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Research Article

DE- and EDP$_M$- compound optimality for the information and probability-based criteria

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Abstract

Several optimality criteria have been considered in the literature as information-based criteria. The probability- based criteria have been recently proposed for maximizing the probability of a desired outcome. However, designs that are optimal for the information- based criteria may be inadequate for probability- based criteria. This paper introduces the DE- and EDP$_M$ – optimum designs for multi aims of optimality for Generalized Linear Models (GLMs). An equivalence theorem is proved for both compound criteria. Finally, two numerical examples are given to illustrate the potentiality of the proposed compound criteria.

Keywords: Optimum design, E-optimality, D-optimality, P-optimality, Compound criteria, Generalized linear models.

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1. Introduction

An optimality criterion is a criterion, which summarizes how good the design is, and it is maximized or minimized by an optimal design. Information-based criteria is one of the popular types of optimality criteria that related to the Fisher information matrix of the design. This type included many common optimality criteria such as; D-, G-, I-, A- and E-optimality.

The most important and popular design criterion for parameter estimation is D-optimality. It has been central to work on optimum experimental designs. Several publications on D-optimality can be seen in *Atkinson et al.* [3]. D-optimal designs are mainly

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intended to obtain efficient parameter estimation by the way of minimizing the generalized variance of the estimated regression coefficients by maximizing the determinant of the Fisher information matrix.

E-optimal design is devoted to minimize the maximum variance of all possible normalized linear combinations of the parameter estimation via maximizing the smallest eigenvalue of the information matrix.

Most of the literature concentrates on D-optimal designs but much less attention has been paid to E-optimal designs in nonlinear regression models (see *Dette and Haines* [11]; *Dette and Wong* [13]. *Dette et al.* [12] established that in the exponential regression models the E-optimal designs are usually more efficient for estimating individual parameters than D-optimal designs. Moreover, E-optimal designs usually behave substantially more reliably with respect to minimize the variances of the parameter estimates than do D-optimal designs. However, the problem of determining E-optimal designs is substantially harder than the D-optimal design problem.

The probability-based optimality criterion was initially introduced by *McGree and Eccleeston* [13] that maximizing a probability of a particular event that assess an importance to the experimenter. Moreover, the DP- compound optimality criterion was proposed and discussed.

Some designs could be adequate for optimality criterion but inadequate for others hence, the motivation of constructing compound criteria is to satisfy multi objective aims of optimality. Many authors have developed optimality criteria which are applicable to the multiple objective problems (for example *Clyde and Chaloner* [6], *McGree et al.* [20], *Atkinson* [2], *Denman et al.* [8], *Kilany*[17], *Kilany et al.*[18] and *Mwan et al.*[21]).

The main objective of this paper is to construct new compound criteria via E-,D-, and P_M - optimality criterion to achieve the multi optimality problem of efficient parameter estimation, minimizing the maximum variance of all possible normalized linear combinations of the parameter estimation and obtaining the maximum probability of a desired outcome.

The paper is organized as follows; Section 2 is devoted to represent the optimum design background. In Section 3, E-, D- P_M- and DP_M-optimum designs are recalled. Section 4 is dedicated to propose the DE-optimality and EDP_M- optimality criteria. The equivalence theorem is stated and proved for both. Finally, Section 5 is devoted to introduce the applications for the offered criteria.

2. Optimum Design Background

Throughout this paper, the generalized linear models (GLMs) are considered. GLMs extend normal theory of regression to encompass non-normal response distributions belonging to the one-parameter exponential family. As well as the normal, this includes gamma, Poisson, and binomial distributions, all of which are important in the analysis of data. GLMs relate the random term (the independent response Y) to the systematic term to the linear predictor ($X\theta$) via a link function $g(.)$, see *Agresti* [1].

Consider the generalized linear model GLMs

$$g\left(E\left(y\right)\right) = X\boldsymbol{\theta}$$

Three components are involved:

(1) *Random component*, which describes the response variable y and its probability distribution. The observations of $y = (y_1, \ldots .y_n)^T$ are independent.
(2) *A link function* $g\left(.\right)$ that is applied to each component of $E\left(y\right)$.
(3) *Linear Predictor* is $X\boldsymbol{\theta}$ for the parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots .\theta_p)^T$ and a $n \times p$ model matrix X involved p explanatory variables for n observations.

GLMs are commonly used to model binary or count data. Some common link functions are used such that the identity, logit, log and probit link to induce the traditional linear regression, logistic regression, Poisson regression models.

An approximate (continuous) design is represented by the probability measure ξ over δ. If the design has trials at n distinct points in δ , it can be written as

$$\xi = \left\{ \begin{array}{lll} x_1 & x_2 \ldots\ldots & x_n \\ w_1 & w_2 \ldots\ldots & w_n \end{array} \right\}$$

A design ξ defines, for $i = 1, \ldots, n$, the vector of experimental conditions $x_i \in \chi$ related to y_i, where χ is a compact experimental domain and the experimental weights w_i corresponding to each x_i, where $\sum_{i=1}^{n} w_i = 1$. The design space can be then expressed as

$$\delta = \{\xi_i \in X^n \times [0,1]^n : \sum_{i=1}^{n} w_i = 1\}$$

The cornerstone in optimal design is the Fisher information matrix. The Fisher information matrix $M(\boldsymbol{\theta}, \xi)$ is defined as

$$M(\boldsymbol{\theta}, \xi) = -E[\frac{\partial^2 l(\theta; y)}{\partial\theta\partial\theta^T}]$$

where $l(\boldsymbol{\theta}; y)$ is the log-likelihood function. The inverse of $M(\boldsymbol{\theta}, \xi)$ is the variance-covariance matrix of the unbiased parameter $\boldsymbol{\theta}$. From this point, $M(\boldsymbol{\theta}, \xi)$ is used to measure the amount of information that y carries about the parameter $\boldsymbol{\theta}$.

Due to *Atkinson et al.* [3], for the continuous design ξ, the information matrix is

$$M(\boldsymbol{\theta}, \xi) = \int_{\chi} f(x) f^T(x) \xi(dx) = \sum_{i=1}^{n} w_i f(x_i) f^T(x_i)$$

where $f^T(x_i)$ is the ith row of X.

Consider a Bernoulli random variable b_i. The likelihood for it is $L(\theta; b_i) = \pi_i^{b_i}(1-\pi_i)^{1-b_i}$, $b_i = 0, 1$. For logistic link function, $log(\frac{\pi_i}{1-\pi_i}) = x_i\theta$ where $E(\pi_i) = b_i$. In the case of logistic model, the Fisher information matrix $M(\boldsymbol{\theta}, \xi)$ is defined as

$$M(\boldsymbol{\theta}, \xi) = X^T W X$$

where W is diag of $(w_1\pi_1(1-\pi_1) \ldots\ldots.. w_n\pi_n(1-\pi_n))$.

3. E-,D-, P$_M$- and DP$_M$-Optimality

3.1. E-optimality. E-optimality was firstly introduced by *Ehrenfeld* [14]. *Heiligers* [15] derived the E-optimal polynomial regression designs and presented several numerical examples for some efficiency functions. *Pukelsheim and Studden* [22] determined the E-optimal design for the polynomial regression model on the interval [-1,1] where the variances of different observations are assumed to be constant and also investigated the relationship between E- and c-optimality. *Dette* [9] generalized the results of *Pukelsheim and Studden* [22] for polynomial regression models with non-constant variances proportional to specific functions. *Dette and Studden* [10] studied the geometry of E-optimality. E-optimal designs for polynomial regression without intercept was introduced by *Chang and Heiligers* [5], also E-optimal designs for polynomial spline regression were presented by *Heiligers* [16].

The E-optimality criterion determines the design such that the minimal eigenvalue, say $\lambda_{min}(M(\boldsymbol{\theta}, \xi))$, of the information matrix $M(\boldsymbol{\theta}, \xi)$ is maximal. This corresponds to the minimization of the worst variance of the least squares estimator for the linear combination of parameter estimation. It can be expressed as the following form

$$\Phi_{\mathrm{E}}(\xi) = max\lambda_{min}(M(\boldsymbol{\theta}, \xi))$$

Following *Pukelsheim and Studden* [22], the equivalence theorem for E-optimal design is stated that ξ_{E}^* is the E-optimum design if and only if there exists a nonnegative definite matrix A^* such that trA^*=1 and,

$$\max_{\lambda_{min}(M(\boldsymbol{\theta},\xi))} f^{\mathrm{T}}(x) A^* f(x) \leq \lambda_{min}(M(\boldsymbol{\theta}, \xi))$$

The matrix A^* can be represented as $A^* = \sum_{i=1}^{s} k_i p_{(i)} {p_{(i)}}^T$, where s is the multiplicity of the minimal eigenvalue, $k_i \geq 0$, $\sum_{i=1}^{s} k_i = 1$, $\{p_{(i)}\}_{i=1,2,\ \ldots,s}$ is a system of orthonormal eigenvectors corresponding to the minimal eigenvalue. The E- efficiency of a design ξ relative to the optimum design ξ_E^* is given by

$$Eff_E(\xi) = \frac{\lambda_{min}(M(\boldsymbol{\theta}, \xi))}{\lambda_{min}(M(\boldsymbol{\theta}, \xi_E^*))} \tag{3.1}$$

3.2. D-optimality. D-optimality is the vital design criterion, introduced by *Wald* [23], which interested of the efficient parameter estimates. The idea of D-optimality depends on maximization of logarithm the determinant of the information matrix $M(\theta,\xi)$,$log\,|M(\boldsymbol{\theta},\xi)|$, or equivalently, minimizes logarithm determinant of the inverse of information matrix, $log|M^{-1}(\boldsymbol{\theta},\xi)|$. Hence minimizes the generalized variance of $\widehat{\boldsymbol{\theta}}$, the BLUE of $\boldsymbol{\theta}$ is obtained.

A design ξ_D^* is a D-optimum design iff $d(x, \xi_D^*) \leq q$, $x \in \chi$, where

$$d(x, \xi_D) = f^T(x) M^{-1}(\boldsymbol{\theta}, \xi_D^*) f(x)$$

and q is the number of parameters for each model. The D-efficiency of any design ξ is given by

$$Eff_D(\xi) = \left(\frac{|M(\boldsymbol{\theta}, \xi)|}{|M(\boldsymbol{\theta}, \xi_D^*)|}\right)^{1/q} \tag{3.2}$$

3.3. P$_M$-optimality. *McGree* and *Eccleston* [19] proposed two types of probability - based optimality criteria that applied for GLMs. One of the forms of P-optimality criteria is $\mathrm{P_M}$-optimality criterion that defined as a maximization of the minimum probability of success. The form of this criterion is as follows:

$$\Phi_{P_M}(\xi) = \min\{\pi_i(\boldsymbol{\theta}, \xi_i)\}\ ,\ i = 1, 2, \ldots .n$$

where, $\pi_i(\boldsymbol{\theta}, \xi_i)$ is the i-th probability of success given by ξ_i.

Such a criterion seems useful in situations in which relatively high-expected number of successes are desired across all observations. This means, avoiding design points with a low to moderate probability of success.

A design $\xi_{P_M}^*$ is a P_M-optimum design for high probability of success iff $\psi_{P_M}\left(x, \xi_{P_M}^*\right) \leq 0$, $x \in \chi$, where

$$\psi_{P_M}\left(x, \xi_{P_M}^*\right) = \frac{\Phi_{\mathrm{P}_M}(x) - \Phi_{\mathrm{P}_M}\left(\xi_{P_M}^*\right)}{\Phi_{\mathrm{P}_M}\left(\xi_{P_M}^*\right)}$$

is the directional derivative of $\Phi_{P_M}(\xi)$. The P_M- efficiency of design ξ relative to the optimum design $\xi_{P_M}^*$ is

$$Eff_{P_M}(\xi) = \frac{\min\{\pi_i(\boldsymbol{\theta}, \xi_i)\}}{\min\{\pi_i\left(\boldsymbol{\theta}, \xi_{P_M}^*\right)\}}\ , \qquad i = 1, 2, \ldots, n \tag{3.3}$$

3.4. DP$_M$-optimality. For the aim of obtaining efficient parameter estimation and maximizing the minimum probability of success, *McGree and Eccleston* [19] have proposed DP$_M$-optimality criterion to combine D- and P_M-optimality criteria. In order to obtain design for both D- and P$_M$-optimality, consider a maximization of a weighted product of the efficiencies:

$$\left(\frac{|M(\boldsymbol{\theta},\xi)|}{|M(\boldsymbol{\theta},\xi_D^*)|}\right)^{\alpha/q}\left(\frac{\min\{\pi_i\left(\boldsymbol{\theta},\xi_i\right)\}}{\min\{\pi_i\left(\boldsymbol{\theta},\xi_{P_M}^*\right)\}}\right)^{1-\alpha} \tag{3.4}$$

where, the coefficients $0 \leq \alpha \leq 1$ .Taking the logarithm of (3.4) yields,

$$\Phi_{DP_M}(\xi) \ = \frac{\alpha}{q}\log|M(\boldsymbol{\theta},\xi)| + (1-\alpha)\log\min\{\pi_i\left(\boldsymbol{\theta},\xi_i\right)\} \tag{3.5}$$

The terms containing ξ_D^* and $\xi_{P_M}^*$ have been ignored, since they are constants when a maximization is taken over ξ. A DP$_M$-optimum design, $\xi_{DP_M}^*$, maximizes $\Phi_{DP_M}(\xi)$. The derivative function for $\Phi_{DP_M}(\xi)$ is given by

$$\psi_{DP_M}\left(x,\xi_{DP_M}^*\right) = \frac{\alpha}{q}f^{\mathrm{T}}(x)M^{-1}\left(\boldsymbol{\theta},\xi_{DP_M}^*\right)f(x) + (1-\alpha)\times \left(\frac{\Phi_{\mathrm{P}_M}(x) - \Phi_{\mathrm{P}_M}\left(\xi_{DP_M}^*\right)}{\Phi_{\mathrm{P}_M}\left(\xi_{DP_M}^*\right)}\right)$$

4. DE- and EDP$_M$- Compound Design Criteria

Several competing objectives may be relevant in the experimental design. The compound design criterion, which defined as a geometric weighted mean of efficiencies is contributed to achieve the possible requirement objectives.

In this section, we will introduce two new compound criteria; namely DE- and EDP$_M$-optimality. DE-optimality criterion aimed to obtain the dual goal of efficient parameter estimation and minimum variance. On the other hand, the EDP$_M$- optimality criterion can satisfy the triple objectives of DE-optimality criterion in addition to maximum probability. An approach to these design problems is to weight each criterion and find the design that optimizes the weighted average of the criteria.

4.1. DE-optimality. To combine D- and E-optimality, we need a common scale of comparison, as they are different completely in the behavior. In this case, the efficiencies of both criteria can be used. In other words, the weighted product of the efficiencies are maximized as

$$\left(\frac{|M(\boldsymbol{\theta},\xi)|}{|M(\boldsymbol{\theta},\xi_D^*)|}\right)^{\alpha/q}\left(\frac{\lambda_{min}(M(\boldsymbol{\theta},\xi))}{\lambda_{min}(M(\boldsymbol{\theta},\xi_E^*))}\right)^{1-\alpha} \tag{4.1}$$

where, the coefficients $0 \leq \alpha \leq 1$. Taking logarithm of (4.1),

$$\frac{\alpha}{q}\log|M(\boldsymbol{\theta},\xi)| - \frac{\alpha}{q}\log|M\left(\boldsymbol{\theta},\xi_D^*\right)| + (1-\alpha)\log\lambda_{min}\left(M\left(\boldsymbol{\theta},\xi\right)\right) - (1-\alpha)\log\lambda_{min}\left(M\left(\boldsymbol{\theta},\xi_E^*\right)\right)$$

which can be reduced to

$$\Phi_{DE}\left(\xi\right) = \frac{\alpha}{q}\log|M(\boldsymbol{\theta},\xi)| + (1-\alpha)\log\lambda_{min}\left(M\left(\boldsymbol{\theta},\xi\right)\right) \tag{4.2}$$

As the terms involving ξ_D^* and ξ_E^* are constants when a maximum is taken over ξ. Design maximized $\Phi_{DE}\left(\xi\right)$ are called DE-optimum and denoted by ξ_{DE}^*. The equivalence theorem for DE-criterion can be stated as follows:

4.1. Theorem. *A design ξ^*_{DE} is DE-optimal if and only if it satisfy the following inequality,*

$$\psi_{DE}\left(x, \xi^*_{DE}\right) \leq 1, x \in \chi$$

where the derivative function

$$\psi_{DE}\left(x, \xi^*_{DE}\right) = \frac{\alpha}{q} f^{\mathrm{T}}\left(x\right) M^{-1}\left(\boldsymbol{\theta}, \xi^*_{DE}\right) f\left(x\right) + \left(1-\alpha\right) \frac{f^{\mathrm{T}}\left(x\right) A^* f\left(x\right)}{\lambda_{min}\left(M\left(\boldsymbol{\theta}, \xi^*_{DE}\right)\right)} \tag{4.3}$$

Moreover, the upper bound of $\psi_{DE}\left(x, \xi^*_{DE}\right)$ is attained at the support points of the DE-optimum design.

Proof. Since $0 \leq \alpha \leq 1$, the criterion in (4.2) is a convex combination of two functions. The first one is D-optimality criterion which is concave optimality criterion. The second term is the logarithm of minimum eigenvalue of information matrix *M*. Since the information matrix $M = X^T X$ is real symmetric matrix, then its minimal eigenvalues can be written as follows:

$$\lambda_{min}(M) = \min_{\|\nu\|=1} < M\nu, \nu >$$

where, ν is a fixed vector and $< M\nu, \nu >$ is a linear function of *M*. From the fact that the minimum of any family of linear functions is concave, thus $\lambda_{min}(M)$ is concave function. Moreover, since M is symmetric matrix with positive diagonal elements, then M is positive definite matrix and therefore all its eigenvalues are positive. From convex analysis (see *Boyd and Vandenberghe* [4]), we can conclude that, $\log\lambda_{min}\left(M\left(\boldsymbol{\theta}, \xi\right)\right)$ is concave function of concave design criterion. Thus, the ED-criterion is a convex combination of two concave functions and therefore satisfies the conditions of convex optimum design theory and the proof is done. □

4.2. EDP$_M$-Optimum Designs. The formula of EDP$_M$- optimality can be derived using the weighted geometric mean of efficiencies design for E- , D- and P$_M$- optimum design as follows:

$$\left\{Eff_E\left(\xi\right)\right\}^{\alpha(1-\alpha)} \left\{Eff_D\left(\xi\right)\right\}^{(\alpha-1)^2} \left\{Eff_{\mathrm{P}_M}\left(\xi\right)\right\}^{\alpha} \tag{4.4}$$

Without loss of generality, powers can be taken to sum to one. The form of equation (4.4) is not unique; the powers can be changed to obtain different designs. We obtain the criterion by taking the logarithm of (4.4):

$$\begin{aligned}\Phi_{\mathrm{EDP}_M}\left(\xi\right) =& \alpha\left(1-\alpha\right)\log\lambda_{min}\left(M\left(\boldsymbol{\theta}, \xi\right)\right) + \\ & (\alpha-1)^2 \log\left|M(\boldsymbol{\theta}, \xi)\right| + \alpha\log(\min\{\pi_i\left(\boldsymbol{\theta}, \xi_i\right)\})\end{aligned} \tag{4.5}$$

The terms containing ξ^*_E, ξ^*_D and $\xi^*_{\mathrm{P}_M}$ have been ignored, since they are constants when a maximum is taken over ξ. Design maximizing $\Phi_{\mathrm{EDP}_M}\left(\xi\right)$ is called *EDP$_M$*-optimum design and denoted by $\xi^*_{\mathrm{EDP}_M}$. This optimum design satisfies the following general equivalence theorem:

4.2. Theorem. *For EDP$_M$-optimal design,$\xi^*_{\mathrm{EDP}_M}$, the following three statements are equivalent:*

(1) *A necessary and sufficient condition for a design* $\xi^*_{\mathrm{EDP}_M}$ *to be* EDP_M*-optimum is fulfillment of the inequality,* $\psi_{\mathrm{EDP}_M}\left(x,\xi^*_{\mathrm{EDP}_M}\right)\leq 1, \quad x\in\chi$*, where,*

$$\psi_{\mathrm{EDP}_M}\left(x,\xi^*_{\mathrm{EDP}_M}\right) = \alpha\left(1-\alpha\right)\ \frac{f^{\mathrm{T}}\left(x\right)A^{*}f\left(x\right)}{\lambda_{min}\left(M\left(\boldsymbol{\theta},\xi^*_{\mathrm{EDP}_M}\right)\right)} + \frac{\left(\alpha-1\right)^2}{q}f^{\mathrm{T}}\left(x\right)M^{-1}\left(\boldsymbol{\theta},\xi^*_{\mathrm{EDP}_M}\right)f\left(x\right) + \alpha\left(\frac{\Phi_{\mathrm{P}_M}\left(x\right)-\Phi_{\mathrm{P}_M}\left(\xi^*_{\mathrm{EDP}_M}\right)}{\Phi_{\mathrm{P}_M}\left(\xi^*_{\mathrm{EDP}_M}\right)}\right) \tag{4.6}$$

is the directional derivative of the criterion function (4.5)*.*

(2) *The upper bound of* $\psi_{\mathrm{EDP}_M}\left(x,\xi^*_{\mathrm{EDP}_M}\right)$ *is attained at the points of the optimum design.*

(3) *For any non-optimum design* ξ*, that is a design for which* $\Phi_{\mathrm{EDP}_M}(\xi)<\Phi_{\mathrm{EDP}_M}(\xi^*_{\mathrm{EDP}_M})$*,* $sup_{x\in\chi}\,\psi_{EDP_M}\left(x,\xi^*_{\mathrm{EDP}_M}\right)>1$

Proof. Since $0\leq\alpha\leq 1$ and the sum of coefficients $\alpha\left(1-\alpha\right)$, $\left(\alpha-1\right)^2$ and α equals one, the criterion in (4.5) is a convex combination of three functions. The first and the second one for E- and D- criterion, respectively, are concave function (see proof of Theorem 1). Since, the third function of the convex combination (4.5) is the logarithm of minimum probability of success and $\pi_i\left(\boldsymbol{\theta},\xi_i\right)\geq 0$, so that, $log(\min\{\pi_i\left(\boldsymbol{\theta},\xi_i\right)\})$ is concave function. Thus, the EDP_M- criterion is a convex combination of three concave functions and therefore satisfies the conditions of convex optimum design theory. In addition, the upper bound of $\psi_{\mathrm{EDP}_M}\left(x,\xi^*_{\mathrm{EDP}_M}\right)$ over $x\in\chi$ is one achieved at the points of the optimum design because the terms in (4.6) have been scaled. Thus, the theorem has been proved. □

5. Applications

In the following sections two separate illustrative examples are considered for logistic GLMs.

5.1. Application of the DE-Optimum Design. In this section, the DE - optimality criterion is applied to Logistic GLMs for binary data. By using the simulated designs (given in *Corana et al.* [7]), the DE - compound criterion can achieve the dual goal of obtaining efficient parameter estimation and minimizing the maximum variance of all possible normalized linear combinations of the parameter estimation.

The considering model has two main factor effects besides the interaction with initial parameter estimates $\theta=[1,-2,1,-1]^T$ as follows.

Consider the Logistic GLM;

$$\log\left(\frac{\pi}{1-\pi}\right)=1-2x_1+x_2-x_1x_2 \tag{5.1}$$

DE-optimal designs and their D- and E-efficiencies for α= 0.25, 0.5, 0.75 , 1 are obtained and presented in Table 1.

Table 1 shows the design that maximize the DE-criterion. It can be noticed that there is little changes in the design points with high variation in design weights. Figure 1 illustrates the E- and D-efficiencies for $\alpha=0.25,\ 0.5,\ 0.75$ *and* 1. The dot-dashed line represents the D-efficiency of the designs, and the solid line shows their E-efficiencies. The E-optimal design has a D-efficiency of 0.6383 and the D-optimal design has E-efficiency of 0.691145. By using the compound DE-criterion and compute $\Phi_{DE}(\xi)$ corresponding

Table 1. DE-optimum design and their E-and D- efficiencies for different values of α.

α	x_1	x_2	w_i	π	E_{eff}	D_{eff}
0.25	1.0000	-1.000	0.0835	0.2689	1	0.6383
	0.8020	1.000	0.0999	0.3999		
	-1.000	-1.000	0.1983	0.7311		
	-0.3980	1.000	0.6182	0.9596		
0.5	**1.000**	**-1.000**	**0.1570**	**0.2689**	**0.953696**	**0.9054**
	1.000	**1.000**	**0.1600**	**0.2889**		
	-1.000	**-1.000**	**0.2802**	**0.7311**		
	-0.1059	**1.000**	**0.4028**	**0.9103**		
0.75	1.000	1.000	0.2121	0.2689	0.802473	0.9864
	1.000	-1.000	0.2121	0.2689		
	-1.000	-1.000	0.2740	0.7311		
	0.0148	1.000	0.3017	0.8761		
1	1.000	-1.000	0.2500	0.2689	0.691145	1
	1.000	1.000	0.2500	0.2689		
	-1.000	-1.000	0.2500	0.7311		
	0.0680	1.000	0.2500	0.8577		

to those values of α, we will prefer the optimality criterion with the largest common efficiency for the dual aim, *i.e.* choosing $\alpha = 0.5$, the E-efficiency is increased to 0.953696 and achieving a D-efficiency of 0.9054. The DE-optimal design is then

$$\xi^*_{DE} = \left\{ \begin{array}{ccc} 1.000 & -1.000 & 0.1570 \\ 0.9669 & 1.000 & 0.1600 \\ -1.000 & -1.000 & 0.2802 \\ -0.1059 & 1.000 & 0.4028 \end{array} \right\}$$

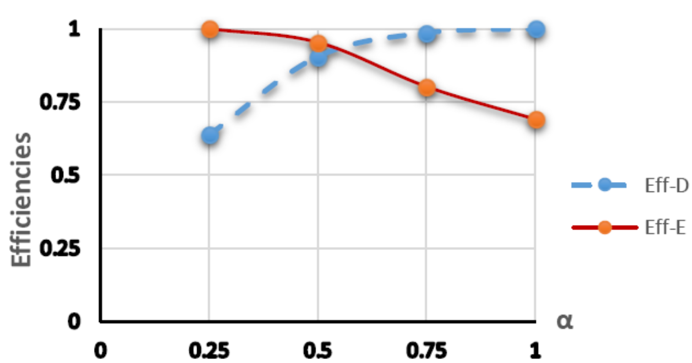

Figure 1. E- and D-efficiencies of DE-optimal designs for different values of α

5.2. Application of EDP$_M$ - Optimum Design. In this section, the EDP$_M$ -optimality criterion is applied to Logistic GLMs for binary data. The EDP$_M$ -compound criterion can provide triple goals of obtaining efficient parameter estimation plus maximizing the minimum eigenvalue of the information matrix and maximizing the minimum probability of a desired outcome. For the GLM which considered in (5.1). Let us consider another simulated designs given in *Corana et al.* [7], the EDP$_M$ -optimal designs and their D- , E- and P$_M$- efficiencies is obtained for $\alpha = 0.25,\ 0.5,\ 0.75\ ,\ 1$. The results are shown in Table 2.

Table 2. EDP_M-optimum designs and their E- D- and P_M-efficiencies for different values of α

α	x_1	x_2	w_i	π	E_{eff}	D_{eff}	$P_{M\,eff}$
0.25	0.3333	1.000	0.2469	0.7311	0.335688	0.3441	0.7360
	-0.1555	-0.2691	0.2500	0.7311			
	-1.000	-1.000	0.2469	0.7311			
	-0.3605	1.000	0.1281	0.9561			
	-1.0000	0.0408	0.1281	0.9561			
0.5	**0.2369**	**-0.8374**	**0.2500**	**0.4718**	**0.712605**	**0.7155**	**0.6507**
	0.7043	**1.000**	**0.2500**	**0.4718**			
	-0.1074	**1.000**	**0.2500**	**0.9107**			
	-1.0000	**-1.000**	**0.2500**	**0.7311**			
0.75	0.0509	1.000	0.2500	0.8638	0.97064	0.9790	0.5467
	0.9672	1.000	0.2500	0.2887			
	0.9016	-1.000	0.2500	0.2887			
	-1.0000	-1.000	0.2500	0.7311			
1	1.000	-1.000	0.2500	0.2689	1	1	0.2707
	1.000	1.000	0.2500	0.2689			
	-1.000	-1.000	0.2500	0.7311			
	0.0680	1.000	0.2500	0.8577			

Searching for the most higher common efficiencies for the three criteria, it is found that at $\alpha = 0.5$, where the E-efficiency is 0.7126, D-efficiency is 0.7155 and P$_M$-efficiency is 0.6507 as illustrated in Figure 2. Hence, the EDP$_M$ optimal design is then

$$\xi^*_{EDP_M} = \left\{ \begin{array}{cccc} 0.2369 & -0.8374 & 0.2500 & 0.4718 \\ 0.7043 & 1.000 & 0.2500 & 0.4718 \\ -0.1074 & 1.000 & 0.2500 & 0.9107 \\ -1.000 & -1.000 & 0.2500 & 0.7311 \end{array} \right\}$$

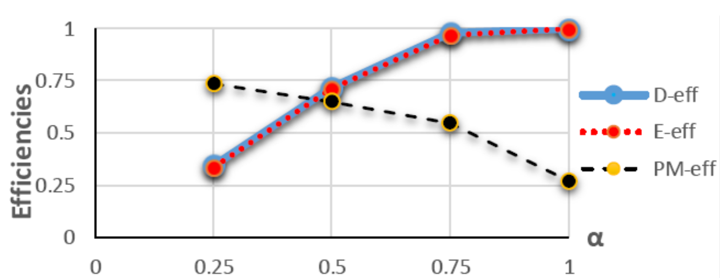

Figure 2. E- and D- and P_M efficiencies of EDP_M -optimal designs for different values of α

6. Conclusion

Most experimenters are interested in designing the experiments, which satisfy different goals. This requires developments of the field of constructing the compound optimality criteria. Hence, in this paper, two compound criteria named by DE and EDP_M are proposed. They offered multi-objective optimality properties of having efficient parameter estimation, minimizing the maximum variance of all possible normalized linear combinations of the parameter estimation and obtaining the maximum probability of a desired outcome. By applying these designs on logistic GLM, the largest common efficiency for the multi aim described above is achieved which indicated the benefits of using the proposed compound criteria.

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