Set-Valued Additive Functional Equations

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ABSTRACT. In this paper, we introduce set-valued additive functional equations and prove the Hyers-Ulam stability of the set-valued additive functional equations by using the fixed point method.

Keywords: Hyers-Ulam stability; set-valued additive functional equation; fixed point.

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1. INTRODUCTION AND PRELIMINARIES

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering papers by Aumann [4] and Debreu [11] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [2], McKenzie [24], the monographs by Hindenbrand [18], Aubin and Frankowska [3], Castaing and Valadier [7], Klein and Thompson [22] and the survey by Hess [17].

The stability problem of functional equations originated from a question of Ulam [37] concerning the stability of group homomorphisms. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [35] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

The functional equation $f(x + y) = f(x) + f(y)$ is called an *additive functional equation*. In particular, every solution of the additive functional equation is said to be an *additive mapping*. The functional equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ is called a *Jensen additive functional equation*. In particular, every solution of the Jensen additive functional equation is said to be a *Jensen additive mapping*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [15, 16, 20, 36]).

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [8, 12] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
  (1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
  (2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
  (3) y^* *is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
  (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [9, 10, 13, 26, 32, 34]).

Let Y be a Banach space. We define the following:
2^Y : the set of all subsets of Y;
$C_b(Y)$: the set of all closed bounded subsets of Y;
$C_c(Y)$: the set of all closed convex subsets of Y;
$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.
On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$\begin{aligned} d(x, C) &= \inf\{\|x - y\| : y \in C\}, & x \in Y, \\ s(x^*, C) &= \sup\{\langle x^*, x \rangle : x \in C\}, & x^* \in Y^*. \end{aligned}$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$H(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \qquad C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.1. *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*
  (a) $H(C \oplus C', K \oplus K') \leq H(C, K) + H(C', K')$;
  (b) $H(\lambda C, \lambda K) = \lambda H(C, K)$.

Let $(C_{cb}(Y), \oplus, H)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, H)$ is a complete metric semigroup (see [7]). Debreu [11] proved that $(C_{cb}(Y), \oplus, H)$ is isometrically embedded in a Banach space as follows.

Lemma 1.1. [11] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\| \cdot \|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, H) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*
  (a) $j(A \oplus B) = j(A) + j(B)$;
  (b) $j(\lambda A) = \lambda j(A)$;
  (c) $H(A, B) = \|j(A) - j(B)\|_u$;
  (d) $j(C_{cb}(Y))$ *is closed in $C(B_{Y^*})$*
for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), H)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [6]). In this case, the Debreu integral of f in Ω is the unique element $(D) \int_\Omega f d\nu \in C_{cb}(Y)$ such tha $j((D) \int_\Omega f d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [5, 27, 28, 29, 30, 31, 33]).

Using the fixed point method, we prove the Hyers-Ulam stability of the following set-valued additive functional equations

$$(1.1) \qquad H(F(x + y), F(x) \oplus F(y)) \leq \varphi(x, y)$$

and

$$(1.2) \qquad H\left(2F\left(\frac{x + y}{2}\right), F(x) \oplus F(y)\right) \leq \varphi(x, y).$$

Throughout this paper, let X be a real normed space and Y a real Banach space.

2. STABILITY OF THE SET-VALUED ADDITIVE FUNCTIONAL EQUATION (1.1)

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued additive functional equation (1.1).

Definition 2.1. [23] *Let $F : X \to C_{cb}(Y)$. The set-valued additive functional equation is defined by*

$$F(x + y) = F(x) \oplus F(y)$$

for all $x, y \in X$. Every solution of the set-valued additive functional equation is called a set-valued additive mapping.

Definition 2.2. *Let $F : X \to C_{cb}(Y)$. The set-valued Jensen additive functional equation is defined by*

$$2F\left(\frac{x + y}{2}\right) = F(x) \oplus F(y)$$

for all $x, y \in X$. Every solution of the set-valued Jensen additive functional equation is called a set-valued Jensen additive mapping.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying

$$(2.3) \qquad H(F(x + y), F(x) \oplus F(y)) \leq \varphi(x, y)$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to (C_{cb}(Y), H)$ such that

$$(2.4) \qquad H(F(x), A(x)) \leq \frac{L}{2 - 2L} \varphi(x, x)$$

for all $x \in X$.

Proof. Let $y = x$ in (2.3). Since $F(x)$ is convex, we get

(2.5)
$$H(F(2x), 2F(x)) \leq \varphi(x, x)$$

and so

(2.6)
$$H\left(F(x), 2F\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x, x)$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = inf\{\mu \in (0, \infty) : \ H(g(x), f(x)) \leq \mu\varphi(x, x), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [14, Theorem 2.4] and [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g, f) = \varepsilon$. Then

$$H(g(x), f(x)) \leq \varepsilon\varphi(x, x)$$

for all $x \in X$. Hence

$$H(Jg(x), Jf(x)) \quad = \quad H\left(2g\left(\frac{x}{2}\right), 2f\left(\frac{x}{2}\right)\right) = 2H\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \leq L\varepsilon\varphi(x, x)$$

for all $x \in X$. So $d(g, f) = \varepsilon$ implies that $d(Jg, Jf) \leq L\varepsilon$. This means that

$$d(Jg, Jf) \leq Ld(g, f)$$

for all $g, f \in S$.

It follows from (2.6) that $d(F, JF) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

(2.7)
$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x)$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$H(F(x), A(x)) \leq \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^n F, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 2^n F\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(F, A) \leq \frac{1}{1-L}d(F, JF)$, which implies the inequality

$$d(F, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

$$H(A(x + y), A(x) \oplus A(y)) = \lim_{n \to \infty} 2^n H\left(F\left(\frac{x + y}{2^n}\right), F\left(\frac{x}{2^n}\right) \oplus F\left(\frac{y}{2^n}\right)\right)$$

$$\leq \lim_{n \to \infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in X$. Since $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$, $\operatorname{diam}\left(2^n F\left(\frac{x}{2^n}\right)\right) \leq \frac{2^n}{2^{rn}} M\|x\|^r$ for all $x \in X$ and so $A(x) = 2^n F\left(\frac{x}{2^n}\right)$ is a singleton set and $A(x+y) = A(x) \oplus A(y)$ for all $x, y \in X$. $\square$

Corollary 2.1. *Let $p > 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying*

$$(2.8) \qquad\qquad H(F(x + y), F(x) \oplus F(y)) \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to Y$ satisfying

$$H(F(x), A(x)) \leq \frac{2\theta}{2^p - 2}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $L := 2^{1-p}$ and

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. $\square$

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying (2.3). Let r and M be positive real numbers with $r < 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to (C_{cb}(Y), H)$ such that

$$H(F(x), A(x)) \leq \frac{1}{2 - 2L}\varphi(x, x)$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$H\left(F(x), \frac{1}{2}F(2x)\right) \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.2. *Let $1 > p > 0$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying (2.8). Let r and M be positive real numbers with $r < 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued additive mapping $A : X \to Y$ satisfying*

$$H(F(x), A(x)) \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $L := 2^{p-1}$ and

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$.   □

3. STABILITY OF THE SET-VALUED JENSEN ADDITIVE FUNCTIONAL EQUATION (1.2)

Using the fixed point method, we prove the Hyers-Ulam stability of the set-valued Jensen additive functional equation (1.2).

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying $F(0) = \{0\}$ and

$$(3.9) \qquad H\left(2F\left(\frac{x+y}{2}\right), F(x) \oplus F(y)\right) \leq \varphi(x, y)$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued Jensen additive mapping $A : X \to (C_{cb}(Y), H)$ such that

$$H(F(x), A(x)) \leq \frac{1}{1-L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.9). Since $F(x)$ is convex, we get

$$(3.10) \qquad H\left(F(x), 2F\left(\frac{x}{2}\right)\right) \leq \varphi(x, 0)$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g, f) = inf\{\mu \in (0, \infty) : \ H(g(x), f(x)) \leq \mu\varphi(x, 0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [14, Theorem 2.4] and [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.10) that $d(F, JF) \leq 1$.

The rest of the proof is similar to the proof of Theorem 2.2.   □

Corollary 3.3. *Let $p > 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying $F(0) = \{0\}$ and*

$$(3.11) \qquad H\left(2F\left(\frac{x+y}{2}\right), F(x) \oplus F(y)\right) \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Let r and M be positive real numbers with $r > 1$ and $\operatorname{diam} F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued Jensen additive mapping $A : X \to Y$ satisfying

$$H(F(x), A(x)) \leq \frac{2^p\theta}{2^p - 2}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $L := 2^{1-p}$ and

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X.$. $\square$

Theorem 3.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi \left(\frac{x}{2}, \frac{y}{2} \right)$$

for all $x, y \in X$. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying $F(0) = \{0\}$ and (3.9). Let r and M be positive real numbers with $r < 1$ and $\mathrm{diam}\, F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued Jensen additive mapping $A : X \to (C_{cb}(Y), H)$ such that

$$H(F(x), A(x)) \leq \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.11) that

$$H\left(F(x), \frac{1}{2}F(2x) \right) \leq \frac{1}{2}\varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.4. $\square$

Corollary 3.4. *Let $0 < p < 1$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $F : X \to (C_{cb}(Y), H)$ is a mapping satisfying $F(0) = \{0\}$ and (3.11). Let r and M be positive real numbers with $r < 1$ and $\mathrm{diam}\, F(x) \leq M\|x\|^r$ for all $x \in X$. Then there exists a unique set-valued Jensen additive mapping $A : X \to Y$ satisfying*

$$H(F(x), A(x)) \leq \frac{2^p\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.5 by taking $L := 2^{p-1}$ and

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. $\square$

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