}{2}}$, then (X, δ) is T_0' or $Pre\overline{T}_2$. But the converse implication is not true. For example, take (X, δ) to be the proximity space in Remark 3.16 (ii). Then (X, δ) is T_0' or $Pre\overline{T}_2$ but it is not $C\Delta T_{3\frac{1}{2}}$.

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