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A Remark on the decay property for the Klein-Gordon equation in anti-de Sitter space time

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Abstract: We consider the inital value problem for the Klein-Gordon equation in anti-de Sitter spacetime. We derive the pointwise decay estimate by using the fundamental solution to the linear Klein-Gordon equation in anti-de Sitter spacetime with source term.

Keywords: Anti-De Sitter spacetime, Klein-Gordon eqution, fundamental solution, pointwise estimate.

1 Introduction

In this article, we consider the decay estimate for the solution of the following Klein-Gordon equation in anti-de Sitter spacetime:

$$\begin{aligned} &\partial_t^2 u - e^{2t}\Delta u + M^2 u = f(x,t), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ &\Phi(x,0) = 0, \qquad \partial_t \Phi(x,0) = 0, \qquad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $f \in C^\infty(\mathbb{R}^{n+1})$. The curved mass M is defined as follows: $M^2 := m^2 - n^2/4$ where $m > 0$ represents the physical mass. We briefly review how the equation in (1) is deduced. The line element in de Sitter spacetime is given by

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

where R is the radius of the universe. By using the Lemaitre-Robertson transformation in [6],

$$r' = \frac{r}{\sqrt{1 - r^2/R^2}} e^{-t/R}, \qquad t' = t + \frac{R}{2}\ln\left(1 - \frac{r^2}{R^2}\right), \qquad \theta' = \theta, \qquad \phi' = \phi,$$

the line element has the following form

$$ds^2 = -dt'^2 + e^{2t'/R}\left(dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2\theta' d\phi'^2\right). \tag{3}$$

Changing the coordinate as

$$t = t', \qquad x_1 = r'\sin\theta'\cos\phi', \qquad x_2 = r'\sin\theta'\sin\phi', \qquad x_3 = r'\cos\theta',$$

we get

$$ds^2 = -dt^2 + e^{2Ht}\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{4}$$

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where we put $H = 1/R$. We may write the line element in the general spatial dimensions as

$$ds^2 = -dt^2 + e^{2Ht}\left(dx_1^2 + ... + dx_n^2\right).$$

For simplicity, we set $H = 1$. Thus the corresponding metric is

$$(g_{ik})_{0\le i,k\le n} := \mathrm{diag}(-1, e^{2t}, ..., e^{2t}).$$

Let $g := \det(g_{ik})_{0\le i,k\le n}$ and $(g^{ik})_{0\le i,k\le n}$ be the inverse matrix of $(g_{ik})_{0\le i,k\le n}$. Then the scalar field Φ in de Sitter spacetime is described by the following equation:

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ik}\frac{\partial\Phi}{\partial x_k}\right) = m^2\Phi + f,$$

where $x_0 := t$. More explicitly, we get the following equation

$$\partial_t^2\Phi + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = f \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}. \tag{5}$$

If we introduce the new unknown function $u = e^{\frac{n}{2}t}\Phi$, the equation (5) takes the form of the linear Klein-Gordon equation in de Sitter spacetime

$$\partial_t^2 u - e^{-2t}\Delta u + M^2 u = f. \tag{6}$$

The time inversion transformation $t \to -t$ reduces the equation (6) to the equation in (1)

$$\partial_t^2 u - e^{2t}\Delta u + M^2 u = f,$$

that is regarded as the equation in anti-de Sitter space time.

In Minkowski spacetime, the initial value problem for the semilinear Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = |u|^\alpha u,$$

has been extensively investigated. The existence of global weak solutions has been obtained by Jörgens [5], Pecher [7], Brenner [2], Ginibre and Velo [3,4]. In order that the total energy is well-defined in the energy space, one needs the assumption $\alpha < 4/(n-1)$.

Turning back to the initial value problem (1), the following theorem obtained by Galstian [9] states the estimate in the Sobolev space $H^s(\mathbb{R}^n)$.

Theorem 1. *Let $u = u(x,t)$ be the solution of the initial value problem*

$$u_{tt} - e^{2t}\Delta u + M^2 u = f, \quad u(x,0) = 0, \quad u_t(x,0) = 0$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, where $f\in C^\infty(\mathbb{R}^{n+1})$. Let l be a nonnegative integer, $m \ge n/2$ and $n \ge 2$. Then there exists a constant $C > 0$ such that

$$\left\|(-\Delta)^{-s}u(.,t)\right\|_{W^{l,q}(\mathbb{R}^n)} \le Ce^{t\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\int_0^t \|f(.,b)\|_{W^{l,p}(\mathbb{R}^n)}(1+t-b)^{1-\mathrm{sgn}M}db \tag{7}$$

for all $t > 0$, *provided that* $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \leq 2s \leq n(\frac{1}{p} - \frac{1}{q}) < 2s+1$. *Here we have set* $M = \sqrt{m^2 - \frac{n^2}{4}}$.

The decay estimate is an important tool to prove the global existence for nonlinear partial differential equations. On the other hand, we are interested in the pointwise decay estimate for the solution of (1). The limiting case $q = \infty$ $(i.e.\ p = 1)$ for the decay estimate is excluded in Theorem 1. We remark that the decay rate for the pointwise decay estimate is faster than the decay rate for the L^2 decay estimate. Therefore, by using the pointwise decay estimate, we prove the following theorem.

Theorem 2. *Let* $u = u(x,t)$ *be the solution of the initial value problem*

$$u_{tt} - e^{2t}\Delta u + M^2 u = f, \quad u(x,0) = 0, \quad u_t(x,0) = 0$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, *where* $f \in C^\infty(\mathbb{R}^{n+1})$. *Let* $m \geq n/2$ *and* $n \geq 2$. *Then there exists a constant* $C > 0$ *such that*

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq C \int_0^t \|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} (1+t-b)^{1-\operatorname{sgn} M} db \tag{8}$$

for all $t > 0$. *Here we have set* $M = \sqrt{m^2 - \frac{n^2}{4}}$.

Here, $W^{k,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \ D^\alpha u \in L^p(\mathbb{R}^n),\ |\alpha| \leq k\}$, denotes a Sobolev space with the norm

$$\|u\|_{W^{k,p}(\mathbb{R}^n)} = \left(\sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |D^\alpha u|^p \right)^{1/p}, \quad (1 \leq p < \infty),$$
$$\|u\|_{W^{k,\infty}(\mathbb{R}^n)} = \sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_{\mathbb{R}^n} |D^\alpha u|.$$

2 Preliminary

Throughout this paper, the positive constants which may change, are denoted by the same letters C. We prepare the inequality for proving Theorem 2. First of all, we introduce the hypergeometric function $F(a,b;c;\zeta)$ and study its property. It is defined by the power series

$$F(a,b;c;\zeta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{\zeta^n}{n!}, \qquad |\zeta| < 1,$$

where $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, ...$, and we denoted

$$\begin{cases} (a)_0 = 1, \\ (a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)...(a+n-1), \qquad n = 1,2,3,.... \end{cases}$$

Here Γ is the gamma function (see e.g. [1]).

Lemma 1. *Assume that* $M \geq 0$. *Then, for* $z > 1$,

$$\int_0^{z-1} ((z+1)^2 - y^2)^{-\frac{1}{2}} \left| F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy \leq C(1+\ln z)^{1-\operatorname{sgn} M} (z-1)(z+1)^{-1}. \tag{9}$$

Proof. First, we consider $M > 0$. From the formulas 15.3.6 and 15.3.10 of chapter 15 in [1], we get

$$\left| F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \xi\right) \right| \leq C,$$

for all $\xi \in [0,1)$. Hence we have

$$\int_0^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)\right| dy \le C\int_0^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} dy$$
$$\le C(z-1)(z+1)^{-1}.$$

Next, we consider $M=0$. If $1<z<N$ with some constant N, then the argument of hypergeometric function is bounded,

$$\frac{(z-1)^2-y^2}{(z+1)^2-y^2} \le \frac{(z-1)^2}{(z+1)^2} \le \frac{(N-1)^2}{(N+1)^2} \le C$$

for all $y \in (0, z-1)$, and

$$\int_0^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)\right| dy \le C\int_0^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} dy$$
$$\le C(z-1)(z+1)^{-1}.$$

Then, we consider the case $z \ge N$. In particular, we choose $N>6$ and split the integral into two parts:

$$\int_0^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)\right| dy$$
$$= \int_0^{\sqrt{(z+1)^2-8z}} ((z+1)^2-y^2)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{4z}{(z+1)^2-y^2}\right)\right| dy$$
$$+ \int_{\sqrt{(z+1)^2-8z}}^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{4z}{(z+1)^2-y^2}\right)\right| dy$$

For the second integral, if $y \ge \sqrt{(z+1)^2-8z}$, then

$$\frac{4z}{(z+1)^2-y^2} \ge \frac{1}{2} \Rightarrow 0 < 1-\frac{4z}{(z+1)^2-y^2} \le \frac{1}{2}$$

implies

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{4z}{(z+1)^2-y^2}\right)\right| \le C.$$

Hence, we get

$$\int_{\sqrt{(z+1)^2-8z}}^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{4z}{(z+1)^2-y^2}\right)\right| dy \le C\int_0^{z-1} ((z+1)^2-y^2)^{-\frac{1}{2}} dy$$
$$\le C(z-1)(z+1)^{-1}.$$

For the first integral, $y \le \sqrt{(z+1)^2-8z}$ and $z \ge N \ge 6$ imply $8z \le (z+1)^2-y^2$. It follows

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{4z}{(z+1)^2-y^2}\right)\right| \le C\left|\ln\left(\frac{4z}{(z+1)^2-y^2}\right)\right| \le C(1+\ln z).$$

Hence, we get

$$\int_0^{\sqrt{(z+1)^2-8z}}((z+1)^2-y^2)^{-\frac{1}{2}}\left|F\left(\frac{1}{2},\frac{1}{2};1;1-\frac{4z}{(z+1)^2-y^2}\right)\right|dy \le C(1+\ln z)\int_0^{z-1}((z+1)^2-y^2)^{-\frac{1}{2}}dy$$
$$\le C(1+\ln z)(z-1)(z+1)^{-1}.$$

This completes the proof.

3 Fundamental solution of the Klein-Gordon equation

It was shown in [8] that the solution $u=u(x,t)$ of the initial value problem

$$u_{tt}-e^{2t}\Delta u+M^2u=f, \quad u(x,0)=0, \quad u_t(x,0)=0, \quad (x,t)\in\mathbb{R}^n\times(0,\infty), \tag{10}$$

with $f\in C^\infty(\mathbb{R}^n)$ is given by

$$u(x,t)=2\int_0^t db\int_0^{e^t-e^b}dr v(x,r;b)(4e^{b+t})^{iM}\left((e^t+e^b)^2-r^2\right)^{-\frac{1}{2}-iM}F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^b-e^t)^2-r^2}{(e^b+e^t)^2-r^2}\right), \tag{11}$$

where $v(x,t;b)$ is the solution to the following initial value problem for the wave equation

$$v_{tt}-\Delta v=0, \quad v(x,0;b)=f(x,b), \quad v_t(x,0;b)=0, \quad (x,t)\in\mathbb{R}^n\times(0,\infty), \tag{12}$$

where $b>0$.

4 Proof of theorem 2

The solution $v(x,t)$ of the initial value problem (12) satisfies

$$\|v(.,r;b)\|_{L^\infty(\mathbb{R}^n)}\le C(1+r)^{-\frac{n-1}{2}}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}, \tag{13}$$

for all $r>0$, if $n\ge 2$ (see e.g. [10]). By using (13), we have

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^n)}\le C\int_0^t\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}db$$
$$\times\int_0^{e^t-e^b}(1+r)^{-\frac{n-1}{2}}((e^t+e^b)^2-r^2)^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^b-e^t)^2-r^2}{(e^b+e^t)^2-r^2}\right)\right|dr$$
$$\le C\int_0^t\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}db$$
$$\times\int_0^{e^t-e^b}((e^t+e^b)^2-r^2)^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^b-e^t)^2-r^2}{(e^b+e^t)^2-r^2}\right)\right|dr.$$

If we change the variable by $r=e^by$, then we obtain

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^n)}\le C\int_0^t\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}db$$
$$\times\int_0^{e^{t-b}-1}((e^{t-b}+1)^2-y^2)^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+iM,\frac{1}{2}+iM;1;\frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2}\right)\right|dy. \tag{14}$$

Then, we apply Lemma 1 with $z = e^{t-b}$ to the second integrand of the right-hand side of the inequality (14);

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq C \int_0^t \|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} (e^{t-b}-1)(e^{t-b}+1)^{-1}(1+t-b)^{1-\operatorname{sgn} M} db,$$

and we obtain (8). This completes the proof of Theorem 2.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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