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The B-spline Collocation Approach for Coupled Klein-Gordon Equation

İkili Klein-Gordon Denklemi İçin B-spline Kollokasyon Yaklaşımı

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Abstract

This research presents a new approach for obtaining numerical solutions of Coupled Klein Gordon equation using the collocation method which based on cubic B-spline base functions and finite element approximation. The main advantage of the collocation method is that the structure of the method is simple and the computational cost is low. It also provides an easy and simpler procedure for solving various problems involving differential equations that model real-world phenomena. In the current research, the temporal and spatial partial derivatives are discretized with using approximate solution which is formed linear combination of B-spline basis and time dependent parameters. With the help of the idea that approximate solution satisfy the PDE at collocation points, a new numerical scheme is constructed. The newly obtained numerical scheme tested on a model problem. Numerical results are compared with exact solution with the aid of the error norms L_2 and L_∞ presented via tables. Additionally, graphical simulations of numerical solutions are presented.

Keywords: Coupled Klein Gordon equation, Collocation, cubic B-spline basis, Finite element method

Öz

Bu çalışma, kübik B-spline baz fonksiyonları ve sonlu eleman yaklaşımına temellenen kollokasyon yöntemi kullanılarak ikili Klein-Gordon denkleminin nümerik çözümlerini elde etmek için yeni bir yaklaşım sunmaktadır. Kollokasyon yönteminin başlıca avantajı, yöntemin yapısının basit ve hesaplama maliyetinin düşük olmasıdır. Ayrıca, gerçek dünya olgularını modelleyen diferansiyel denklemleri içeren çeşitli problemlerin çözümünde kolay ve daha basit bir prosedür elde edilmesini sağlar. Mevcut çalışmada, zamansal ve konumsal kısmi türevler, B-spline bazların ve zamana bağlı parametrelerin doğrusal birleşiminden oluşan yaklaşık çözüm kullanılarak ayrıştırılır. Yaklaşık çözümün kısmi diferansiyel denklemi kollokasyon noktalarında sağlaması fikrinin yardımı ile yeni bir sayısal şema oluşturulur. Yeni elde edilen şema bir model problem üzerinde test edilir. Sayısal sonuçlar L_2 *ve* L_∞ *hata normları yardımı ile tam çözümlerle karşılaştırılır ve tablolar aracılığı ile sunulur. Ayrıca sayısal çözümlerin grafik benzetimleri sunulur.*

***Anahtar kelimeler**: İkili Klein Gordon denklemi, Kollokasyon, Kübik B-spline bazları, Sonlu eleman yöntemi*

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1. Introduction

It is necessary to utilise mathematical modelling to understand the nature, science and to improve engineering. The models allow us to predict future behaviours or unseen results of the natural phenomena. Most of process in the scenario of real-life are formulated by partial differential equations and with the aid of the solutions science are developed. Thus, for centuries many academics and practitioners have focused on solving PDEs which are important for theory or applications with numerous methods including numerical and exact solution techniques. However, encountering large-scale computational problems arising in science leads to the development of computer and efficient computational algorithms. Finite element method is one of the foremost techniques for computational solution of the PDEs. The basic principle lying under finite element approach is to divide problem domain into assembled several elements and uses an piecewise approximations with basis functions to the solution function.

Our main interest here is coupled Klein Gordon equation (Alagasen et al., 2004; Doha et al., 2014) which reads as:

$$\begin{aligned} &u_{tt} = u_{xx} - u + 2u^3 + 2uv \\ &v_t + 4u\,u_t = v_x \end{aligned} \tag{1}$$

and the boundary-initial conditions pertaining with the equation are ;

$$\begin{aligned} &u(a,t) = u_0, && v(a,t) = v_0, \\ &u(b,t) = u_1, && v(b,t) = v_1 \\ &u(x,0) = f(x), && v(x,0) = h(x) \\ &u_t(x,0) = g(x), && \end{aligned} \tag{2}$$

The coupled Klein Gordon equation is one of the vital important equations seen in theoretical physics, solid state physics, nonlinear optics and optical solitons (Malomed et al.,2005; Mihalache, 2012). In the literature, there are some study on the coupled Klein Gordon equation. One can find knowledge in (Porsezian et al., 1995; Khusnutdinova, 2003; Liu et al., 2004; Biswas et al., 2014).

The objective of the present paper is to present a new perspective to obtain numerical solutions for coupled Klein Gordon Equation using a combination of collocation method and finite element approximation. For this study, cubic B-spline basis functions are chosen as piecewise approximation. In order to demonstrate the handiness of collocation method numerical experiments are discussed and the the error norms L_2 and L_2 are calculated and presented by tables. Additionally simulations of numerical results are provided with graphically.

2. Collocation Finite Element Formulation for Coupled Klein Gordon Equation

To begin constructing collocation finite element scheme, let the interval $[a,b]$ be partitioned by

$$a = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = b$$

with a uniform mesh discretaization

$$h = x_{i+1} - x_i \quad (i = 0,1,2,\ldots,\ell)$$

where $\{x_i\}_{i=0}^{\ell}$ denotes nodal points of subintervals. Assume that we denote approximate solutions by $u_h(x,t), v_h(x,t)$ and the exact solutions by $u(x,t), v(x,t)$, respectively. The approximate solution can be defined using cubic B-spline basis (Kutluay et al., 2016) over the interval by

$$\begin{aligned} u_h(x,t) &= \sum_{j=-1}^{\ell+1} \phi_j(x)\delta_j(t) \\ v_h(x,t) &= \sum_{j=-1}^{\ell+1} \phi_j(x)\sigma_j(t) \end{aligned}$$

where j is the number of nodal points or we can say "edges of our elements", $\delta_j(t)$ and $\sigma_j(t)$ are shape parameter functions and $\phi_j(x)$ are cubic B-spline basis functions (Dag et al., 2005; Prenter, 2008; Esen et al., 2015). Constructing the FEM formulation which is easier and using a systematic procedure , we make moving to a local coordinate system by $\xi = x_m - x, (0 \le \xi \le h)$. Now, we can denote all elements by a emblematic element such as $[x_m, x_{m+1}]$. Transformed cubic B-spline basis rewritten follows as

$$\begin{aligned} \phi_{m-1} &= (h-\xi)^3 / h^3, \\ \phi_m &= \left(h^3 + 3h^2(h-\xi) + 3h(h-\xi)^2 - 3(h-\xi)^3\right) / h^3, \\ \phi_{m+1} &= \left(h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3\right) / h^3, \\ \phi_{m+2} &= \xi^3 / h^3. \end{aligned} \tag{3}$$

Thus, we get the approximate solution with local coordinate given as

$$u_h(\xi) = \sum_{j=m-1}^{m+2} \phi_j(\xi)\delta_j(t), \quad v_h(\xi) = \sum_{j=m-1}^{m+2} \phi_j(\xi)\sigma_j(t). \tag{4}$$

Evaluation of u_h and v_h ,its necessary derivatives seen in the Eq. (1) at nodal points x_i in terms of $\delta(t)$ and $\sigma(t)$ parameters as follows:

$$\begin{aligned}
(u_h)_m &= \delta_{m-1} + 4\delta_m + \delta_{m+1} \\
(u_h)'_m &= \frac{3}{h}(\delta_{m+1} - \delta_{m-1}) \\
(u_h)''_m &= \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1})
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
(v_h)_m &= \sigma_{m-1} + 4\sigma_m + \sigma_{m+1} \\
(v_h)'_m &= \frac{3}{h}(\sigma_{m+1} - \sigma_{m-1}).
\end{aligned} \tag{6}$$

On substituting the approximate solutions into their places u_h, v_h from Eqs.(5) -(6) and for linearization taking $u = z_m$ at the Eq. (1), yields

$$\begin{aligned}
&\ddot{\delta}_{m-1} + 4\ddot{\delta}_m + \ddot{\delta}_{m+1} = \frac{6}{h^2}\left(\delta_{m-1} - 2\delta_m + \delta_{m+1}\right) - \left(\delta_{m-1} + 4\delta_m + \delta_{m+1}\right) \\
&\qquad + 2z_m^2\left(\delta_{m-1} + 4\delta_m + \delta_{m+1}\right) + 2z_m\left(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}\right) \\
&\left(\dot{\sigma}_{m-1} + 4\dot{\sigma}_m + \dot{\sigma}_{m+1}\right) + 4z_m\left(\dot{\delta}_{m-1} + 4\dot{\delta}_m + \dot{\delta}_{m+1}\right) = \frac{3}{h}\left(\sigma_{m+1} - \sigma_{m-1}\right)
\end{aligned}$$

where $\dot{\sigma}$ and $\ddot{\delta}$ denotes first and second order derivative respect to time parameter, respectively. Then, if the central difference formula for $\ddot{\delta}$, Crank-Nicolson formula for $\dot{\sigma}$ and $\dot{\delta}$, at the forward finite difference approximation for δ and σ are used, respectively

$$\ddot{\delta} = \frac{\delta^{n+1} - 2\delta^n + \delta^{n-1}}{(\Delta t)^2} \qquad \dot{\sigma} = \frac{\sigma^{n+1} - \sigma^n}{\Delta t}$$

$$\dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t} \qquad \sigma = \frac{\sigma^{n+1} + \sigma^n}{2}$$

$$\delta = \frac{\delta^{n+1} + \delta^n}{2}$$

and the terms are collacted, we get the following differential equations given in terms of parameters δ and σ

$$\begin{cases}
\left(1 - (\Delta t)^2\left(\frac{3}{h^2} - \frac{1}{2} + z_m^2\right)\right)\delta_{m-1}^{n+1} + \left(4 + (\Delta t)^2\left(\frac{6}{h^2} + 2 - 4z_m^2\right)\right)\delta_m^{n+1} + \left(1 - (\Delta t)^2\left(\frac{3}{h^2} - \frac{1}{2} + z_m^2\right)\right)\delta_{m+1}^{n+1}\left(-z_m(\Delta t)^2\right)\sigma_{m-1}^{n+1} + \left(-4z_m(\Delta t)^2\right)\sigma_m^{n+1} \\
+\left(-z_m(\Delta t)^2\right)\sigma_{m+1}^{n+1} = \left(2 + (\Delta t)^2\left(\frac{3}{h^2} - \frac{1}{2} + z_m^2\right)\right)\delta_{m-1}^n + \left(8 - (\Delta t)^2\left(\frac{6}{h^2} + 2 - 4z_m^2\right)\right)\delta_m^n + \left(2 + (\Delta t)^2\left(\frac{3}{h^2} - \frac{1}{2} + z_m^2\right)\right)\delta_{m+1}^n \\
+z_m(\Delta t)^2\left(\sigma_{m-1}^n + 4\sigma_m^n + \sigma_{m+1}^n\right) - \left(\delta_{m-1}^{n-1} - 2\delta_m^{n-1} + \delta_{m+1}^{n-1}\right) \\
4z_m\delta_{m-1}^{n+1} + 16z_m\delta_m^{n+1} + 4z_m\delta_{m+1}^{n+1} + \left(1 + \frac{3\Delta t}{2h}\right)\sigma_{m-1}^{n+1} + 4\sigma_m^{n+1} + \left(1 - \frac{3\Delta t}{2h}\right)\sigma_{m+1}^{n+1} = 4z_m\left(\delta_{m-1}^n + 4\delta_m^n + \delta_{m+1}^n\right) + \left(1 + \frac{3\Delta t}{2h}\right)\sigma_{m-1}^n 4\sigma_m^n + \left(1 - \frac{3\Delta t}{2h}\right)\sigma_{m+1}^n
\end{cases} \tag{7}$$

where $\Delta t = \dfrac{T}{n}$ is time step and $h = \dfrac{b-a}{N}$ is space step. Also, values of z_m at nodal points are $z_m = \delta_{m-1}^n + 4\delta_m^n + \delta_{m+1}^n$.

Therefore; a linear differential equation system including $(2N+2)$ equations and $(2N+6)$ unknown parameters $(\delta_j \text{ and } \sigma_j)$ is obtained for $m = 0,1,2,\ldots,N$. For a solvable system, we need to eliminate four unknown parameters from Eq. (7) using boundary conditions as

$$\delta_{-1} = u(a = x_0, t) - 4\delta_0 - \delta_1$$
$$\delta_{N+1} = u(b = x_N, t) - \delta_{N-1} - 4\delta_N$$
$$\sigma_{-1} = v(a = x_0, t) - 4\sigma_0 - \sigma_1$$
$$\sigma_{N+1} = v(b = x_N, t) - \sigma_{N-1} - 4\sigma_N$$

with the help of the elimination of the parameters, the above system reduces to $(2N+2)$ equations and $(2N+2)$ unknown parameters. Before starting to solve the reduced system, an initial vector should be get. For this situation, initial conditions of the problem and approximate solution will help us. Using values of initial conditions $u(x,0)$ and $v(x,0)$ together with approximate solution u_h and v_h at the initial time

$t = 0,$ we get

$$u(x,0) = \delta_{m-1}^0 + 4\delta_m^0 + \delta_{m+1}^0$$
$$v(x,0) = \sigma_{m-1}^0 + 4\sigma_m^0 + \sigma_{m+1}^0 \tag{8}$$

after elinating $\delta_{-1}, \sigma_{-1}, \delta_{N+1}$ and σ_{N+1} from the (8).

Now we have a solvable system again. Then, firstly we can obtain initial vector δ_j^0 and σ_j^0 parameters solving (8), then δ_j^{n+1} and σ_j^{n+1} unknown parameters can be obtain with the help of δ_j^n and σ_j^n known parameters solving (7) via any algorithm, iteratively.

However, when we check the system of differential equations given in Eq. (7), we encounter an imaginary time given as δ_j^{-1}. In order to deal with this imaginary time, we are going to use the initial condition given with the fist derivative respect to time together with centrel difference formula for first derivative such as

$$u_t(x,0) = u_h(x,0) = \frac{\delta^{n+1} - \delta^{n-1}}{2\Delta t}.$$

Thus, using the above calculations yields to express time step δ_j^{-1} in terms of δ_j^1 at initial time $t = 0.$

3. Numerical Results for Coupled Klein Gordon Equation

In the previous section, we have obtained a new numerical scheme for the coupled Klein Gordon equation with the aid of collocation finite element method. Now, we are going to obtain numerical results for the problem. As the exact solution is known, it leads to calculate absolute and maximum errors i. e. L_2 and L_∞ with given formula

$$L_2 = \|u - U_N\|_2 = \sqrt{h\sum_{j=0}^{N}\left|u_j - (U_N)_j\right|^2},$$
$$L_\infty = \|u - U_N\|_\infty = \max_{0\le j\le N}\left|u_j - (U_N)_j\right|. \tag{9}$$

Consider the coupled Klein Gordon equation given in (1) with initial-boundary conditions (1). In equation (2),

$$u_0 = \sqrt{\frac{1+c}{1-c}}\,\mathrm{sech}\left(\frac{a-ct}{\sqrt{1-c^2}}\right) \quad v_0 = \frac{-2c}{1-c}\mathrm{sech}^2\left(\frac{a-ct}{\sqrt{1-c^2}}\right),$$
$$u_1 = \sqrt{\frac{1+c}{1-c}}\,\mathrm{sech}\left(\frac{b-ct}{\sqrt{1-c^2}}\right) \quad v_1 = \frac{-2c}{1-c}\mathrm{sech}^2\left(\frac{b-ct}{\sqrt{1-c^2}}\right)$$
$$f(x) = \sqrt{\frac{1+c}{1-c}}\,\mathrm{sech}\left(\frac{x}{\sqrt{1-c^2}}\right) \quad h(x) = \frac{-2c}{1-c}\mathrm{sech}^2\left(\frac{y}{\sqrt{1-c^2}}\right)$$
$$g(x) = \frac{c\sqrt{\frac{1+c}{1-c}}}{\sqrt{1-c^2}}\mathrm{sech}\left(\frac{x}{\sqrt{1-c^2}}\right)\tanh\left(\frac{x}{\sqrt{1-c^2}}\right) \tag{10}$$

and exact solutions of the such equation are

$$u(x,t) = \sqrt{\frac{1+c}{1-c}}\,\mathrm{sec}h\left(\frac{x-ct}{\sqrt{1-c^2}}\right)$$
$$v(x,t) = \frac{-2c}{1-c}\mathrm{sec}h^2\left(\frac{x-ct}{\sqrt{1-c^2}}\right).$$

The domain of the problem is given as $x \in [0,1]$ and final time is $T = 1.$ The error norms L_2 and L_∞ are presented for various values of space step and time step and $c = 0.5$ in Tables 1-4 respectively. The first and the second tables are prepared for $u(x,t),$ and the others are for $v(x,t)$. In tables from 1 to 4, the comparisons of numerical values with exact ones are shown for different partition numbers varying from $N = 2$ to 100 and time step size for from $\Delta t = 0.05$ to 0.001. It is observed from the error norms presented in tables that numerical results are matching with exact results. As a result of using collocation method, increasing of collocation points leads to improved convergence. Together with decreasing of time step the best solutions can be obtained when $N = 100$ and $\Delta t = 0.001$.

Table 1: The error norms L_2 of $u(x,t)$ for various values of Δt and h

Δt	0.05	0.025	0.01	0.001
2	2.1117483×10^{-2}	2.1426286×10^{-2}	2.1650394×10^{-2}	2.180175×10^{-2}
4	5.549822×10^{-3}	5.631996×10^{-3}	5.753750×10^{-3}	5.855224×10^{-3}
8	2.504682×10^{-3}	1.827506×10^{-3}	1.577408×10^{-3}	1.534394×10^{-3}
10	2.325349×10^{-3}	1.472896×10^{-3}	1.086567×10^{-3}	9.89403×10^{-4}
20	2.205067×10^{-3}	1.166034×10^{-3}	5.40442×10^{-4}	2.57481×10^{-4}
40	2.193843×10^{-3}	1.131230×10^{-3}	4.70152×10^{-4}	8.2750×10^{-5}
80	2.191563×10^{-3}	1.124399×10^{-3}	4.60123×10^{-4}	5.1228×10^{-5}
100	2.191318×10^{-3}	1.123533×10^{-3}	4.59071×10^{-4}	4.9008×10^{-5}

Table 2: The error norms L_∞ of $u(x,t)$ for various values of Δt and h

Δt	0.05	0.025	0.01	0.001
2	2.9864630×10^{-2}	3.0301350×10^{-2}	3.0618280×10^{-2}	3.083234×10^{-2}
4	7.867893×10^{-3}	8.272647×10^{-3}	8.588344×10^{-3}	8.803457×10^{-3}
8	4.352221×10^{-3}	2.930330×10^{-3}	2.288971×10^{-3}	2.175901×10^{-3}
10	4.141703×10^{-3}	2.580601×10^{-3}	1.647240×10^{-3}	1.385919×10^{-3}
20	3.869133×10^{-3}	2.054752×10^{-3}	9.58485×10^{-4}	3.66860×10^{-4}
40	3.804672×10^{-3}	1.938773×10^{-3}	8.11771×10^{-4}	1.36876×10^{-4}
80	3.800903×10^{-3}	1.915686×10^{-3}	7.77534×10^{-4}	9.0673×10^{-5}
100	3.802503×10^{-3}	1.913958×10^{-3}	7.73942×10^{-4}	8.5674×10^{-5}

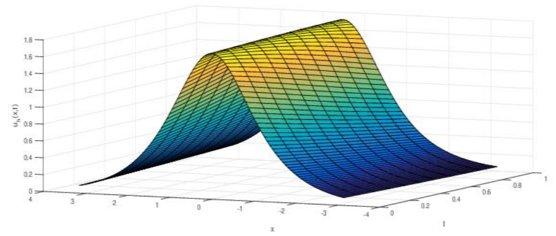
Table 3: The error norms L_2 of $v(x,t)$ for various values of Δt and h

Δt	0.05	0.025	0.01	0.001
2	10.3601227×10^{-2}	10.5591671×10^{-2}	10.6966514×10^{-2}	10.7875384×10^{-2}
4	29.279325×10^{-2}	7.241041×10^{-3}	3.2369624×10^{-3}	3.3519732×10^{-2}
8	1.2139186×10^{-2}	8.449216×10^{-3}	8.137684×10^{-3}	8.985218×10^{-3}
10	1.1747715×10^{-2}	6.680046×10^{-3}	5.184316×10^{-3}	5.759141×10^{-3}
20	1.2597028×10^{-2}	6.411526×10^{-3}	2.545129×10^{-3}	1.377738×10^{-3}
40	1.3153466×10^{-2}	6.863859×10^{-3}	2.749699×10^{-3}	3.52275×10^{-4}
80	1.3370683×10^{-2}	1.0113778×10^{-2}	2.882678×10^{-3}	2.68306×10^{-4}
100	1.3407588×10^{-2}	7.062596×10^{-3}	2.902427×10^{-3}	2.76644×10^{-4}

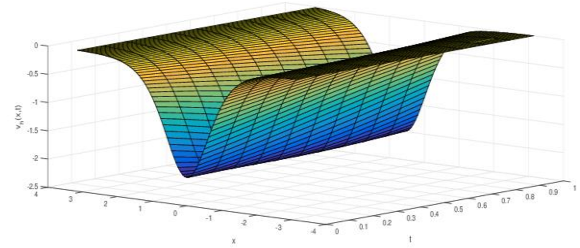
Table 4: The error norms L_∞ of $v(x,t)$ for various values of Δt and h

Δt	0.05	0.025	0.01	0.001
2	14.65143×10^{-2}	14.93292×10^{-2}	15.12735×10^{-2}	15.25588×10^{-2}
4	4.0917020×10^{-2}	1.2582690×10^{-2}	4.7002710×10^{-2}	5.0253780×10^{-3}
8	2.3677950×10^{-2}	1.5536220×10^{-2}	1.2350990×10^{-2}	1.4263810×10^{-3}
10	2.0550620×10^{-2}	1.2550750×10^{-2}	7.819947×10^{-3}	9.439125×10^{-3}
20	2.3965290×10^{-2}	1.2097710×10^{-2}	4.611566×10^{-3}	2.274541×10^{-3}
40	2.5724430×10^{-2}	1.3491740×10^{-2}	5.399549×10^{-3}	6.62613×10^{-4}
80	2.6256080×10^{-2}	1.7316930×10^{-2}	5.701815×10^{-3}	5.12884×10^{-4}
100	2.6373460×10^{-2}	1.3948110×10^{-2}	5.741665×10^{-3}	5.42177×10^{-4}

Additionally, 3D graphics of u and v are exhibited in figure 1 for parameters $\Delta t = 0.1, N = 100$ and $c = 0.5$. The plotting domain of the problem is chosen as $[-\pi, \pi]$. Initially, the peak of the waves related with $u(x,t)$, $v(x,t)$ is 1.732051 and -2.0, respectively. Over time, each wave moves toward to right side of the x axis with a negligible reduction for u and a negligible raise for v . At time $T = 1$, the waves are located at nearly $x = 0.502656$ and $x = 0.565488$ with their peak points are 1.715227 and -1.984599, respectively.



a) $u_h(x,t)$



b) $v_h(x,t)$

Figure 1: Numerical solutions of u and v values

4. Conclusion

As a conclusion, in this study a numerical technique is outlined for obtaining the numerical solutions of coupled Klein Gordon equation. The approximate solutions are produced using cubic B-spline piecewise basis functions and collocation finite element method. The newly calculated error norms and figures of numerical resul show that collocation FEM is quite suitable, admissible and efficient tool for solving such problems. It also has wide applicability to different partial differential equations arising in various fields of science and engineering.

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