A STUDY ON MEASURING COMPLEXITY IN MUQARNAS PATTERNS

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ABSTRACT
Throughout history, mathematicians from different cultures and places traced other scholars' work to make a contribution and extend the knowledge of the geometry field. In the Islamic world, artisans combine the theoretical knowledge about geometry and making skills to build more sophisticated and complicated geometric patterns. Today, the underlying principles of geometric patterns are still a research subject of many physicists and mathematicians. Geometric patterns are used both in design and construction phases of muqarnas which is a specialized spatial element in Islamic Architecture. In the scope of this study, after a comprehensive analysis, new muqarnas patterns are generated by using algorithms.

Keywords: Muqarnas, Islamic Geometric Pattern, Geometry, Algorithmic Design, Complexity

MUKARNAS ÖRÜNTÜLERİNİN KARMAŞIKLIĞININ ÖLÇÜLMESİ ÜZERİNE BİR ÇALIŞMA

ÖZ
Tarih boyunca farklı coğrafyalardan ve kültürlerden matematikçiler birbirlerinden ilan olarak geometri alanına katkı yapmak için căbalamıştır. İslam Dünyası’nda sanatçılar geometri hakkında elde edilmiş teorik bilgiyi kullanmış ve daha sofistike geometrik örüntüler üretmek için yeteneklerini geliştirmiştir. Günümüzde, matematik ve fizik alanındaki araştırmacılar geometrik örüntülerin altyapısını oluşturan prensipleri araştırmaktadır. Öte yandan, geometrik örüntüler İslam Mimarisi’nin özeleşmiş bir mekânsal elemanı olan mukarnasların tasarım ve üretim süreçlerinde kullanılmaktadır. Bu çalışmanın kapsamında, kapsamlı bir analiz sonrasında algoritmik yöntemler kullanılarak yeni mukarnas örtüntülerini üretmiştir.

Anahtar Kelimeler: Mukarnas, İslami Geometrik Öruntüler, Geometri, Algoritmik Tasarım, Karmaşıklık

ATTEMPTS TO MEASURE THE UNCOUNTABLE UNITIES
Nature evolves into disorder and the total entropy of the universe constantly increases as it is accepted in the second law of thermodynamics. Although the notion of entropy has first appeared for measuring physical disorder of substance, in 1949 for measuring the disorder in information it was rediscovered by Shannon (1949). In the Shannon’s information theory, quantity of information that is carried with a
message is dependent on the number of probabilities of outcome that can be created by that message. In the case of only one probable output, it is not possible to obtain new information from the message. However, in dynamic systems of the real world, it is not easy to measure complexity through one way. For example, an entity with a minimum physical complexity might provide high level of visual complexities. Therefore, while the sum of sub-components of an object can perform one outcome, the inner relations of the components might lead more complex organisations (Klinger & Salingaros, 2000).

This notion entropy and the effort of measuring the complexity of entities/artefacts have been also influential in the fields of art and architecture since late 1970s. The built environments, buildings or abstract objects have been examined in the studies, addressed as sources of information and entropy values were calculated according to measurable physical factors (Arnheim, 1971; Krampen, 1979; Stamps, 1998; Stamps, 2004; Crompton, 2012). One can assert it is impossible to reduce any whole into measured qualities. Although Shannon’s theory has been mostly implemented in the content-independent measurements. The action of measurement is depended to the initial assumptions of the measurer. At this point, following concepts might be helpful to open further discussions: level of abstraction, and generative affordances of the encoded items. At which extent, one can encode any whole with minimum elements with the concern of minimum information loss?

In some cases, the action of measuring entropy might lead potentially conflicting results such as visual and physical complexity. Measurement would be only possible with the measurable qualities of a whole. It is yet discussable that whether the measured qualities might provide a holistic view or not. In each turn, measuring might neglect indivisible qualities. However, the basic motivation of this study lies on searching minimum number of indivisible tokens in a whole to constitute new variations, including the initial state. To mention Kant’s terminology, “manifold” is related with the combination of different representations which have potential to result with a “synthetic unity” (Kant, 1998). The ability of to lead new variations with minimum elements refers to generative affordances. In this sense, in the scope of this study, the generative affordances of muqarnas patterns will be investigated.

GEOMETRY BY MEANS OF TRIGGERING OF NEW IDEAS
By etymology, the origins of ‘geometry’ can be traced back in the action of measuring the earth in Ancient Greek (Url-1). Topology, the sum of “τόπος” and “-λογία”, merging the meanings of ‘locality’ and ‘study of a branch of knowledge’(Url-2). Therefore, in the very beginning the measuring action it can be assumed that there had been a relationship with the ‘observable context’ and the geometric inferences. Einstein (1921) states this reciprocity as the relationship between “intuitive content” and “logical-formal” axiomatics. By the time, the assumptions and the findings on geometry are transformed into formal and logical rules departing from the initial contexts. Related to this problematic, Einstein (1921) points out the necessity of complementary potential of “practical geometry”, apart from the “purely axiomatic geometry”.

Pythagoras (6th century BC) is commonly accepted one of the well-known mathematicians of Ancient Greek, introducing the formula for 3-4-5 right triangle. Further, his ideas would be influential in Euclid’s Theorem. However, the knowledge of creating a right triangle from sides with 3-4-5 unit proportions was present in Egypt before Pythagoras, while it had been calculated through rope and sticks located on the ground during ancient Egypt period (Osserman, 1995). Euclid’s (4th century BC) geometrical theorems were required an imagination and understanding of planar surfaces. In the 10th century, there had been an intensive translation activity from Greek manuscripts into Arabic. Euclid’s The Elements, The Data, The Optics, Archimedes’ Sphere and Cylinder, Measurement of the Circle, Heptagon in the Circle, Dioaphantos’ Arithmetics and Menelaos’ Sphrerica can be listed among those translated books (Berggren, 2017).
The emergence of geometrical patterns in art and architecture of Medieval Islam overlaps with the intensive mathematical translations and the investigations of that time. As Berggren states (2017) the strong tradition of geometrical design had been continued since ancient Egypt and Greek times. Although, the first implementations of geometric patterns of Islamic art and architecture were seen in the 9th and 10th century, both sophisticated explorations of 2D geometric patterns and also 3D spatial interpretations have gained acceleration in 11th and 12th century (Abdullahi & Embi, 2013). The introduction of Algebra by al-Khwârizmî (780-850) and new map projections by al-Bîrûnî (973-1048) had augmented the studies on mathematic. Abu’l-Wafa (940-998) introduced a manuscript on Geometric Constructions which were necessary for the artisans (Özdural, 2000). Umar al-Khayyâmî (1048-1131), the astronomist poet and mathematician have studies all kinds of cubic equations, conic sections, construction of the roots of these equations as line segments, apart from his manuscript titled Explanation of the Difficulties in the Postulates of Euclid in 1077 (Berggren, 2017). The definition of parabola was already known and studied before Al Kashi (1380-1429), who had contributed to arithmetics, measurements in astronomy and relatively precise calculation of pi with 16 decimal places after comma.

Figure 1: Left: Apollonios’ Parabola, 3rd century BC (Berggren, 2017; pp: 86); Right: Ibrâhîm b. Sinân’s Parabola Method, 10th century AD (Berggren, 2017, pp: 97).

DEBATES ON THE ROOTS OF GEOMETRY IN ISLAMIC ART

Historical foundations of development and dominance of symbolic expressions in Islamic Culture is a long story. Prohibition of figurative/pictorial expressions in art is not unique for Islamic Culture. Apart from the general distance to the figurative/pictorial expression in Judaism, in Christianity the prohibition of figurative/pictorial made a peak during the Iconoclastic movement in the 8th century in the Byzantine period had been resulted with destruction of many art works and figurative ornaments (Url-3). However, the longer term prohibition in Islamic Culture has triggered the search of new ways of representation through symbolic and geometric expressions. As a consequence of replacement of pictorial expression with symbolic representation in Islamic Art, geometric explorations have gained a crucial role becoming an experimental and dialogical field between pure geometry and symbols. Hence, both in level of surface decoration and also the spatial components, different variations of patterns have been mushroomed. It can be asserted that as a “generator”, geometry has been reflected into decorative forms and patterns and also design of structural and spatial organisations, distinctively between 12th-16th centuries. For example, during the Mamluk period (1250-1517), artisans and architects were encouraged to design sophisticated and unique ornaments. Furthermore, during Seljuk period, The Seljuk Empire (1037-1194) and Anatolian Seljuk Sultanate (and also known as The Sultanate of Rûm; 1075-1308) floral and figural ornaments have been transformed into characterized geometric patterns. During Ottoman Empire period (1299-1922), in terms of geometry in architecture,
architects and artisans mainly focused on form explorations and master planning, instead of merely extensive use of geometrical patterns as a decorative element (Abdullahi & Embi, 2013).

In a broader sense, art, architecture and built environment can be considered both as initiator and reflector of the cultural, social and economic life of the period they have been constituted. However, there are always risks in the retrospective look at to the past. This is because in each look, the observer needs a focus and a scope. Literally and conceptually, the action of focused looking neglects the other peripheral parts of the whole. When it comes to artefacts and the built environment which have been formed under countless forces throughout the time, consisting of -to name but a few- cultural, social, economic, physical, spatial, topological, topographical dimensions; it is not easy to acquire a holistic picture. However, the artefacts and the built environments still have potential to provide some clues about the spirit and the level of scientific knowledge of their own age. In relation with the geometric patterns, we argue that the visual and the spatial content from 11th, 12th and 13th centuries might be an opportunity to explore implicit and embedded geometries and relations, which might enable new interpretations.

Although, there is no clear evidence whether mathematical knowledge had influenced the development of the tradition of geometrical patterns or not in the Islamic Art, a comprehensive overview can be seen in Özdural’s (2000) article. Contrariwise, to mention a few of them, there are remarkable amount of studies on Islamic patterns which have been a source of inspiration for mathematicians (Castera, 1999; Castera, 2016; Harmsen, 2006; Cromwell, 2009) and physicians (Lu & Steinhardt, 2007; Arık & Sancak, 2007). Penrose’s 1979 article “Pentaplexity” provided a mathematical foundation for the studies on periodic and aperiodic tiling and the studies on Islamic patterns (Lu & Steinhardt, 2007; Arık & Sancak, 2007). Arık and Sancak (2007) developed a new tile set based on Penrose tiles which also include 2 tiles to cover the surface. Another method to create geometric pattern is called Girih Tiles. The set of Girih Tiles includes decagon, pentagon, hexagon, bowtie and rhombus shapes. Girih Tiles of the 15th century were studied to generate complex aperiodic patterns (Güzelci & Güzelci, 2015; Lu & Steinhardt, 2007).

Figure 2: Octagon patterns from Jameh Mosque in Isfahan (Photos by Authors).

Figure 3: Right: Pattern based on 14-gons and twinned heptagons (Hankin, 1905); Left: Pattern based on 16-gons, 12-gons and heptagons (Hankin, 1905).
In comparison to the 2D geometric patterns, there are limited number of research on the potentialities. Consisting of implicit and explicit geometrical relations, different composition patterns (such as bi-axial and radial), its affordances allowing fractal-like recursive generations through inflation and deflation methods, the potentiality to be represented with minimum amount of elements, involving topological relations between parts, muqarnas provides a reach research field for further investigations. This is why, this study can be considered as one of the attempts for gaining deeper understanding on and from muqarnas tectonics.

Yet, we keep in mind the risks of reduction which might lead departures from the context and result with misunderstandings, risks of looking back to past with the novel structures (language, concept or tools). As a spatial architectural element muqarnas, which can be seen in a widespread geographic area from Spain to India (Figure 4) becomes the main focus of this study for geometrical, topological and logical investigations.

**MUQARNAS: TRANSITION FROM SQUARE TO CIRCLE, FROM FINITE TO INFINITE**
Throughout history, the transition from a 2D square layout to a 3D circular shell has been a challenging topic. During construction of a dome the transitions from a square, a rectangular, a trapezoidal or a warped polygonal plan to a shell have been a field of experimentation, especially in masonry. This accumulation of construction knowledge resulted in the emergence of specific archetypes such as dome, muqarnas, pendentives, Turkish Triangle and squinches. As an architectural element muqarnas has been one of the prolific design elements which has been used in entrance portals, cornices, transition element, mihrab niches, domes, minarets, iwans (Ödekan, 1988) in various geometrical organisations.

There is not a consensus between researchers on the origin of muqarnas. It is commonly said that the word muqarnas refers to a stalactite vault, which conveys the relationship between the word and the potential meanings of formation. There are different debates on how, where and when the root of the word was derived. *The Encyclopaedia of Islam* defines muqarnas as a type of decoration typical for Islamic architecture all over the central and eastern parts of the Muslim world; for its counterpart in the Muslim West, see mukarbas (same as mocarabe) (Houtsma et al, 1934). *Diccionario de la lengua españa* explains the word mocarabe as ‘the formation by geometric combination of interlocking prisms, externally cut in concave surfaces and used as decoration in vaults, cornices, etc’. Among these semantic or phonetic similarities, the verb ‘qarnes’ in Syriac means to ‘hammer’, with its passive participle being ‘muqarnas’ (Heinrichs,1997). One of the notable resources on the traditional making techniques of muqarnas is Al Kashi’s manuscripts which were completed and presented in 1427. Al Kashi’s manuscript provides shape based vocabulary elements such as square, rhombus, half-rhombus, almond, small biped, jug, large biped and barley kernel; topological relation based vocabulary elements such as units, cells, roofs, filling elements. Apart from these, parametric and algorithmic descriptions of traditional muqarnas making were introduced, involving geometrical parameters such as angles, dimensions, adjacency and detailed instructions for artisans such as mathematical
definitions, construction rules (Al Kashi, 1977; Özdural, 2000). These earlier assumptions, 
descriptions and definitions of the 15th century have been insightful even for the novel studies (Dold-

Figure 5: Composing a square from five square (Özdural, 2000).

While mathematicians had been teaching muqarnas making techniques to artisans, the logic of 
“dividing and assembling”, ”cut and paste” were employed in the 10th century. Citing to the 10th 
century manuscript “On the Geometric Constructions Necessary for the Artisan” by Abu’l-Wafa’s 
(940-998), Özdural (2000) provides general explanations and specific techniques for generating 
geometric patterns from the sum of squares. It is known that, in traditional muqarnas making, it was a 
common tendency to begin with 2D pattern generation and afterwards building the 3D form through 
plane projections. The arrangements of muqarnas elements were mostly symmetrical and the niches 
were also equal (Alaçam et al. 2017).

Figure 6: Left: Northwest Iwan, Jameh Mosque in Isfahan. Middle: Southeast Niche, Jameh Mosque 
in Isfahan. Right: Southwest Iwan, Jameh Mosque in Isfahan (Photos by Authors); Patterns 
regenerated based on Shiro Takahashi’s Drawings (Url-4).
REVISITING MUQARNAS BY USING ALGORITHMS

“The intellectual power of an algorithm lies in its ability to infer new knowledge and to extend certain limits of the human intellect.” (Terzidis, 2003). The term algorithm is related with transformation of a process into a procedure with finite number of steps. Constitution of an algorithm is considered as a way of thinking, instead of merely translating a process into computer environment. Before computers, Al-Kashi’s detailed instructions for artisans on muqarnas making can be accepted as algorithms. Cache suggests considering Vitruvius’s descriptions on how to settle on a hill as algorithms. Whether computers are used or not, an algorithm consists of deduction, induction, abstraction, generalization, and structured logic (Terzidis, 2003). There are several reasons about the selection of muqarnas as a source of intuition for algorithmic inquiries. To name but a few, the relationship between the part and the whole, the changing tension between its visual and mathematical complexity, consisting of repetitive and iterative elements, involving explicit geometric characteristics apart from being open to new and multiple explorations can be listed.

Repetition and Iteration

In order to extract qualities for constructing assumptions and rules derived from a muqarnas pattern, reductions are required at some extent. Indivisible tokens such as numbers (angle, side, distance), shapes (point, line, polygon) and operations (rotate, translate, fold, rescale, mirror) can be used by means of decomposition of the whole. Similarities, differences and proximity among different parts, thresholds, transformations from one state to another (simple to complex, one shape to another, one direction to another, etc.) can be considered helpful faculties for the algorithmic investigations.

Figure 7: Complex muqarnas patterns. Left: Southeast Iwan, Jameh Mosque in Isfahan. Right: Main Entrance Portal, Jameh Mosque in Isfahan (Photos by Authors).

Figure 8: (a) Pattern regenerated based on Shiro Takahashi’s Drawings (Url-4) (b) Photo of Muqarnas Dome in Mevlana Dergah/Tekke & Museum (Url-5) (c) Geometrical analyses of geometrical relations (d) Basic principle and components of the pattern.
In the scope of this study, the muqarnas organisation of Mevlana Dergah/Tekke & Museum (Figure 8b) was selected to be analysed. At first look, the 3D muqarnas organisation might seem too complex to an observer. After a while, some similarities, differences and transitions between parts (whatever perceived as a part) might appear. Apart from the real-time experience on the site, the 2D projection drawing (Figure 8a) might be also helpful to explore further details such as, an emphasis on the center, four star-like octagons indicating new centers in between the initial center and the corners of the space, the layers and the transition between them, so on. We analysed the 2D drawing of the muqarnas (Figure 8c), concerning whether it is possible to constitute the muqarnas pattern of Mevlana Dergah/Tekke & Museum without any mathematical knowledge through traditional cut-paste methods. The analyses show that (Figure 8c, 8d), if one has a square shape, it is possible to achieve a 2D layout for generating pattern based on cut-paste and rotate operations (Figure 9a). If the length (L) of a square plan is given as an initial input, for the calculation of the length of the square unit we needed a quadratic equation (Figure 8d). Further to achieving a square-based layout, it would be possible for an artisan to explore new 2D compositions, develop the pattern and make decisions for the layers to construct the 3D muqarnas.

Figure 9: Patterns based on cut-paste logic derived from muqarnas pattern of Mevlana Dergah/Tekke & Museum. Asymmetrical details were intentionally left for showing the potentials of differentiation.
Parametric Inquiry and Recursion

The way the procedures are specified and the initial assumptions have influence on the overall process. All the decisions taken in the earlier phases affect the outputs resulted by the algorithms. The same shape can be represented through different number or variables and procedures. To exemplify, while drawing an equilateral polygon on a planar surface, if the number of the sides and the dimension of one side is given as an initial parameter, the angular relations can be calculated based on simple equations. In this case, a pivot point might be needed as an initiator for shape generation. However, if the number of the equilateral polygon is known, only a pivot point and dimension of the side would be sufficient for a code structure.

The decisions such as defining angular relations, topological relations, selection of initial shapes, the ways the shapes are defined, number of the parameters, data type of the parameters, dimensions between different reference points such as center or corner of the shapes will form the potentialities of the procedure. Moreover the operations, repetition of the operations, the functions, recursive relations between different parts of the algorithm will be influential, apart from the inputs and assumptions.

Figure 10: A computer generated visual composition generated through coding in Processing 3.3.6.

Figure 11: Computer generated pattern explorations in Processing 3.3.6.
The compositions shown in Figure 10 and Figure 11 are based on 2D array logic in changing the locations of the shapes. The same code structure allows generation of different equilateral polygons. The generic code for equilateral polygons accepts four parameters to begin drawing the shape: An X value and Y value for a pivot/initiator point, a length value (L) and number of the sides (n). The number of sides are taken from the distance from the origin. It is a simple example for drawing a equilateral polygons by using algorithms. However, depending on the initial assumptions and shape definition, the overall composition can be generated towards different level of complexity.

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