Bernstein Series Solution of the Heat Equation in 2-D

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Abstract: A broad class of steady-state physical problems can be reduced to finding the harmonic functions that satisfy certain boundary conditions. The Dirichlet problem for the Laplace equation is one of the above mentioned problems. In this paper, a numerical matrix method is developed for numerically solving the Heat equation in 2-D. The method converts the heat equation in 2-D to a matrix equation, which corresponds to a system of linear algebraic equations. Error analysis is included to demonstrate the validity and applicability of the technique. Finally, the effectiveness of the method is illustrated in the heat equation for a cut ring region.

Keywords: Heat equation, collocation method; Bernstein series solution; error analysis.

1 Introduction

Today, the theory of complex variables is used to solve problems of heat flow, fluid mechanics, aerodynamics, electromagnetic theory and practically every other field of science and engineering. A broad class of steady-state physical problems can be reduced to finding the harmonic functions that satisfy certain boundary conditions. The Dirichlet problem is to find a function $U(z)$ that is harmonic in a bounded domain $D \subset R^2$, is continuous up to the boundary ∂D of D, assumes the specified values $U_0(z)$ on the boundary ∂D , where $U_0(z)$ is a continuous function on ∂D. Let D be a rectangular region and ∂D is the boundary of D and can be formulated as

$$\nabla^2 U = 0, \ \ z \in D, \ \ U|_{z \in \partial D} = U_0(z)$$

Here, for a point (x, y) in the plane R^2, one takes the complex notation $z = x + yi$, $U(z) = U(x, y)$ and $U_0(z) = U_0(x, y)$ are real functions and $\nabla^2 U = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. Similarly the Dirichlet problem for the Poisson equation can be formulated as

$$\nabla^2 U = h(z), \ \ z \in D, \ \ U|_{z \in \partial D} = U_0(z)$$

The Green function of the Dirichlet problem for the Laplace differential equation in a triangle region was expressed in terms of elliptic functions and the solution of problem was based on the Green function, and therefore on elliptic functions by Kurt and Sezer [9,10]. Solution of the two-dimensional heat equation in a square region was given by Kurt [11]. Analytic solution was given two-dimensional heat equation for some regions by Baykuş Savaşaneril et al. [2,3,5,7]. The Chebyshev tau technique for the solution of Laplace's equation [1] and Chebyshev tau matrix method for Poisson-type equations in irregular domain [8] were studied by Ahmadi et al and Kong et al. Error analysis of the Chebyshev collocation method for linear second-order partial differential equations was expressed by Yüksel et al. [13,14,15].

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In this paper, we have solved a matrix method which is based on Bernstein polynomials and collocation points. Let we consider the Dirichlet problem for the Poisson equation on $D = [0,a] \times [0,b]$ can be formulated as

$$\nabla^2 U = G(x,y) \tag{1}$$

with the conditions defined at the points $x = \alpha_k$, $y = \beta_k$

$$\sum_{k=1}^{t} \sum_{i=0}^{1} \sum_{j=0}^{1} a_{i,j}^k U^{(i,j)}(\alpha_k, \beta_k) = \lambda_t$$

for $(\alpha_k, \beta_k) \in \partial D$, $a_{i,j}^k$, $k = 1,...,t$ and λ_t are constants. Here $G(x,y)$ are functions defined on D. We will find an approximated solution, namely Bernstein series solution, of (1) such that

$$U_{n,n}(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} \mathbf{B}_{i,n}(x) \mathbf{B}_{j,n}(y) \tag{2}$$

where $\mathbf{B}_{k,n}$, $0 \leq k \leq n$ are Bernstein polynomials.

In this study, we will find a matrix form for each term in (1). Substituting these matrix forms into (1) gives a matrix equation fundemental matrix equation. We then obtain a linear system by using collocation points. Moreover, this operation gives a new approximate solution.

2 The conform mapping of a cut-ring domain

The conformal mapping of the cut-ring in the z-plane, onto the rectangle in the w-plane can be written as [3].
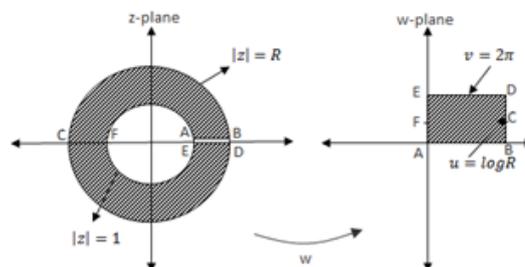
$$w = \log z$$

which is illustrated in Fig. 1. [4]

Fig. 1: Conformal mapping of the cut-ring in the z-plane onto the rectangular in the w-plane

Now let the function $t = \log z$, with $1 \leq |z| \leq 10^{2K}$; $k = 3 - 2\sqrt{2}$; $K' = 2K$; $K' \simeq \pi$ [12], maps the cut-ring onto the rectangle of the t-plane maps the rectangle with $0 < Re(t) < 2K$, $0 < Im(t) < 2K'$ onto the rectangle which is seen Fig.2
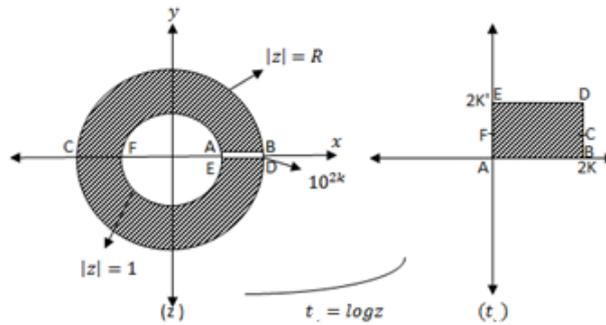
Fig. 2: Conformal mapping of the cut-ring in the z-plane onto the rectangular in the w-plane

3 Fundamental relation

Let $U_{n,n}$ be Bernstein series solution of (1). Also Let us find the matrix form of $U_{n,n}$ and $U_{n,n}^{(i,j)} = \frac{\partial^{i+j}U_{n,n}}{\partial x^i \partial y^j}$. $U_{n,n}$ can be written as

$$U_{n,n}(x,y) = \mathbf{B}_n(x)\mathbf{Q}_n(y)\mathbf{A} \tag{3}$$

where

$$\mathbf{B}_n(x) = \begin{bmatrix} B_{0,n}(x) \ B_{1,n}(x) \ \cdots \ B_{n,n}(x) \end{bmatrix}$$

$$\mathbf{Q}_n(y) = \begin{bmatrix} B_n(y) & 0 & \cdots & 0 \\ 0 & B_n(y) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n(y) \end{bmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} a_{00} \ a_{01} \ \cdots \ a_{0n} \ a_{10} \ a_{11} \ \cdots \ a_{1n} \ \cdots \ a_{n0} \ a_{n1} \cdots \ a_{nn}. \end{bmatrix}$$

Therefore, $U_{n,n}^{(i,j)}$ can be written as

$$U_{n,n}^{(i,j)}(x,y) = \mathbf{B}_n^i(x)\mathbf{Q}_n^j(y)\mathbf{A}. \tag{4}$$

on the other hand, $\mathbf{B}_n^{(i)}(x)$ can be written as [6]

$$\mathbf{B}_n^i(x) = \mathbf{X}^{(i)}(x)\mathbf{D}^T \tag{5}$$

where

$$\mathbf{D} = \begin{bmatrix} d_{00} \ d_{01} \ \cdots \ d_{0n} \\ d_{10} \ d_{11} \ \cdots \ d_{1n} \\ \vdots \ \vdots \ \ddots \ \vdots \\ d_{n0} \ d_{n1} \ \cdots \ d_{nn} \end{bmatrix}, \quad d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^j} \binom{n}{j} \binom{n-i}{j-i}, & i \le j \\ 0 & , & i > j \end{cases}$$

$$\mathbf{X}(x) = \begin{bmatrix} 1 \ x \ \cdots \ x^n \end{bmatrix}$$

for $\mathbf{X}^{(i)}(x)$, the relation

$$\mathbf{X}^{(k)} = \mathbf{X}(x)\mathbf{B}^k \tag{6}$$

is obtained where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Substituting (6) into (5) yields

$$\mathbf{B}_n^{(i)} = \mathbf{X}(x)\mathbf{B}^i\mathbf{D}^T \tag{7}$$

If a similar procedure is carried out, $\mathbf{Q}_n(y)$, the relation $\mathbf{Q}_n(y) = \overline{\mathbf{Y}}(y)\overline{\mathbf{D}}$ will be obtained where

$$\overline{\mathbf{Y}}(y) = \begin{bmatrix} Y(y) & 0 & \cdots & 0 \\ 0 & Y(y) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y(y) \end{bmatrix}, \quad \mathbf{Y}(y) = \begin{bmatrix} 1 & y & \cdots & y^n \end{bmatrix}$$

$$\overline{\mathbf{D}} = \begin{bmatrix} D^T & 0 & \cdots & 0 \\ 0 & D^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^T \end{bmatrix}$$

Thus, $\overline{\mathbf{Y}}^{(j)}(y)$ can be written as

$$\overline{\mathbf{Y}}^{(j)}(y) = \overline{\mathbf{Y}}(y)\overline{\mathbf{B}}^j \tag{8}$$

where

$$\overline{\mathbf{B}} = \begin{bmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{bmatrix}$$

Putting (7) and (8) into (4) yield the the matrix form of $U_{n,n}^{(i,j)}(x,y)$ as

$$U_{n,n}^{(i,j)}(x,y) = \mathbf{X}(x)\mathbf{B}^i\mathbf{D}^T\overline{\mathbf{Y}}(y)\overline{\mathbf{B}}^j\overline{\mathbf{D}}\mathbf{A} \tag{9}$$

By substituting (3) and (9) in (1), we obtain fundamental matrix equation as

$$[\mathbf{X}(x)\mathbf{B}^2\mathbf{D}^T\overline{\mathbf{Y}}(y)\overline{\mathbf{D}} + \mathbf{X}(x)\mathbf{D}^T\overline{\mathbf{Y}}(y)\overline{\mathbf{B}}^2\overline{\mathbf{D}}]\mathbf{A} = G(x,y) \tag{10}$$

By using the collocation points $\{(x_i,y_j) : 0 \le i,j \le n\}$ in (9), one obtains a matrix $\mathbf{W}_{(n+1)^2 \times (n+1)^2}$ whose m-th row, $1 \le m \le (n+1)^2$, comes from (x_k,y_l), $k = [|\frac{m}{n+1}|]$, $l = m - k(n+1) - 1$. Similarly, $\mathbf{G}$ is column matrix such as

$$[\mathbf{G}]_{1m} = G(x_t,y_l), \quad t = [|\frac{m}{n+1}|] \quad l = m - t(n+1) - 1$$

Thus, a linear system is obtained as

$$\mathbf{WA} = \mathbf{G} \tag{11}$$

By using (3) and (9), matrix relations are obtained for the conditions

$$\mathbf{CA} \sum_{k=0}^{t} \sum_{i=0}^{1} \sum_{j=0}^{1} a_{i,j}^{k} \mathbf{X}(\alpha_t) \mathbf{B}^i \mathbf{D}^T \overline{\mathbf{Y}}(\beta_t) \overline{\mathbf{B}}^j \overline{\mathbf{D}} \mathbf{A} = \lambda_t \tag{12}$$

respectively. Let us write it given in (12) as, respectively,

$$\mathbf{CA} = \mathbf{G}_1 \tag{13}$$

where $[\mathbf{G}_1]_{t1} = \lambda_t$. Combining $[\mathbf{W},\mathbf{G}]$ and $[\mathbf{C},\mathbf{G}_1]$ and yields a new system $[\tilde{\mathbf{W}},\tilde{\mathbf{G}}]$:

$$[\tilde{\mathbf{W}},\tilde{\mathbf{G}}] = \begin{bmatrix} \mathbf{W},\mathbf{G} \\ \mathbf{C},\mathbf{G}_1 \end{bmatrix} \tag{14}$$

By using the Gauss elimination method, and them by removing zero rows of augmented matrix $[\tilde{\mathbf{W}},\tilde{\mathbf{G}}]$, we obtain $[\tilde{\mathbf{W}},\tilde{\mathbf{G}}]$. If $\tilde{\mathbf{W}}$ is a square matrix, then the unknown matrix A is obtained as

$$\mathbf{A} = \tilde{\mathbf{W}}^{-1}\tilde{\mathbf{G}} \tag{15}$$

Otherwise, the collocation points should be changes such that $dim(\tilde{\mathbf{W}}) = (n+1)^2$. Also, if the columns of $\tilde{\mathbf{W}}$ are linearly independent, then the matrix A can be calculated by the pseudo inverse method; that is,

$$\tilde{\mathbf{W}}^{\dagger} = (\tilde{\mathbf{W}}^*\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{W}}^* \tag{16}$$

where $\tilde{\mathbf{W}}^*$ is the transpose of $\tilde{\mathbf{W}}$.

4 Accuracy of the solution and error analysis

We can easily check the accuracy of the method. Since the truncated Bernstein series (3) is an approximate solution of Eq.(1), when the function $U(x,y)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is, for $(x,y) = (x_q,y_q) \in \{0 \le x_q \le a,\ 0 \le y_q \le b\}$ $q = 0,1,2,...$ $E(x_q,y_q) = |D(x_q,y_q) - \lambda I(x_q,y_q)| \cong 0$ and $E(x_q,y_q) \le 10^{-k_q}$ (k_q positive integer). If max $10^{-k_q} = 10^{-k}$(k positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_q,y_q)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand, the error can be estimated by the function

$$E_N = \sum_{r=0}^{N} \sum_{s=0}^{N} a_{r,s} T_{r,s}(x,y) - g(x,y) - I(x,y) \tag{17}$$

If $E_N(x,y) \to 0$ when N is sufficiently large enough, then the error decreases.

5 Numerical example

In this section, an illustrative example is presented. We have performed numerical computation a computer program written in Maple.

5.1 Example

A thing cut-plate homogeneous thermally conducting plate lies in the *xy*-plane and occupies the cut-ring with $|z| = 10^{2K}$ and $|z| = 1$. The faces of the plate are insulated and no internal sources or sinks are present. The boundaries of the inner

periphery of the cut-ring $100°C$ and the boundaries of the outer periphery of the ring $0°C$. The problem is to find the steady temoperature $U(x,y)$ in the plate, the steady state temoperature U must satisfy

$$U_{xx}(x,y) + U_{yy}(x,y) = 0 \quad (1 < x^2 + y^2 < 10^{2K})$$

The boundary conditions are

$$U(x, +\sqrt{10^{2K} - x^2}) = 0°C \quad U(x, -\sqrt{10^{2K} - x^2}) = 100°C \quad (1 < x < 10^{2K})$$
$$U(x, +\sqrt{1 - x^2}) = U(x, -\sqrt{1 - x^2}) = 0°C \quad (1 < y < 10^{2K})$$

(18)

We can solving, the conformal mapping of the cut-ring in the z-plane onto the rectangle in the w-plane can be written as (Fig.3).
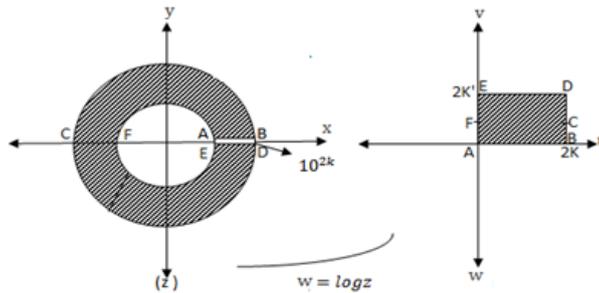

Fig. 3

$(|z| < 10^{2K} \wedge |z| < 1) \, k = 3 - \sqrt{2}, K' = 2K$ and $K = 1,582760 \, K' = 3,155879 \cong \pi$ According to this problem, we will get back to conformal mappings.
If the problem is rewritten, (19) is obtained

$$U_{xx} + U_{yy} = 0 \quad (0 < x < 2K, \quad 0 < y < 4K)$$
$$U(x,0) = 0°C \quad U(x,4K) = 100°C \quad (0 < x < 2K)$$
$$U(2K,y) = U(0,y) = 0°C \quad (0 < y < 4K)$$

(19)

And then the fundamental matrix equation for (19) is obtained as

$$[\mathbf{X}(x)\mathbf{B}^2\mathbf{D}^T\overline{\mathbf{Y}}(y)\overline{\mathbf{D}} + \mathbf{X}(x)\mathbf{D}^T\overline{\mathbf{Y}}(y)\overline{\mathbf{B}}^2\overline{\mathbf{D}}]\mathbf{A} = 0$$

Let the collocation points be the Chebyshev interpolation nodes

$$\left\{ (x_i, y_j) : 0 < i, j < n, \, x_i = \frac{1}{2} + \frac{1}{2}cos(\frac{2i-1}{2n})\pi, \, y_j = \frac{1}{2} + \frac{1}{2}cos(\frac{2i-1}{2n})\pi \right\}$$

or equidistant nodes. Then, $\mathbf{W}$ is a matrix rows are

$$\mathbf{X}(x_i)\mathbf{B}^2\mathbf{D}^T\overline{\mathbf{Y}}(y_j)\overline{\mathbf{D}} + \mathbf{X}(x_i)\mathbf{D}^T\overline{\mathbf{Y}}(y_j)\overline{\mathbf{B}}^2\overline{\mathbf{D}}$$

and **G** is a zero matrix. The condition matrices for $U(x,0) = 0°C$, $U(x,4K) = 100°C U(2K,y) = 0°C, U(0,y) = 0°C$ are obtained as

$$U_{n,n}(x_i,0) = \mathbf{X}(x_i)\mathbf{D}^T\overline{\mathbf{Y}}(0)\overline{\mathbf{D}}\mathbf{A} = 0 \qquad\qquad i = 0,1,..,n$$
$$U_{n,n}(x_i,4K) = \mathbf{X}(x_i)\mathbf{D}^T\overline{\mathbf{Y}}(4K)\overline{\mathbf{D}}\mathbf{A} = 100 \qquad\qquad i = 0,1,..,n$$
$$U_{n,n}(0,y_j) = \mathbf{X}(0)\mathbf{D}^T\overline{\mathbf{Y}}(y_j)\overline{\mathbf{D}}\mathbf{A} = 0 \qquad\qquad i = 0,1,..,n$$
$$U_{n,n}(2K,y_j) = \mathbf{X}(2K)\mathbf{D}^T\overline{\mathbf{Y}}(y_j)\overline{\mathbf{D}}\mathbf{A} = 100 \qquad\qquad i = 0,1,..,n$$

Combining these matrices gives the augmented matrix $[\tilde{\mathbf{W}}, \tilde{\mathbf{G}}]$. By calculating the coefficient matrix **A**, Bernstein series solutions are obtained for different n values.

for $N = 5$

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 \end{bmatrix}_{1\times 6}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6}, \quad \overline{\mathbf{B}} = \begin{bmatrix} B & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & B \end{bmatrix}_{36\times 36}$$

$$\mathbf{D} = \begin{bmatrix} 1 & \frac{(-1)^1}{(4K)^1}\binom{5}{0}\binom{5}{1} & \frac{(-1)^2}{(4K)^2}\binom{5}{0}\binom{5}{2} & \frac{(-1)^3}{(4K)^3}\binom{5}{0}\binom{5}{3} & \frac{(-1)^4}{(4K)^4}\binom{5}{0}\binom{5}{4} & \frac{(-1)^5}{(4K)^5}\binom{5}{0}\binom{5}{5} \\ 0 & \frac{(-1)^0}{(4K)^1}\binom{5}{1}\binom{4}{0} & \frac{(-1)^1}{(4K)^2}\binom{5}{1}\binom{4}{1} & \frac{(-1)^2}{(4K)^3}\binom{5}{1}\binom{4}{2} & \frac{(-1)^3}{(4K)^4}\binom{5}{1}\binom{4}{3} & \frac{(-1)^4}{(4K)^5}\binom{5}{1}\binom{4}{4} \\ 0 & 0 & \frac{(-1)^0}{(4K)^2}\binom{5}{2}\binom{3}{0} & \frac{(-1)^1}{(4K)^3}\binom{5}{2}\binom{3}{1} & \frac{(-1)^2}{(4K)^4}\binom{5}{2}\binom{3}{2} & \frac{(-1)^3}{(4K)^5}\binom{5}{2}\binom{3}{3} \\ 0 & 0 & 0 & \frac{(-1)^0}{(4K)^3}\binom{5}{3}\binom{2}{0} & \frac{(-1)^1}{(4K)^4}\binom{5}{3}\binom{2}{1} & \frac{(-1)^1}{(4K)^1}\binom{5}{4}\binom{2}{2} \\ 0 & 0 & 0 & 0 & \frac{(-1)^0}{(4K)^4}\binom{5}{4}\binom{1}{0} & \frac{(-1)^1}{(4K)^5}\binom{5}{4}\binom{1}{1} \\ 0 & 0 & 0 & 0 & 0 & \frac{(-1)^0}{(4K)^5}\binom{5}{5}\binom{0}{0} \end{bmatrix}_{6\times 6}$$

$$\overline{\mathbf{D}} = \begin{bmatrix} D^T & 0 & 0 & 0 & 0 & 0 \\ 0 & D^T & 0 & 0 & 0 & 0 \\ 0 & 0 & D^T & 0 & 0 & 0 \\ 0 & 0 & 0 & D^T & 0 & 0 \\ 0 & 0 & 0 & 0 & D^T & 0 \\ 0 & 0 & 0 & 0 & 0 & D^T \end{bmatrix}_{36\times 36}, \quad \mathbf{Y}(y) = \begin{bmatrix} 1 & y & y^2 & y^3 & y^4 & y^5 \end{bmatrix}_{1\times 6}$$

$$\overline{\mathbf{Y}}(y) = \begin{bmatrix} Y(y) & 0 & 0 & 0 & 0 & 0 \\ 0 & Y(y) & 0 & 0 & 0 & 0 \\ 0 & 0 & Y(y) & 0 & 0 & 0 \\ 0 & 0 & 0 & Y(y) & 0 & 0 \\ 0 & 0 & 0 & 0 & Y(y) & 0 \\ 0 & 0 & 0 & 0 & 0 & Y(y) \end{bmatrix}_{6\times 36}$$

and then, as a result, we get the following error analysis graphics and tables.

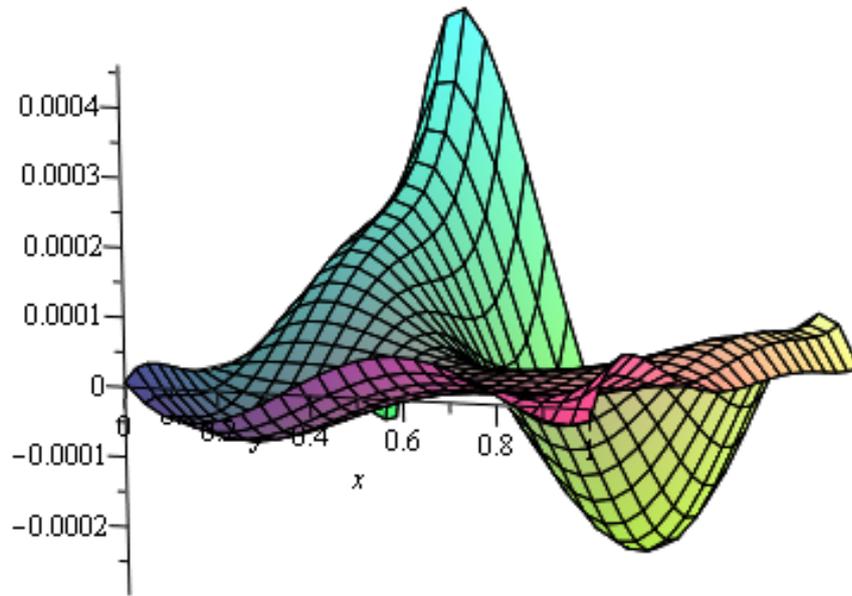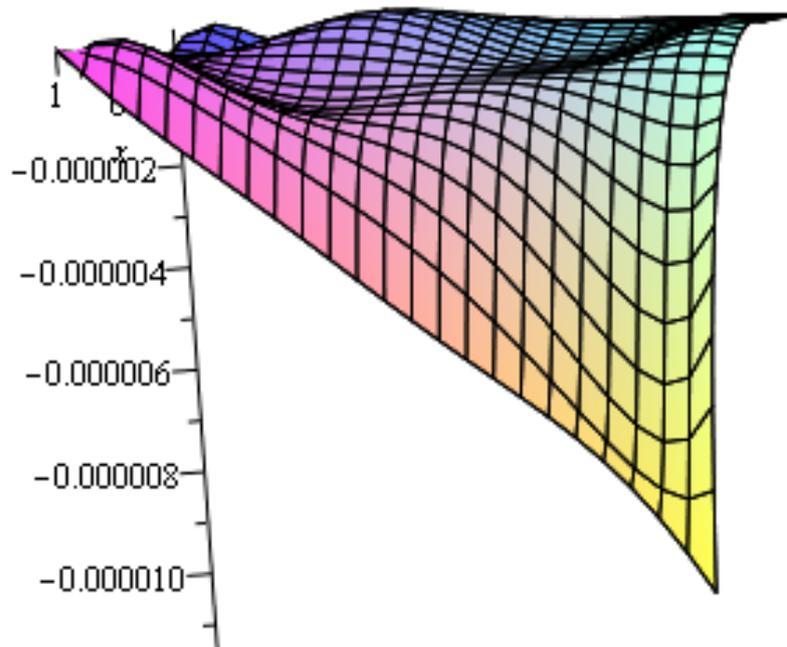**Fig. 4:** Error analysis for $N = 5$.

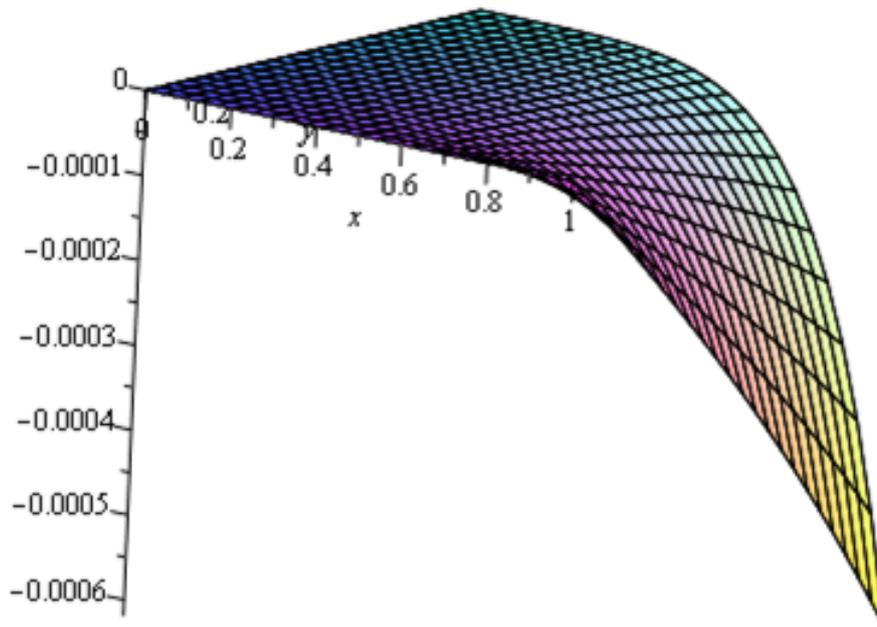**Fig. 5:** Error analysis for $N = 7$.
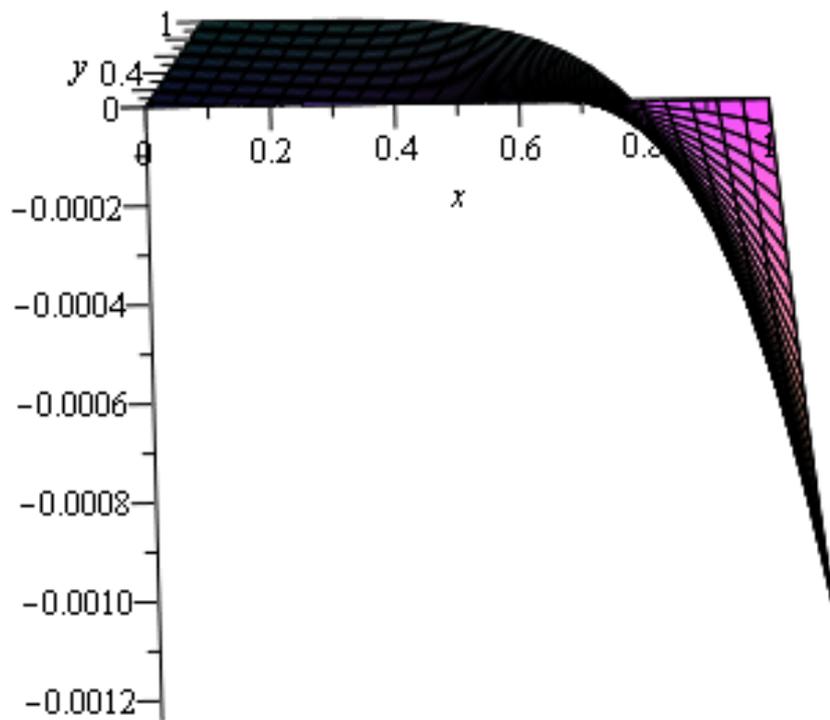
Fig. 6: Error analysis for $N = 9$.

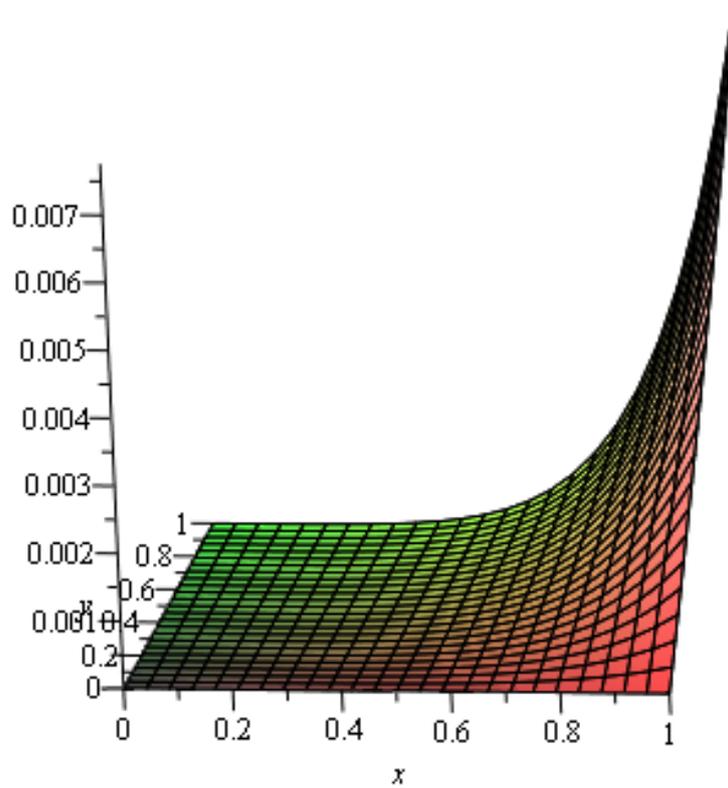**Fig. 7:** Error analysis for $N = 10$.

Fig. 8: Error analysis for $N = 12$.

Table 1: Comparison of the error analysis on ∂D that is the boundary of D for different values of N for example. $D : (0 < x < 2K, \; 0 < y < 4K)$

x	y	$N = 5$	$N = 7$	$N = 9$	$N = 10$	$N = 12$
0	1	-1.3153 10^{-4}	8.7700 10^{-9}	-1.5367 10^{-8}	1.1485 10^{-9}	-2.5990 10^{-8}
0	0.9	-1.1069 10^{-4}	2.5152 10^{-8}	-8.2612 10^{-9}	-7.3865 10^{-9}	-8.6300 10^{-9}
0	0.8	6.9917 10^{-6}	4.3672 10^{-8}	-8.1740 10^{-11}	-8.5024 10^{-13}	5.6959 10^{-15}
0	0.7	-5.9596 10^{-6}	-1.7412 10^{-7}	2.3310 10^{-10}	-7.4260 10^{-11}	-1.6508 10^{-8}
0	0.6	1.4245 10^{-5}	-2.5563 10^{-7}	6.2384 10^{-9}	1.3858 10^{-9}	-1.8501 10^{-8}
0	0.5	1.1422 10^{-5}	-1.5566 10^{-7}	1.3010 10^{-8}	-6.4459 10^{-9}	-1.2439 10^{-8}
0	0.4	4.1614 10^{-7}	2.6020 10^{-8}	2.7589 10^{-10}	-2.7536 10^{-9}	-1.3180 10^{-8}
0	0.3	2.7813 10^{-6}	2.4781 10^{-8}	-5.8942 10^{-9}	-4.6241 10^{-9}	6.7503 10^{-9}
0	0.2	9.2210 10^{-6}	-1.6962 10^{-7}	1.9069 10^{-8}	-5.1181 10^{-10}	-3.3228 10^{-9}
0	0.1	2.5483 10^{-5}	-2.6172 10^{-7}	-1.2031 10^{-9}	-2.8188 10^{-10}	-1.6020 10^{-9}
0	0	6.9917 10^{-6}	4.3672 10^{-8}	-8.1740 10^{-11}	-8.5024 10^{-13}	5.6959 10^{-15}

From Table 1, it is obvious that the results get better as N increase.

Table 2: Comparison of the error analysis in domain $D : (0 < x < 2K, \ 0 < y < 4K)$ for $N = 5, 7, 9, 10, 12$.

x	y	$N = 5$	$N = 7$	$N = 9$	$N = 10$	$N = 12$
1	1	-2.8334 10^{-5}	-9.7000 10^{-6}	-3.7732 10^{-4}	-2.3036 10^{-3}	4.0000 10^{-4}
0.5	0.5	-2.1443 10^{-6}	-3.8410 10^{-7}	-9.3046 10^{-6}	-3.1379 10^{-5}	1.6407 10^{-4}
0.2	0.8	-4.2706 10^{-5}	-1.5081 10^{-7}	-1.7677 10^{-11}	-1.2346 10^{-11}	-5.4833 10^{-15}
0.1	0.7	1.6544 10^{-4}	1.1086 10^{-7}	-3.0081 10^{-8}	-3.9190 10^{-8}	-1.2399 10^{-7}
0.6	0.6	-4.5004 10^{-5}	-1.2848 10^{-6}	-2.0034 10^{-5}	-1.4436 10^{-4}	3.1793 10^{-4}
0.3	0.2	-6.4769 10^{-5}	-5.7562 10^{-7}	1.8806 10^{-7}	-6.7845 10^{-7}	1.8098 10^{-6}
1	0.4	-5.8261 10^{-6}	-4.2100 10^{-6}	-1.4227 10^{-4}	-9.8134 10^{-4}	3.6700 10^{-4}
0.8	0.3	-5.7720 10^{-6}	-8.0000 10^{-7}	-5.9895 10^{-5}	-9.2438 10^{-5}	5.8856 10^{-4}
0.2	0.9	2.3544 10^{-4}	-1.0430 10^{-7}	-3.5920 10^{-7}	-4.0026 10^{-7}	2.6980 10^{-6}
0.5	0.7	-1.2069 10^{-4}	-1.2074 10^{-6}	-1.3524 10^{-5}	-4.4375 10^{-5}	2.3671 10^{-4}

The some calculating values of the error analysis give in Table 2 that is clearly shown when N values increase, error function values rapidly decrease for $N = 5, 7, 9, 10$ and 12.

6 Conclusion

The method of conformal mapping is a more flexible and far-reaching tool for the Laplace equation the plane. In a certain sense conformal mapping provides the analoque for elliptic differential equations of the method of characteristic developed for hyperbolic differential equations [4]. To solve partial differential equations numerically, we introduce a new matrix method depending on Bernstein polynomials and collocation points. Present method provides two main advantes: it is very simple to construct the main matrix equations and it is very easy for computer programming.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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