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Research Article

Nonlocal boundary value problems for nonlinear toppled system of fractional differential equations

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Abstract

The aim of this paper is to study multiplicity results for the solutions of a coupled system of fractional differential equations. The problem under consideration is subjected to nonlocal boundary conditions involving Riemann-Liouville integrals and derivatives of fractional order. Necessary and sufficient conditions are established for the existence of at least one and more solutions by using various fixed point theorems of cone type. Moreover sufficient conditions for uniqueness is also discussed by using a concave type operator for the considered problem. Further, the conditions are also provided under which the considered system has no positive solution. The results are demonstrated by providing several examples.

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1. Introduction

Differential equations of arbitrary order are the best tools in the mathematical modeling of several phenomena in various disciplines of science and engineering; for some of these application we refer to [3, 9, 14, 15, 18]. Therefore much attention has been given to the subject of differential equations of arbitrary order, (see [13, 14, 26, 27] and the references therein). The area devoted to the existence of positive solutions to fractional differential equations and their systems, especially coupled systems have been studied by many authors, for details see [5, 6, 19, 20]. In recent years, much attention has been given to the study of multiplicity results of boundary value problems for nonlinear differential equations of fractional order; for details we refer to [4, 7, 8, 10, 11] and the references therein. Similarly the iterative schemes for approximating the solutions of nonlinear fractional differential equations have been studied in many articles, (see [1, 21, 24] and the references therein). In most of these articles the results were obtained by using classical fixed point theorems such as the Banach contraction theorem, the Leray- Schauder fixed point theorem and fixed point theorems of cone type. The area devoted to nonlocal boundary value problems of differential equations of classical as well as of fractional order can be more interesting and applicable in the modeling of various phenomenons of applied sciences as studied in [12, 22, 23]. Nonlocal boundary value problems are used

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to study mathematical epidemiology in terms of fractional order models. This is due to the fact that fractional order models of epidemic disease are more realistic and provide greater information as compared to integer order models. In last few decades, fractals and chaos are also studied by using the tools of nonlocal boundary value problems for fractional differential equations. Besides from the above discussion differential equations of arbitrary order with nonlocal boundary conditions have also many applications in the fields of aerodynamics, fluid dynamics, physical chemistry, economics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, etc. According to [16, 17], nonlocal multi-point boundary value problems are helpful to establish some spectral properties which are used to prove a Rabinowitz-type global bifurcation theorem to address bifurcation problems related to physical and biological models.

In this article, we study the following nonlinear toppled system of fractional ordered differential equations with nonlocal boundary conditions provided by

$$\begin{cases} \mathcal{D}^p w(t) + \theta(t, w(t), z(t)) = 0; \quad t \in J; \ 2 < p \leq 3, \\ \mathcal{D}^q z(t) + \phi(t, w(t), z(t)) = 0; \quad t \in J; \ 2 < q \leq 3, \\ \mathcal{J}^{3-p} w(t)|_{t=0} = \mathcal{D}^{p-2} w(t)|_{t=0} = 0, \ w(1) = w(\eta), \\ \mathcal{J}^{3-q} z(t)|_{t=0} = \mathcal{D}^{q-2} z(t)|_{t=0} = 0, \ z(1) = z(\xi), \end{cases} \tag{1.1}$$

where $\eta, \xi \in (0,1), J = [0,1], \ \theta, \ \phi : J \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are nonlinear continuous functions and $\mathcal{D}, \mathcal{J}$ are used as differential and integral operator in Riemann-Liouville sense. The considered toppled system involved Riemann-Liouville fractional integral and derivative at its boundary conditions. We obtain necessary and sufficient conditions for the existence of at least one solution to system (1.1) by using the nonlinear Leray-Schauder alternative theorem. Also by using a concave type operator, with an increasing or decreasing property, to prove uniqueness of solution to the considered problem. With the help of Krasnosel'skii's fixed point theorem, we establish required conditions for multiplicity results of positive solutions to the problem under consideration. Moreover some conditions are also provided under which the considered boundary value problem (1.1) has no solution. We also provide various examples to illustrate our main results.

2. Preliminaries

In the current section, we recall some basic notions and results as given in [2,5,9,15,25].

Definition 2.1. The Riemann-Liouville integral of non-integer order $p > 0$ of a function $w : (0,\infty) \to \mathbb{R}$ is defined by

$$\mathcal{J}^p w(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} w(s) ds,$$

such that the integral on the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $p > 0$ of a function $w \in C[0,1]$ is defined by

$$\mathcal{D}^p w(t) = \frac{1}{\Gamma(m-p)} \left(\frac{d}{dt}\right)^m \int_0^t (t-s)^{m-p-1} w(s) ds, \ \text{where } m = [p] + 1.$$

Here $[p]$ represents integer part of p.

The following results are needed in this paper:

Theorem 2.3 ([19])**.** *Let $p > 0$, $w(t) \in C[0,1] \cap L^1[0,1]$, then for arbitrary $C_i \in$ $mathbbR$, $i = 0,1,2,\dots,m$, $m = [p]+1$, the homogenous differential equation*

$$\mathcal{D}^\alpha w(t) = 0$$

has a solution given by

$$w(t) = C_1 t^{p-1} + C_2 t^{p-2} + \ldots + C_m t^{p-m}.$$

Definition 2.4 ([2,21])**.** For the ordered Banach space $\mathcal{B}$ a cone $\mathbf{C} \subset \mathcal{B}$ induces a partial order such that $w \preceq z$ if and only if $z - w \in \mathbf{C}$. Similarly if $\mathbf{C} \neq \emptyset$ is a closed and convex subset of $\mathcal{B}$, it will be a cone if it satisfies:

(1) $w \in \mathbf{C}, \mu \geq 0 \Rightarrow \mu w \in \mathbf{C}$;
(2) $w, -w \in \mathbf{C} \Rightarrow w = 0_{\mathcal{B}} \in \mathbf{C}$, where $0_{\mathcal{B}}$ is the zero element of $\mathcal{B}$.

A cone $\mathbf{C}$ is normal if and only if for all

$$w, z \in \mathcal{B},\ 0 \preceq w \preceq z \Rightarrow \|w\| \leq \mu \|z\|, \mu > 0.$$

For every $w, z \in \mathcal{B}$, $w \sim z$ means that there exist $\alpha_1, \beta_1 > 0$ such that $\alpha_1 w \preceq z \preceq \beta_1 z$. It is obvious that the relation $\sim$ is an equivalence relation, for which we define a set $\mathcal{C}_h = \{w \in \mathcal{B} : w \sim h\}$. Also, we can prove that $\mathcal{C}_h \subset \mathbf{C}$ for $h \succ 0$.

Definition 2.5. For all $\alpha \in (0,1)$ and $w \in \mathbf{C}$, with $0 < \lambda < 1$, the operator $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ is said to be λ-concave if and only if $\mathcal{T}(\alpha w) \succeq \alpha^{\lambda} \mathcal{T} w$.

Definition 2.6. If $w, z \in \mathbf{C}$ and $w \preceq z$ implies that $\mathcal{T} w \preceq \mathcal{T} z$, then $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ is said to be an increasing operator.

Theorem 2.7 ([2,21])**.** *Let $\mathbf{C}$ be a normal cone in a real Banach space $\mathcal{B}$ and $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ be an increasing α-concave operator with $w \succ 0$ such that $\mathcal{T} w \in \mathcal{C}_h$. Then, $\mathcal{T}$ has a unique fixed point $w \in \mathcal{C}_h$.*

Theorem 2.8 ([19,21])**.** *Let $\mathcal{B}$ be a Banach space with $\mathbf{C} \subseteq \mathcal{B}$ closed and convex. Let $\mathcal{E}$ be a relatively open subset of $\mathbf{C}$ with $0 \in \mathcal{E}$ and $\mathcal{T} : \overline{\mathcal{E}} \to \mathcal{E}$ be a continuous and compact operator. Then the operator $\mathcal{T}$ has a fixed point in $\overline{\mathcal{E}}$, or there exist $w \in \partial \mathcal{E}$ and $\lambda \in (0,1)$ with $w = \lambda \mathcal{T} w$.*

Theorem 2.9 ([19,21])**.** *For a real Banach space $\mathcal{B}$, the set $\mathbf{C}$ be a cone with two bounded subset Ω_1 and Ω_2 in $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Moreover the operator $\mathcal{T} : \mathbf{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathbf{C}$ is completely continuous. If one of the two conditions given by*

(1) $\|\mathcal{T} w\| \leq \|w\|$ *for all* $w \in \mathbf{C} \cap \partial \Omega_1$; $\|\mathcal{T} w\| \geq \|w\|$, *for all* $w \in \mathbf{C} \cap \partial \Omega_2$;
(2) $\|\mathcal{T} w\| \geq \|w\|$ *for all* $w \in \mathbf{C} \cap \partial \Omega_1$; $\|\mathcal{T} w\| \leq \|w\|$, *for all* $w \in \mathbf{C} \cap \partial \Omega_2$,

holds, then $\mathcal{T}$ has a fixed point in $\mathbf{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

Theorem 3.1. *Let $x \in C(J)$ and $\delta_1 = 1 - \eta^{p-1} < 1, 0 < q_1 < 1, 1 < p_1 < 2$. Then the unique solution of the nonlocal boundary value problem*

$$\begin{aligned} &\mathcal{D}^p w(t) + x(t) = 0,\ t \in J,\ 2 < p \leq 3, \\ &\mathcal{I}^{3-p} w(t)|_{t=0} = \mathcal{D}^{p-2} w(t)|_{t=0} = 0, w(1) = w(\eta), \end{aligned} \tag{3.1}$$

is given by

$$w(t) = \int_0^1 \mathcal{H}_1(t,s) x(s) ds, \tag{3.2}$$

where $\mathcal{H}_1(t,s)$ is the Green's function defined by

$$\mathcal{H}_1(t,s)=\frac{1}{\Gamma(p)}\begin{cases}\frac{t^{p-1}}{\delta_1}\left[(1-s)^{p-1}-(\eta-s)^{p-1}\right]-(t-s)^{p-1}, \\ \qquad\qquad 0\le s\le t\le\eta\le 1,\\ \frac{t^{p-1}}{\delta_1}\left[(1-s)^{p-1}-(\eta-s)^{p-1}\right],\ 0\le t\le s\le\eta\le 1,\\ \frac{t^{p-1}}{\delta_1}(1-s)^{p-1}-(t-s)^{p-1},\ 0\le\eta\le s\le t\le 1,\\ \frac{t^{p-1}}{\delta_1}(1-s)^{p-1},\ 0\le\eta\le t\le s\le 1.\end{cases} \tag{3.3}$$

Proof. Thanks to Theorem 2.3, (3.1) yields

$$w(t)=-\mathcal{I}^p x(t)+C_1t^{p-1}+C_2t^{p-2}+C_3t^{p-3}, \tag{3.4}$$

which upon application of fractional integration of order $3-p$ yields

$$\mathcal{I}^{3-p}w(t)=-\mathcal{I}x(t)+C_1\frac{\Gamma(p)t^2}{\Gamma(3)}+C_2\frac{\Gamma(p-1)t}{\Gamma(2)}+C_3.$$

Then using $\mathcal{I}^{3-p}w(t)|_{t=0}=0$ implies that $C_3=0$, because as $t\to 0$, we deal with a singularity. Thus (3.4) becomes

$$w(t)=-\mathcal{I}^p x(t)+C_1t^{p-1}+C_2t^{p-2}. \tag{3.5}$$

Now taking the $p-2$ order fractional order derivative of (3.5) and then using $\mathcal{D}^{p-2}w(t)|_{t=0}=0$ gives $C_2=0$ due to the fact that as $t\to 0$, we again deal with a singularity. Hence (3.4) becomes

$$w(t)=-\mathcal{I}^p x(t)+C_1t^{p-1}. \tag{3.6}$$

Now using the boundary condition $w(1)=w(\eta)$ in (3.6), we get $C_1=\frac{1}{\delta_1}\left[\mathcal{I}^p x(1)-\mathcal{I}^p x(\eta)\right]$. Hence, we get the solution as

$$\begin{aligned} w(t) &= \frac{t^{p-1}}{\delta_1}\left[\mathcal{I}^p x(1)-\mathcal{I}^p x(\eta)\right]-\mathcal{I}^p x(t)\\ &= \frac{t^{p-1}}{\delta_1\Gamma(p)}\left[\int_0^1(1-s)^{p-1}x(s)ds-\int_0^\eta(\eta-s)^{p-1}x(s)ds\right]\\ &\quad -\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}x(s)ds\\ &= \int_0^1\mathcal{H}_1(t,s)x(s)ds, \end{aligned} \tag{3.7}$$

where $\mathcal{H}_1(t,s)$ is the Green's function given in (3.3), which can easily be obtained. □

In view of Theorem 3.1, considering the Toppled system (1.1) of differential equations of fractional order is equivalent to the Toppled system of Fredholm integral equations given by

$$\begin{cases} w(t)=\int_0^1\mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds,\\ z(t)=\int_0^1\mathcal{H}_2(t,s)\phi(s,w(s),z(s))ds, \end{cases} \tag{3.8}$$

where $\mathcal{H}_2(t,s)$ is the Green's function for the second equation in system (1.1), which is obtained like $\mathcal{H}_1(t,s)$, and both are continuous on $J\times J$ and satisfy the following properties:

(*i*) $\mathcal{H}_j(t,s)\ge 0$ for every $t,s\in J$, where $j=1,2$;
(*ii*) $\max_{t\in J}\mathcal{H}_j(t,s)=\mathcal{H}_j(1,s)$, for all $s\in J$, where $j=1,2$;

(*iii*) $\min_{t\in[\vartheta,1-\vartheta]} \mathcal{H}_1(t,s) \geq \gamma_1 \mathcal{H}_1(1,s)$ and $\min_{t\in[\vartheta,1-\vartheta]} \mathcal{H}_2(t,s) \geq \gamma_2 \mathcal{H}_2(1,s)$ for each $\vartheta \in (0,1), s \in (0,1)$,
where $\gamma_1 = \vartheta^{p-1},\ \gamma_2 = \vartheta^{q-1}$.

We will use

$$\gamma = \min\{\gamma_1, \gamma_2\}$$

thoughtout the rest of this paper.

Let us define $\mathcal{B} = \{w(t)|w \in C(J)\}$ endowed with the norm $\|w\| = \max_{t\in J} |w(t)|$. Further the norm for the product space, we define as $\|(w,z)\| = \|w\| + \|z\|$. Obviously $(\mathcal{B}\times\mathcal{B}, ||\cdot||)$ is a Banach space. We define the cone $\mathbf{C} \subset \mathcal{B}\times\mathcal{B}$ by

$$\mathbf{C} = \{(w,z) \in C[0,1]\times C[0,1] : \min_{t\in[\vartheta,1-\vartheta]} [w(t)+z(t)] \geq \gamma||(w,z)||\}.$$

Further for existence of fixed points, we define an operator $\mathcal{T} : \mathcal{B}\times\mathcal{B} \to \mathcal{B}\times\mathcal{B}$ as

$$\begin{aligned} \mathcal{T}(w,z)(t) &= \Big(\int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds,\ \int_0^1 \mathcal{H}_2(t,s)\phi(s,w(s),z(s))ds\Big) \\ &= \Big(\mathcal{T}_1(w,z), \mathcal{T}_2(w,z)\Big)(t). \end{aligned} \tag{3.9}$$

The solutions of the Toppled system (1.1) and the fixed points of the operator $\mathcal{T}$ in (3.9) coincide with each other.

Theorem 3.2. *If $\theta,\phi : J\times[0,\infty)\times[0,\infty) \to [0,\infty)$ are continuous functions, then $\mathcal{T}(\mathbf{C}) \subset \mathbf{C}$ and $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ is a completely continuous operator.*

Proof. To show $\mathcal{T}(\mathbf{C}) \subset \mathbf{C}$, let $(w,z) \in \mathbf{C}$. Then, we have

$$\begin{aligned} \mathcal{T}_1(w,z)(t) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds \\ &\geq \gamma_1 \int_0^1 \mathcal{H}_1(1,s)\theta(s,w(s),z(s))ds. \end{aligned} \tag{3.10}$$

Also, we obtain

$$\begin{aligned} \mathcal{T}_1(w,z)(t) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds \\ &\leq \int_0^1 \mathcal{H}_1(1,s)\theta(s,w(s),z(s))ds. \end{aligned} \tag{3.11}$$

Thus from (3.10) and (3.11), we have

$$\mathcal{T}_1(w,z)(t) \geq \gamma\|\mathcal{T}_1(w,z)\|, \text{ for all } t \in J.$$

Similarly, one can write that

$$\mathcal{T}_2(w,z)(t) \geq \gamma\|\mathcal{T}_2(w,z)\|, \text{ for all } t \in J.$$

Thus,

$$\begin{gathered} \mathcal{T}_1(w,z)(t) + \mathcal{T}_2(w,z)(t) \geq \gamma\|(w,z)\|, \text{ for all } t \in J, \\ \min_{t\in J}[\mathcal{T}_1(w,z)(t) + \mathcal{T}_2(w,z)(t)] \geq \gamma\|(w,z)\|. \end{gathered}$$

Hence, we have $\mathcal{T}(w,z) \in \mathbf{C} \Rightarrow \mathcal{T}(\mathbf{C}) \subset \mathbf{C}$.

Let $(w,z) \in \mathbf{C}$ and as $\mathcal{H}_j(t,s) \geq 0$ for $j = 1,2$ and θ, ϕ are also non-negative and continuous functions, and so $\mathcal{T}$ is also a continuous operator.

Assume that $\mathcal{A} \subseteq \mathbf{C}$ is bounded set. So there exists a constant $\mathcal{R} > 0$, such that $\|(w,z)\| \leq \mathcal{R}$, for all $(w,z) \in \mathcal{A}$.
Let

$$\mathbb{K} = \max_{t\in J}\{|\theta(t,w(t),z(t))| + 1 : \quad t \in J, \quad 0 \leq \|(w,z)\| \leq \mathcal{R}\},$$
$$\mathbb{L} = \max_{t\in J}\{|g(t,w(t),z(t))| + 1 : \quad t \in J, \quad 0 \leq \|(w,z)\| \leq \mathcal{R}\}.$$

Then we have

$$\begin{aligned}
|\mathcal{T}_1(w(t),z(t))| &= \left|\int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds\right| \\
&\leq \int_0^1 \mathcal{H}_1(t,s)|\theta(s,w(s),z(s))|ds \\
&\leq \mathbb{K}\int_0^1 \mathcal{H}_1(1,s)ds \\
&\leq \frac{\mathbb{K}}{\delta_1\Gamma(p+1)},
\end{aligned}$$

also

$$\begin{aligned}
|\mathcal{T}_2(w(t),z(t))| &\leq \mathbb{L}\int_0^1 \mathcal{H}_2(1,s)ds \\
&= \frac{\mathbb{L}}{\delta_2\Gamma(q+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
|\|\mathcal{T}(w,z)\| &= \|(\mathcal{T}_1(w,z)\| + \|\mathcal{T}_2(w,z)\| \\
&\leq \frac{\mathbb{K}}{\delta_1\Gamma(p+1)} + \frac{\mathbb{L}}{\delta_2\Gamma(q+1)} \\
&=: \mathbf{d}.
\end{aligned}$$

Thus $\|\mathcal{T}(w,z)\| \leq \mathbf{d}$, which implies that $\mathcal{T}(\mathcal{A})$ is uniformly bounded. Now $\mathcal{H}_j(t,s)$, for $j = 1,2$, are uniformly continuous on $J \times J$. Thus for a fixed $s \in J$ and for $\varepsilon > 0$, there exists a real number $\delta > 0$ with $t_1, t_2 \in J$ such that $t_1 \leq t_2$ and $|t_2 - t_1| < \delta$, then

$$|\mathcal{H}_1(t_2,s) - \mathcal{H}_1(t_1,s)| < \frac{\varepsilon}{2\mathbb{K}}$$

and

$$|\mathcal{H}_2(t_2,s) - \mathcal{H}_2(t_1,s)| < \frac{\varepsilon}{2\mathbb{L}}.$$

Then

$$\begin{aligned}
|\mathcal{T}_1(w,z)(t_2) - \mathcal{T}_1(w,z)(t_1)| &\leq \mathbb{K}\int_0^1 |\mathcal{H}_1(t_2,s) - \mathcal{H}_1(t_1,s)|ds \\
&< \mathbb{K}\frac{\varepsilon}{2\mathbb{K}}
\end{aligned}$$

$$\text{implies that } |\mathcal{T}_1(w,z)(t_2) - \mathcal{T}_1(w,z)(t_1)| < \frac{\varepsilon}{2}$$

and similarly we can obtain

$$|\mathcal{T}_2(w,z)(t_2) - \mathcal{T}_2(w,z)(t_1)| < \frac{\varepsilon}{2}.$$

Now, for Euclidean space metric d on $\mathbb{R}^2$, if $t_1, t_2 \in J$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned} & d\Big(\mathcal{T}(w,z)(t_2), \mathcal{T}(u,z)(t_1)\Big) \\ & = \sqrt{|\mathcal{T}_1(w,z)(t_2) - \mathcal{T}_1(w,z)(t_1)|^2 + |\mathcal{T}_2(w,z)(t_2) - \mathcal{T}_2(w,z)(t_1)|^2} \\ & < \sqrt{\frac{\epsilon^2}{2}} \end{aligned}$$

implies that $d(\mathcal{T}(w,z)t_2, \mathcal{T}(w,z)t_1) < \frac{\varepsilon}{\sqrt{2}}$.

Using Arzela-Ascoli's theorem, $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ is completely continuous. □

Theorem 3.3. *Under the continuity of* $\theta, \phi : J \times [0,\infty) \times (0,\infty) \to [0,\infty)$, *suppose there exist* $\chi_j, \psi_j, \sigma_j : (0,1) \to [0,\infty), j = 1,2$, *satisfying:*

(H_1) $|\theta(t, w(t),\ z(t))| \le \chi_1(t) + \psi_1(t)|w(t)| + \sigma_1(t)|z(t)|,\ t \in (0,1), w, z \ge 0;$

(H_2) $|\phi(t, w(t),\ z(t))| \le \chi_2(t) + \psi_2(t)|w(t)| + \sigma_2|z(t)|,\ t \in (0,1), w, z \ge 0;$

(H_3) $\mathbb{A}_1 = \int_0^1 \mathcal{H}_1(1,s)\chi_1(s)ds < \infty,\ \mathbb{B}_1 = 2\int_0^1 \mathcal{H}_1(1,s)[\psi_1(s) + \sigma_1(s)]ds < 1;$

(H_4) $\mathbb{A}_2 = \int_0^1 \mathcal{H}_2(1,s)\chi_2(s)ds < \infty,\ \mathbb{B}_2 = 2\int_0^1 \mathcal{H}_2(1,s)[\psi_2(s) + \sigma_2(s)]ds < 1.$

Then the system (1.1) *has at least one solution* (w,z) *in*

$$\mathcal{E} = \left\{(w,z) \in \mathbf{C} : \|(w,z)\| < \max\left\{\frac{2\mathbb{A}_1}{1-2\mathbb{B}_1}, \frac{2\mathbb{A}_2}{1-2\mathbb{B}_2}\right\}\right\}.$$

Proof. Let $\mathcal{E} = \{(w,z) \in \mathbf{C} : \|(w,z)\| < r\}$ with $\max\left\{\frac{2\mathbb{A}_1}{1-2\mathbb{B}_1}, \frac{2\mathbb{A}_2}{1-2\mathbb{B}_2}\right\} < r$. The operator $\mathcal{T} : \overline{\mathcal{E}} \to \mathbf{C}$ is completely continuous operator by Theorem 3.2. Let $(w,z) \in \mathcal{E}$ such that $\|(w,z)\| < r$. Then, we have

$$\begin{aligned} |\mathcal{T}_1(w,z)(t)| &= \max_{t\in J}\left|\int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds\right| \\ &\le \int_0^1 \mathcal{H}_1(1,s)\chi_1(s)ds + \int_0^1 \mathcal{H}_1(1,s)\psi_1(s)|w(s)|ds \\ &+ \int_0^1 \mathcal{H}_p(1,s)\sigma_1(s)|z(s)|ds \\ &\le \int_0^1 \mathcal{H}_1(1,s)\chi_1(s)ds + r\left[\int_0^1 \mathcal{H}_1(1,s)[\psi_1(s) + \sigma_1(s)]ds\right] \end{aligned}$$

which implies that $\|\mathcal{T}_1(w,z)\| \le \mathbb{A}_1 + r\mathbb{B}_1 < \frac{r}{2}$.

Similarly, $\|\mathcal{T}_2(w,z)\| < \frac{r}{2}$, and thus $\|\mathcal{T}(w,z)\| < r$. Therefore, in view of Theorem 2.8, we have $\mathcal{T}(w,z) \in \overline{\mathcal{E}}$. Therefore $\mathcal{T} : \overline{\mathcal{E}} \to \overline{\mathcal{E}}$.

Let there exist $\rho \in (0,1)$ and $(w,z) \in \partial\mathcal{E}$ such that $(w,z) = \rho\mathcal{T}(w,z)$ and $\|(w,z)\| = r$. Then in view of property (ii) of the Green's functions, we obtain for all $t \in J$

$$\begin{aligned}|w(t)| &\le \rho \max_{t\in J} \int_0^1 \mathcal{H}_1(t,s)\Big|\theta(s,w(s),z(s))ds\Big| \\ &\le \rho\Big[\int_0^1 \mathcal{H}_1(1,s)\chi_1(s)ds + \int_0^1 \mathcal{H}_1(1,s)(\psi_1(s)|w(s)| + \sigma_1(s)|z(s)|)ds\Big] \\ &\le \rho(\mathbb{A}_1 + r\mathbb{B}_1)\end{aligned}$$

which implies that $\|w\| < \rho\frac{r}{2}$.

Similarly, we can obtain $\|z\| < \rho\frac{r}{2}$, and so $\|(w,z)\| < \rho r$, which is a contradiction to $(w,z) \in \partial\mathcal{E}$ for all $\rho \in (0,1)$. Thus by means of Theorem 2.8, $\mathcal{T}$ has a fixed point in $\overline{\mathcal{E}}$. □

We now introduce some additional useful assumptions.

- (H_5) The nonlinear functions θ and ϕ are continuous on $J \times [0,\infty) \times (0,\infty) \to [0,\infty)$;
- (H_6) For all $t \in J$, we have $\theta(t,w,z)|_{(w,z)=(0,0)} \ne 0,\ \theta(t,w,z)|_{(w,z)=(1,1)} \ne 0, \phi(t,w,z)|_{(w,z)=(0,0)} \ne 0,\ \phi(t,w,z)|_{(w,z)=(1,1)} \ne 0$;
- (H_7) For every $t \in J$ with $0 \le w \le w_1,\ 0 \le z \le z_1$, we have $\theta(t,w,z) \le \theta(t,w_1,z_1),\ \phi(t,w,z) \le \phi(t,w_1,z_1)$;
- (H_8) There exist $\lambda,\mu \in (0,1)$ such that

$$\text{for } t \in J, \alpha \in (0,1), w,z \ge 0, \text{ we have } \theta(t,\alpha w,\alpha z) \ge \alpha^\lambda \theta(t,w,z);$$
$$\text{for } t \in J, \alpha \in (0,1), w,z \ge 0, \text{ we have } \phi(t,\alpha w,\alpha z) \ge \alpha^\mu \phi(t,w,z).$$

Theorem 3.4. *Under the assumptions* $(H_5)-(H_8)$, *the boundary value problem* (1.1) *has a unique solution in* $\mathcal{C}_h$ *where* $h(t) = (t^{p-1}, t^{q-1})$.

Proof. Let $\kappa = \max\{\lambda,\mu\}$ and $(w,z) \in \mathbf{C}$. For every $t \in J$, using (H_8), we can obtain

$$\begin{aligned}\mathcal{T}_1(\alpha w,\alpha z)(t) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,\alpha w(s),\alpha z(s))ds \\ &\ge \alpha^\lambda \int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds \\ &= \alpha^\lambda \mathcal{T}_1(w,z)(t) \ge \alpha^\kappa \mathcal{T}_1(w,z)(t),\end{aligned}$$

Similarly, we can get $\mathcal{T}_2(\alpha w,\alpha z)(t) \ge \alpha^\kappa \mathcal{T}_2(w,z)(t)$.

Thus, we have

$$\mathcal{T}(\alpha w,\alpha z) \succeq \alpha^\kappa \mathcal{T}(w,z)(t),\ \alpha \in (0,1),\ (w,z) \in \mathbf{C},$$

where $\succeq$ is partial order defined by the cone $\mathbf{C}$. Hence $\mathcal{T}$ is an α-concave operator.

Let $h \in \mathbf{C}$ be defined by $h(t) = (t^{p-1}, t^{q-1}) = (h_1,h_2)(t),\ t \in J$. Let

$$\mathbf{w}_1 = \max\left\{\frac{1}{\Gamma(p)}\int_0^1 \theta(s,1,1)ds, \frac{1}{\Gamma(q)}\int_0^1 \phi(s,1,1)ds\right\},$$
$$\mathbf{w}_2 = \max\left\{\frac{1}{\Gamma(p)}\int_0^1 \mathcal{K}_1(s)\theta(s,0,0)ds, \frac{1}{\Gamma(q)}\int_0^1 \mathcal{K}_2(s)\phi(s,0,0)ds\right\},$$

where from the Green's functions, we have obtained the values

$$\mathcal{K}_1(s) = (1-s)^{p-1}\Big(\frac{1-\delta_1}{\delta_1}\Big),\ \mathcal{K}_2(s) = (1-s)^{q-1}\Big(\frac{1-\delta_2}{\delta_2}\Big). \tag{3.12}$$

Now in view of the nondecreasing properties of θ, ϕ and using (H_7), we get $\mathbf{w}_1 > 0,\ \mathbf{w}_2 > 0$. Thus from (3.12) and using (H_6), we get

$$\begin{aligned}\mathcal{T}_1 h(t) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,h_1(s),h_2(s))ds\\ &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,s^{p-1},s^{q-1})ds\\ &\leq \int_0^1 \mathcal{H}_1(t,s)\theta(s,1,1)ds\\ &\leq \left(\frac{1}{\Gamma(p)}\int_0^1 (1-s)^{p-1}\theta(s,1,1)ds\right)t^{p-1}\\ &\leq \mathbf{w}_1 h_1(t).\end{aligned}$$

Similarly, we get

$$\mathcal{T}_2 h(t) \leq \mathbf{w}_1 h_2(t).$$

Then, we have

$$\mathcal{T}h \preceq \mathbf{w}_1 h. \tag{3.13}$$

Further using (3.12) and (H_8), for all $t \in (0,1)$, one can get

$$\begin{aligned}\mathcal{T}_1 h(t) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,s^{p-1},s^{q-1})ds\\ &\geq \int_0^1 \mathcal{H}_1(t,s)\theta(s,0,0)ds\\ &\geq \left(\frac{1}{\Gamma p}\int_0^1 \mathcal{K}_1(s)\theta(s,0,0)ds\right)t^{p-1}\\ &\geq \mathbf{w}_2 h_1(t),\end{aligned}$$

and in the same fashion, we obtain $\mathcal{T}_2 h(t) \geq \mathbf{w}_2 h_2(t)$. Therefore, we have

$$\mathcal{T}h(t) \succeq \mathbf{w}_2 h. \tag{3.14}$$

Thus from (3.13) and (3.14), we get

$$\mathbf{w}_2 h \preceq \mathcal{T}h \preceq \mathbf{w}_1 h,$$

which implies that $\mathcal{T}h \in \mathbf{C}$. Hence by Theorem 2.7 the operator $\mathcal{T}$ is a concave operator. Therefore $\mathcal{T}$ has a unique fixed point $(u,z) \in \mathbf{C}$ which is the corresponding solution of the Toppled system (1.1). □

The following assumptions are needed hereafter in this paper:

(C_1) $\theta, \phi : J \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are uniformly continuous with respect to t on J;

(C_2) Let $\theta^{\beta} = \lim_{\|(w,z)\|\to\beta} \sup_{t\in J} \dfrac{\theta(t,w,z)}{w+z}$, $\phi^{\beta} = \lim_{\|(w,z)\|\to\beta} \sup_{t\in J} \dfrac{\phi(t,w,z)}{w+z}$, $\theta_{\beta} = \lim_{\|(w,z)\|\to\beta} \inf_{t\in J} \dfrac{\theta(t,w,z)}{w+z}$, $\theta_{\beta} = \lim_{\|(w,z)\|\to\beta} \inf_{t\in J} \dfrac{\theta(t,w,z)}{w+z}$, where $\beta = 0$, or ∞;

(C_3) $\widehat{G}_1 = \max_{t\in J} \int_0^1 \mathcal{H}_1(t,s)ds,\ \widehat{G}_2 = \max_{t\in J} \int_0^1 \mathcal{H}_2(t,s)ds.$

Theorem 3.5. *Under the conditions* $(C_1)-(C_3)$ *and the following two conditions:*

(H_9) $a, b > 0$, *with* $1 \le a+b < \infty$, *are two constants such that*

$$a\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1} < \theta_0 < \infty,\ b\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)^{-1} < \phi_0 < \infty,$$

(H_{10}) λ_1, λ_2, *with* $\lambda_1+\lambda_2 \le 1$, *are two constants such that*

$$0 \le \widehat{G}_1\theta^\infty < \lambda_1, \quad 0 \le \widehat{G}_2\phi^\infty < \lambda_2,$$

the Toppled system (1.1) *has at least one positive solution.*

Proof. Due to (H_9), taking

$$\theta_0 - a\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1} > r_1 > 0,\ \phi_0 - b\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)^{-1} > r_2 > 0,$$

then there exists a constant $\epsilon_1 > 0$ such that

$$\begin{aligned} \theta(t,w,z) &\ge (\theta_0 - r_1)(w+z), t\in J, w,z \in [0,\epsilon_1],\\ \theta(t,w,z) &\ge (\theta_0 - r_2)(w+z), t\in J, w,z \in [0,\epsilon_1]. \end{aligned}$$

We define an open set

$$\mathcal{E}_1 = \{(w,z) \in \mathcal{B}\times\mathcal{B} : \|(w,z)\| < \epsilon_1\}.$$

Let $\tau \in [\vartheta, 1-\vartheta]$. Also from property (iii) of the Green's functions, for any $(w,z) \in \mathbf{C}\cap\partial\mathcal{E}_1$, we obtain that

$$\begin{aligned} \|\mathcal{T}_1(w,z)\| &\ge \mathcal{T}_1(w,z)(\tau)\\ &\ge \int_0^1 \mathcal{H}_1(\tau,s)\theta(s,w(s),z(s))ds\\ &\ge \gamma_1 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)(\theta_0 - r_1)[w(s)+z(s)]ds\\ &\ge (\theta_0 - r_1)\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)\|(w,z)\|\\ &\ge a\|(w,z)\|. \end{aligned}$$

Similarly we obtain

$$\begin{aligned} \|\mathcal{T}_2(w,z)\| &\ge \mathcal{T}_2(w,z)(\tau)\\ &\ge \int_0^1 \mathcal{H}_2(\tau,s)\phi(s,w(s),z(s))ds\\ &\ge \gamma_2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)(\phi_0 - r_2)[w(s)+z(s)]ds\\ &\ge (\phi_0 - r_2)\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)\|(w,z)\|\\ &\ge b\|(w,z)\|. \end{aligned}$$

Therefore

$$\begin{aligned}\|\mathcal{T}(w,z)\| &= \|\mathcal{T}_1(w,z)\| + \|\mathcal{T}_2(w,z)\| \\ &\geq (a+b)\|(w,z)\| \\ &\geq \|(w,z)\|.\end{aligned}$$

Also from (H_{10}), we have

$$\lambda_1 \widehat{G}_1^{-1} - \theta^{\infty} > r_3 > 0, \ \lambda_2 \widehat{G}_2^{-1} - \phi^{\infty} > r_4 > 0.$$

There exists $\epsilon_2 > 0$ such that

$$\begin{aligned}\theta(t,w,z) \leq (\theta^{\infty} + r_3)(w+z), \ t \in J, w, z \geq 0, w+z > \epsilon_2, \\ \theta(t,w,z) \leq (\theta^{\infty} + r_4)(w+z), \ t \in J, w, z \geq 0, w+z > \epsilon_2.\end{aligned}$$

From (H_9), we take

$$M = \sup_{t \in J, w+z \in [0,\epsilon_2]} \theta(t,w,z), \ N = \sup_{t \in J, w+z \in [0,\epsilon_2]} \phi(t,w,z).$$

We have

$$\begin{aligned}\theta(t,w,z) &\leq (\theta^{\infty} + r_3)(w+z) + M, t \in J, w, z \geq 0, \\ \phi(t,w,z) &\leq (\phi^{\infty} + r_4)(w+z) + N, t \in J, w, z \geq 0.\end{aligned}$$

Let

$$\mathcal{E}_2 = \{(w,z) \in \mathcal{B} \times \mathcal{B} : \|(w,z)\| < \epsilon_2\},$$

where

$$\max\{\epsilon_1, M\widehat{G}_1(\lambda_1 - (\theta^{\infty} + r_3)\widehat{G}_1)^{-1}, \ N\widehat{G}_2(\lambda_2 - (\phi^{\infty} + r_4)\widehat{G}_2)^{-1}\} < \epsilon.$$

where $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. From (ii) of the Green's functions and for any $(w,z) \in \mathbf{C} \cap \partial\mathcal{E}_2$, for $t \in J$, we have

$$\begin{aligned}\mathcal{T}_1(w,z)(t) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds \\ &\leq \int_0^1 \mathcal{H}_1(1,s)[(\theta^{\infty} + r_3)(w(s) + z(s) + M)]ds \\ &\leq M\int_0^1 \mathcal{H}_1(1,s)ds + (\theta^{\infty} + r_3)\epsilon \int_0^1 \mathcal{H}_1(1,s)ds \\ &\leq M\widehat{G}_1 + (\theta^{\infty} + r_3)\epsilon\widehat{G}_1 \\ &\leq \lambda_1\epsilon - (\theta^{\infty} + r_3)\epsilon\widehat{G}_1 + (\theta^{\infty} + r_3)\epsilon\widehat{G}_1 \\ &= \lambda_1\epsilon\end{aligned}$$

which implies that $\|\mathcal{T}_1(w,z) \leq \lambda_1\|(w,z)\|$. Similarly, we can obtain for $(w,z) \in \mathbf{C} \cap \partial\mathcal{E}_1$, for $t \in J$

$$\begin{aligned}
\mathcal{T}_2(w,z)(t) &= \int_0^1 \mathcal{H}_2(t,s)\phi(s,w(s),z(s))ds \\
&\leq \int_0^1 \mathcal{H}_2(1,s)[(\phi^{\infty}+r_4)(w(s)+z(s)+N)]ds \\
&\leq N\int_0^1 \mathcal{H}_2(1,s)ds + (\phi^{\infty}+r_4)\epsilon \int_0^1 \mathcal{H}_2(1,s)ds \\
&\leq N\widehat{G}_2 + (\phi^{\infty}+r_4)\epsilon\widehat{G}_2 \\
&\leq \lambda_2\epsilon - (\phi^{\infty}+r_4)\epsilon\widehat{G}_2 + (\phi^{\infty}+r_4)\epsilon\widehat{G}_2 \\
&= \lambda_2\epsilon
\end{aligned}$$

which implies that $\|\mathcal{T}_2(w,z) \leq \lambda_2\|(w,z)\|$. Then we have

$$\begin{aligned}
\|\mathcal{T}(w,z)\| &= \|\mathcal{T}_1(w,z)\| + \|\mathcal{T}_2(w,z)\| \\
&\leq (\lambda_1+\lambda_2)\|(w,z)\| \\
&\leq \|(w,z)\|.
\end{aligned}$$

Thanks to Theorem 3.2, we have that $\mathcal{T} : \mathbf{C} \cap (\overline{\mathcal{E}_2} \setminus \mathcal{E}_1) \to \mathbf{C}$ is a completely continuous operator. Therefore by an application of Theorem 2.9, $\mathcal{T}$ has at least one fixed point $(w,z) \in \mathbf{C} \cap (\bar{\mathcal{E}_2} \setminus \mathcal{E}_1)$, which is the corresponding positive solution to Toppled system (1.1). □

Theorem 3.6. *Assume that* $(C_1) - (C_2)$ *together with the following conditions hold:*

(H_{11}) $\theta_0 = \phi_0 = \infty$;
(H_{12}) $a, b > 0$ *such that* $a + b \leq 1$ *are two constants with* $a \geq \theta^{\infty}\widehat{G}_1 \geq 0,\ b \geq \phi^{\infty}\widehat{G}_2 \geq 0$.

Then the Toppled system (1.1) *has at least one positive solutions.*

Proof. Let $r_1 > \frac{1}{2}\left(\gamma^2 \int_0^1 \mathcal{H}_1(1,s)ds\right)^{-1}$ and $r_2 > \frac{1}{2}\left(\gamma^2 \int_0^1 \mathcal{H}_2(1,s)ds\right)^{-1}$. Then from (H_{12}), there exists a real constant $\epsilon_1 > 0$, such that

$$\theta(t,w,z) \geq r_1(w+z), t \in J, w, z, t \in [0,\epsilon_1],\ \phi(t,w,z) \geq r_2(w+z), t \in J, w, z, t \in [0,\epsilon_1].$$

Assume that

$$\mathcal{E}_1 = \{(w,z) \in \mathcal{B} \times \mathcal{B} : \|(w,z)\| < \epsilon_1\}.$$

Let $\tau \in [\vartheta, 1-\vartheta]$. Under Property (iii) of the Green's functions and for any $(w,z) \in \mathbf{C} \cap \partial\mathcal{E}_1$, we get

$$\begin{aligned}
\|\mathcal{T}_1(w,z)\| &\geq \mathcal{T}_1(w,z)(\tau)\| \\
&\geq \int_0^1 \mathcal{H}_1(\tau,s)\theta(s,w(s),z(s))ds \\
&\geq \widehat{G}_1 \int_\vartheta^{1-\vartheta} \mathcal{H}_1(1,s)[w(s)+z(s)]ds \\
&\geq r_1\Big(\gamma^2 \int_\vartheta^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)\|(w,z)\| \\
&\geq \frac{1}{2}\|(w,z)\|.
\end{aligned}$$

Along the same lines, we can get

$$\begin{aligned}
\|\mathcal{T}_2(w,z) &\geq \mathcal{T}_2(w,z)(\tau) \\
&\geq \int_0^1 \mathcal{H}_2(\tau,s)\theta(s,w(s),z(s))ds \\
&\geq \widehat{G}_2 \int_\vartheta^{1-\vartheta} \mathcal{H}_2(1,s)(\phi_0 - r_2)[w(s)+z(s)]ds \\
&\geq r_2\Big(\gamma^2 \int_\vartheta^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)\|(w,z)\| \\
&\geq \frac{1}{2}\|(w,z)\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\mathcal{T}(w,z)\| &= \|\mathcal{T}_1(w,z)\| + \|\mathcal{T}_2(w,z)\| \\
&\geq \|(w,z)\|.
\end{aligned}$$

Also from (H_{12}) and from Theorem 3.5, there exists $\epsilon_2 < \epsilon_1$ such that

$$\|\mathcal{T}(w,z)\| \leq \|(w,z)\|, (w,z) \in \mathbf{C} \cap \partial\mathcal{E}_2,$$

where

$$\mathcal{E}_2 = \{(w,z) \in \mathcal{B} \times \mathcal{B} : \|(w,z)\| < \epsilon_2\}.$$

By using Theorem 2.9, we get that $\mathcal{T}$ has at least one fixed point $(w,z) \in \mathbf{C} \cap (\bar{\mathcal{E}_2} \setminus \mathcal{E}_1)$, which is the corresponding positive solution to Toppled system (1.1). □

Theorem 3.7. *Assume that* $(C_1)-(C_3)$ *together with the following conditions are satisfied:*

(H_{13}) *for any two constants* $a, b > 0$, *with* $1 \leq a + b < \infty$ *such that*

$$a\Big(\gamma^2 \int_\vartheta^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1} < \theta_\infty < \infty,\ b\Big(\gamma^2 \int_\vartheta^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)^{-1} < \phi_\infty < \infty;$$

(H_{14}) *for any two constants* λ_1, λ_2 *with* $\lambda_1 + \lambda_2 \leq 1$ *such that*

$$0 \leq \widehat{G}_1\theta^0 < \lambda_1, \quad 0 \leq \widehat{G}_2\phi^0 < \lambda_2.$$

Then the Toppled system (1.1) *has at least one positive solution.*

Proof. From (H_{14}) we have

$$\lambda_1 \widehat{G}_1^{-1} - \theta^0 > r_1 > 0,\ \lambda_2 \widehat{G}_2^{-1} - \phi^0 > r_2 > 0.$$

Then there exists $\epsilon > 0$ such that

$$\theta(t,w,z) \le (\theta^0 + r_1)(w+z), t \in J, w, z \in [0,\epsilon],$$

and

$$\theta(t,w,z) \le (\theta^0 + r_2)(w+z), t \in J, w, z \in [0,\epsilon].$$

Let

$$\mathcal{E}_1 = \{(w,z) \in \mathcal{B} \times \mathcal{B} : \|(w,z)\| < \epsilon\}.$$

Inview of Property (iii) of the Green's functions and for any $(w,z) \in \mathbf{C} \cap \partial\mathcal{E}_1, t \in J$, we get

$$\begin{aligned}
\mathcal{T}_1(w,z)(t) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds \\
&\le \int_0^1 \mathcal{H}_1(t,s)(\theta^0 + r_1)(w(s)+z(s))ds \\
&\le (\theta^0 + r_1)\epsilon \int_0^1 \mathcal{H}_1(1,s)ds \\
&\le (\theta^0 + r_1)\epsilon \widehat{G}_1 \\
&\le \lambda_1 \epsilon
\end{aligned}$$

which implies that

$$\|\mathcal{T}_1(w,z) \le \lambda_1 \|(w,z)\|.$$

In a similar way, for $t \in J$, we can obtain $\|\mathcal{T}_2(w,z)\| \le \lambda_2 \|(w,z)\|$. Thus

$$\|\mathcal{T}(w,z)\| = \|\mathcal{T}_1(w,z)\| + \|\mathcal{T}_2(w,z)\| \le (\lambda_1 + \lambda_2)\|(w,z)\| \le \|(w,z)\|.$$

Further from (H_{13}), we obtain

$$\begin{aligned}
\theta_\infty - a\left(\gamma^2 \int_\vartheta^{1-\vartheta} \mathcal{H}_1(1,s)ds\right)^{-1} &> r_3 > 0, \\
\phi_\infty - b\left(\gamma^2 \int_\vartheta^{1-\vartheta} \mathcal{H}_2(1,s)ds\right)^{-1} &> r_4 > 0.
\end{aligned}$$

Then there exist $\bar{\epsilon} > 0$, so that

$$\begin{aligned}
\theta(t,w,z) \ge (\theta_\infty - r_3)(w+z), t \in J, w, z \ge 0, w+z &\ge \bar{\epsilon}, \\
\phi(t,w,z) \ge (\phi_\infty - r_4)(w+z), t \in J, w, z \ge 0, w+z &\ge \bar{\epsilon}.
\end{aligned}$$

Let $\mathcal{E}_2 = \{(w,z) \in \mathcal{B} \times \mathcal{B} : \|(w,z)\| < \epsilon_1\}$, where $\max\{\epsilon, \frac{\bar{\epsilon}}{\gamma}\} < \epsilon_1$. Then for $(w,z) \in \mathbf{C} \cap \partial\mathcal{E}_2$, and for all $s \in J$, we get

$$w(s) + z(s) \ge \min_{t \in J}[w(t) + z(t)] \ge \gamma \|(w,z)\| \ge \gamma\epsilon_1 \ge \bar{\epsilon}.$$

Taking $\tau \in [\vartheta, 1-\vartheta]$ and using (ii), we obtain that

$$\begin{aligned}
\|\mathcal{T}_1(w,z) &\geq \mathcal{T}_1(w,z)(\tau) \\
&\geq \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(\tau,s)\theta(s,w(s),z(s))ds \\
&\geq \gamma_p \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(\tau,s)(\theta_\infty - r_3)(w(s)+z(s))ds \\
&\geq (\theta_\infty - r_3)\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)\|(w,z)\| \\
&\geq a\|(w,z)\|.
\end{aligned}$$

This implies that

$$\|\mathcal{T}_1(w,z) \geq a\|(w,z)\|.$$

In the same fashion, we get

$$\|\mathcal{T}_2(w,z)\| \geq b\|(w,z)\|.$$

which yields

$$\|\mathcal{T}(w,z)\| \geq (a+b)\|(w,z)\| \geq \|(w,z)\|.$$

Thanks to Theorem 2.9 , we get that $\mathcal{T}$ has at least one fixed point $(w,z) \in \mathbf{C} \cap (\overline{\mathcal{E}_2} \setminus \mathcal{E}_1)$, which is the corresponding positive solution of the considered Toppled system (1.1). □

Theorem 3.8. *Assume that $(C_1)-(C_3)$ together with the following conditions are satisfied:*

(H_{15}) $\theta_\infty = \phi_\infty = \infty$;

(H_{16}) *for any two constants $\lambda_1, \lambda_2 > 0$, with $\lambda_1 + \lambda_2 \leq 1$ such that*

$$\lambda_1 \widehat{G}_1^{-1} > \theta^0 > 0, \ \ \lambda_2 \widehat{G}_2^{-1} > \phi^0 > 0.$$

Then the Toppled system(1.1) *has at most one positive solution.*

Proof. We are omitting the proof as it is similar to the proof of Theorem 3.7. □

Theorem 3.9. *Assume that $(C_1)-(C_3)$ (with $\theta_0, \theta_\infty, \phi_0, \phi_\infty$ being extended real values), together with the following assumptions are satisfied:*

(H_{17}) *there are $a, b > 0$ with $a+b \geq 1$ such that*

$$\theta_0 > a\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1}, \ \theta_\infty > a\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1},$$

$$\phi_0 > b\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)^{-1}, \ \phi_\infty > b\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)^{-1};$$

(H_{18}) *there exist $r > 0, c, d > 0$ with $c+d < 1$ such that*

$$\max_{t \in J, (w,z) \in \partial\Omega_r} \theta(t,w,z) \leq \frac{cr}{\widehat{G}_1}, \max_{t \in J, (w,z) \in \partial\Omega_r} \phi(t,w,z) \leq \frac{dr}{\widehat{G}_2}.$$

Then Toppled system (1.1) *has at least two positive solutions (w,z) and $(\bar{w},\bar{z})$, which satisfy*

$$0 < \|(w,z)\| < r < \|(\bar{w},\bar{z})\|.$$

Proof. Consider ϵ, R with $0<\epsilon<r<R$, such that

$$\infty \geq \theta_0 > a\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1} \text{ and } \infty \geq \phi_0 > b\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)^{-1},$$

hold. Then as in the proof of Theorem 3.5, we have

$$\|\mathcal{T}(w,z)\| \geq (a+b)\|(w,z)\| \geq \|(w,z)\|, \text{ for all } (w,z) \in \mathbf{C} \cap \partial\Omega_\epsilon. \tag{3.15}$$

If

$$\infty \geq \theta_\infty > a\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1} \text{ and } \infty \geq \phi_\infty > b\Big(\gamma^2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1,s)ds\Big)^{-1}$$

hold, then as in the proof of Theorem 3.6, we have

$$\|\mathcal{T}(w,z)\| \geq (a+b)\|(w,z)\| \geq \|(w,z)\|, \text{ for all } (w,z) \in \partial\Omega_R. \tag{3.16}$$

Also together with (H_{17}), for $(w,z) \in \mathbf{C} \cap \partial\Omega_r$, we get

$$\begin{aligned} \mathcal{T}_1(w,z) &= \int_0^1 \mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds \\ &\leq \int_0^1 \mathcal{H}_1(1,s)\theta(s,w(s),z(s))ds \\ &\leq \frac{cr}{\widehat{G}_1} \int_0^1 \mathcal{H}_1(1,s)ds \\ &= cr \end{aligned}$$

which implies

$$\mathcal{T}_1(w,z) \leq cr. \tag{3.17}$$

Similarly we can get

$$\mathcal{T}_2(w,z) \leq dr. \tag{3.18}$$

Hence, from (3.17) and (3.18), we have $\|\mathcal{T}(w,z)\| \leq (c+d)r < r$. Which implies that

$$\|\mathcal{T}(w,z)\| < r = \|(w,z)\|, \text{ for all } (w,z) \in \mathbf{C} \cap \partial\Omega_r. \tag{3.19}$$

Thus, an application of Theorem 2.9, to (3.15) and (3.16), gives that $\mathcal{T}$ has a fixed point $(w_1,z_1) \in \mathbf{C} \cap (\bar{\Omega_r} \setminus \Omega_\epsilon)$. Applying Theorem 2.9 to (3.16) and (3.19) gives that $\mathcal{T}$ has a fixed point $(w_2,z_2) \in \mathbf{C} \cap (\bar{\Omega_R} \setminus \Omega_r)$. Further from (3.19), we have that $\|(w_1,z_1)\| \neq r$ and $\|(w_2,z_2)\| \neq r$. Thus

$$0 < \|(w_1,z_1)\| < r < \|(w_2,z_2)\|.$$

□

Theorem 3.10. *Let $(C_1)-(C_3)$ together with the following conditions hold:*

(H_{19}) *if $a,b,c,d>0$ be constants such that $a+b \leq 1, c+d \leq 1$ with*

$$\frac{a}{\widehat{G}_1} > \theta^0 \geq 0, \frac{b}{\widehat{G}_2} > \phi^0 \geq 0, \textit{and } \frac{c}{\widehat{G}_1} > \theta^\infty \geq 0, \frac{d}{\widehat{G}_2} > \phi^\infty \geq 0;$$

(H_{20}) *for constants $\lambda, \mu > 0$ with $\lambda+\mu>1$ there exist $\rho>0$ such that*

$$\min_{t\in[\vartheta,1-\vartheta],(w,z)\in[0,\rho]\times[0,\rho]} \theta(t,w(t),z(t)) > \rho\lambda\Big(\gamma_1 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1,s)ds\Big)^{-1},$$

and

$$\min_{t\in[\vartheta,1-\vartheta],(w,z)\in[0,\rho]\times[0,\rho]} \phi(t,w(t),z(t)) > \rho\mu\Big(\gamma_2\int_{\vartheta}^{1-\vartheta}\mathcal{H}_2(1,s)ds\Big)^{-1}.$$

Then the BVP (1.1) *has at least two positive solutions* (w_1,z_1) *and* (w_2,z_2) *which satisfy*

$$0<\|(w_1,z_1)\|<\rho<\|(w_2,z_2)\|.$$

Proof. Assume that $0<\epsilon<\rho<R$ and construct open balls as in the proof of Theorem 3.9,

$$\mathcal{E}_\epsilon=\{(w,z)\in\mathcal{B}\times\mathcal{B}:\|(w,z)\|<\epsilon\} \text{ and } \mathcal{E}_R=\{(w,z)\in\mathcal{B}\times\mathcal{B}:\|(w,z)\|<R\}.$$

Now from (H_{19}), for $\frac{a}{\widehat{G}_1}>\theta^0\geq 0$, $\frac{b}{\widehat{G}_2}>\phi^0\geq 0$, and from the proof of Theorem 3.8, we get

$$\|\mathcal{T}(w,z)\|\leq\|(w,z)\|, \text{ for all } (w,z)\in\mathbf{C}\cap\partial\mathcal{E}_\epsilon. \tag{3.20}$$

Further for $\frac{c}{\widehat{G}_1}>\theta^\infty\geq 0$, $\frac{d}{\widehat{G}_2}>\phi^\infty\geq 0$, and from the proof of Theorem 3.5, we have

$$\|\mathcal{T}(w,z)\|\leq\|(w,z)\|, \text{ for all } (w,z)\in\mathbf{C}\cap\partial\mathcal{E}_R. \tag{3.21}$$

On the other hand, define an open ball $\mathcal{E}_\rho=\{(w,z)\in\mathcal{B}\times\mathcal{B}:\|(w,z)\|<\rho\}$, and from the assumption (H_{20}), if $(w,z)\in\mathbf{C}\cap\partial\mathcal{E}_\rho$, and for any $\tau\in J$, we have

$$\begin{aligned}\|\mathcal{T}_1(w,z)\geq\mathcal{T}_1(w,z)(\tau)&\geq\int_0^1\mathcal{H}_1(t,s)\theta(s,w(s),z(s))ds\\&\geq\lambda\rho\Big(\gamma_1\int_{\vartheta}^{1-\vartheta}\mathcal{H}_1(1,s)ds\Big)\Big(\gamma_1\int_{\vartheta}^{1-\vartheta}\mathcal{H}_1(1,s)ds\Big)^{-1}=\lambda\rho.\end{aligned}$$

Similarly we have $\|\mathcal{T}_2(w,z)\|\geq\mu\rho$, which implies that

$$\|\mathcal{T}(w,z)\|=\|\mathcal{T}_1(w,z)\|+\|\mathcal{T}_2(w,z)\|\geq\rho(\lambda+\mu)>\rho=\|(w,z)\|.$$

Thus we have from these

$$\|\mathcal{T}(w,z)\|>\rho=\|(w,z)\|, \text{ for all } (w,z)\in\mathbf{C}\cap\partial\mathcal{E}_\rho. \tag{3.22}$$

An application of Theorem 2.9 to (3.20) and (3.22) yields that $\mathcal{T}$ has a fixed point $(w_1,z_1)\in\mathbf{C}\cap(\bar{\mathcal{E}}_\rho\setminus\mathcal{E}_\epsilon)$ and similarly applying Theorem 2.9 to (3.21) and (3.22), we get that $\mathcal{T}$ has a fixed point $(w_2,z_2)\in\mathbf{C}\cap(\bar{\mathcal{E}}_R\setminus\mathcal{E}_\rho)$. Also we know from (3.22) that $\rho\neq\|(w_1,z_1)\|$ and $\rho\neq\|(w_2,z_2)\|$. Therefore

$$0<\|(w_1,z_1)\|<\rho<\|(w_2,z_2)\|.$$

□

Theorem 3.11. *Assume that* $(C_1)-(C_3)$ *hold. If there exist* $2n$ *natural numbers* $\mathbf{x}_i,\widehat{\mathbf{x}}_i$, $i=1,2\ldots n$ *with* $\mathbf{x}_1<\gamma_1\widehat{\mathbf{x}}_1<\widehat{\mathbf{x}}_1<\mathbf{x}_2<\gamma_1\widehat{\mathbf{x}}_2<\widehat{\mathbf{x}}_2\ldots\mathbf{x}_n<\gamma_1\widehat{\mathbf{x}}_n<\widehat{\mathbf{x}}_n$ *and* $\mathbf{x}_1<\gamma_2\widehat{\mathbf{x}}_1<\widehat{\mathbf{x}}_1<\mathbf{x}_2<\gamma_2\widehat{\mathbf{x}}_2<\widehat{\mathbf{x}}_2\ldots\mathbf{x}_n<\gamma_2\widehat{\mathbf{x}}_n<\widehat{\mathbf{x}}_n$ *such that*

(H_{21}) $\theta\geq\mathbf{x}_i\Big(\gamma_1\int_0^1\mathcal{H}_1(1,s)ds\Big)^{-1}$, *for* $(t,w,z)\in J\times[\gamma_1\mathbf{x}_i,\mathbf{x}_i]\times[\gamma_2\mathbf{x}_i,\mathbf{x}_i]$, *and*

$$\theta\leq\widehat{G}_1^{-1}\widehat{\mathbf{x}}_i, \textit{for } (t,w,z)\in J\times[\gamma_1\widehat{\mathbf{x}}_i,\widehat{\mathbf{x}}_i]\times[\gamma_2\mathbf{x}_i,\mathbf{x}_i], i=1,2\ldots n,$$

(H_{22}) $\phi\geq\mathbf{x}_i\Big(\gamma_2\int_0^1\mathcal{H}_2(1,s)ds\Big)^{-1}$, *for* $(t,w,z)\in J\times[\gamma_2\mathbf{x}_i,\mathbf{x}_i]\times[\gamma_1\mathbf{x}_i,\mathbf{x}_i]$, *and*

$$\phi\leq\widehat{G}_2^{-1}\widehat{\mathbf{x}}_j, \textit{for } (t,w,z)\in J\times[\gamma_1\mathbf{x}_j,\mathbf{x}_i]\times[\gamma_2\widehat{\mathbf{x}}_j,\widehat{\mathbf{x}}_j],\ j=1,2\ldots n.$$

Then the Toppled system (1.1) *has at least n positive solutions* (w_j, z_j)*, satisfying* $\mathbf{x}_j \leq \|(w_j, z_j)\| \leq \widehat{\mathbf{x}}_j,\ j = 1, 2 \dots n$.

Proof. The proof is very much like the proofs of Theorems 3.9 and 3.10, and so we omit it. □

Theorem 3.12. *Suppose that* $(C_1) - (C_3)$ *hold. Let there exist* $2n$ *nonnegative integer* $\mathbf{x}_j, \widehat{\mathbf{x}}_j,\ j = 1, 2 \dots n$*, with* $\mathbf{x}_1 < \widehat{\mathbf{x}}_1 < \mathbf{x}_2 < \widehat{\mathbf{x}}_2 \dots < \mathbf{x}_n < \widehat{\mathbf{x}}_n$ *such that*

(H_{23}) θ *and* ϕ *are non-decreasing on* $[0, \widehat{\mathbf{x}}_n]$ *for all* $t \in J$;

(H_{24}) $\theta(t, w, z) \geq \mathbf{x}_j \left(\gamma_1 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_1(1, s) ds \right)^{-1},\ \theta(t, w, z) \leq \widehat{G}_1^{-1} \widehat{\mathbf{x}}_j,\ j = 1, 2 \dots n,$

$$\phi(t, w, z) \geq \mathbf{x}_j \left(\gamma_2 \int_{\vartheta}^{1-\vartheta} \mathcal{H}_2(1, s) ds \right)^{-1},\ \phi(t, w, z) \leq \widehat{G}_2^{-1} \widehat{\mathbf{x}}_i,\ j = 1, 2 \dots n.$$

Then the Toppled system (1.1) *has at least n positive solutions* (w_j, z_j)*, satisfying* $\mathbf{x}_j \leq \|(w_j, z_j)\| \leq \widehat{\mathbf{x}}_j,\ j = 1, 2 \dots n$.

Proof. The proof is very much like the proofs of Theorems 3.9 and 3.10, and so we omit it. □

Next, we derive some results under which the considered Toppled system (1.1) has no solution.

Theorem 3.13. *If*$(C_1) - (C_3)$ *are satisfied and there exist two constants* $a, b > 0$*, such that* $a + b < 1$*, and* $\|\theta(t, w, z)\| < a\|(w, z)\|\widehat{G}_1^{-1}$ *and* $\|\phi(t, w, z)\| < b\|(w, z)\|\widehat{G}_2^{-1}$*, for all* $t \in J, w, z > 0$*, then the Toppled system* (1.1) *has no positive solution.*

Proof. On contrary let (w, z) be a positive solution of the Toppled system (1.1). Then $(w, z) \in \mathbf{C}$ for $0 < t < 1$ and

$$\begin{aligned} \|(w, z)\| &= \max_{t \in J} |(w(t)| + \max_{t \in J} |z(t)| \\ &= \max_{t \in J} |\mathcal{T}_1(w, z)(t)| + \max_{t \in J} |\mathcal{T}_2(w, z)(t)| \\ &\leq \max_{t \in J} \int_0^1 \mathcal{H}_1(t, s) |\theta(s, w(s), z(s))| ds \\ &\quad + \max_{t \in J} \int_0^1 \mathcal{H}_2(t, s) |\theta(s, w(s), z(s)| ds \\ \text{which implies that } \|(w, z)\| &< \int_0^1 \mathcal{H}_1(1, s) a \|(w, z)\| \widehat{G}_1^{-1} ds \\ &\quad + \int_0^1 \mathcal{H}_2(1, s) b \|(w, z)\| \widehat{G}_2^{-1} ds \\ &< a\|(w, z)\| + b\|(w, z)\| \\ &= (a + b)\|(w, z)\| \\ &< \|(w, z)\| \end{aligned}$$

which implies that

$$\|(w, z)\| < \|(w, z)\|,$$

which is a contradiction. Therefore the Toppled system (1.1) has no positive solution. □

Theorem 3.14. *Let* $(C_1)-(C_3)$ *hold and*

$$\begin{aligned}\theta(t,w(t),z(t)) &> \lambda\|(w,z)\|\Big(\gamma_1^2\int_{\vartheta}^{1-\vartheta}\mathcal{H}_1(1,s)ds\Big)^{-1},\\ \phi(t,w(t),z(t)) &> \mu\|(w,z)\|\Big(\gamma_2^2\int_{\vartheta}^{1-\vartheta}\mathcal{H}_2(1,s)ds\Big)^{-1},\end{aligned}$$

for all $t\in J, w,z>0$. *Moreover, if* $\lambda,\mu>0$ *are the two constants with* $\lambda+\mu>1$, *then the Toppled system* (1.1) *has no positive solution.*

Proof. We can prove this result as in the proof of Theorem 3.13. □

4. Examples

In this section we provide some examples to demonstrate our results.

Example 4.1. Consider the Toppled system with nonlocal boundary conditions provided by

$$\begin{cases}\mathcal{D}^{\frac{5}{2}}w(t)+(1+t^2)+\sqrt[3]{w(t)z(t)}=0,\ t\in(0,1),\\ \mathcal{D}^{\frac{7}{3}}z(t)+1+t+\sqrt[4]{w(t)z(t)}=0,\ t\in(0,1),\\ \mathcal{I}^{\frac{1}{2}}w(t)|_{t=0}=\mathcal{D}^{\frac{1}{2}}w(t)|_{t=0}=0,\ w(1)=w\Big(\frac{1}{4}\Big),\\ \mathcal{I}^{\frac{2}{3}}z(t)|_{t=0}=\mathcal{D}^{\frac{1}{3}}z(t)|_{t=0}=0,\ z(1)=z\Big(\frac{1}{3}\Big).\end{cases}\tag{4.1}$$

It is easy to see that

$$\begin{aligned}\theta(t,w,z)|_{(w,z)=(0,0)}\neq0,\ \phi(t,w,z)|_{(w,z)=(0,0)} &\neq 0,\\ \theta(t,w,z)|_{(w,z)=(1,1)}\neq0,\ \phi(t,w,z)|_{(w,z)=(1,1)} &\neq 0.\end{aligned}$$

By simple computation, we can prove that θ,ϕ are nondecreasing for all $t\in(0,1)$. Also for $\alpha\in(0,1)$ and $t\in(0,1), w,z\geq0$, we have $\max\{\frac{1}{4},\frac{1}{3}\}=\frac{1}{3}$. Therefore

$$\begin{aligned}\theta(t,\alpha w,\alpha z) &\geq \alpha^{\frac{1}{3}}\theta(t,w,z),\\ \phi(t,\alpha w,\alpha z) &\geq \alpha^{\frac{1}{3}}\phi(t,w,z).\end{aligned}$$

Thus all the assumption of Theorem3.4 are satisfied, so the Toppled system (4.1) has a unique positive solution in $\mathcal{C}_h$, where $h(t)=(t^{\frac{3}{2}},t^{\frac{4}{3}})$.

Example 4.2. Consider the Toppled system of nonlocal boundary value problems

$$\begin{cases}\mathcal{D}^{\frac{5}{2}}w(t)+\sqrt[4]{w(t)+z(t)}=0,\ \mathcal{D}^{\frac{7}{3}}z(t)+\sqrt[3]{w(t)+z(t)}=0,\ t\in(0,1),\\ \mathcal{I}^{\frac{1}{2}}w(t)|_{t=0}=\mathcal{D}^{\frac{1}{2}}w(t)|_{t=0}=0,\ w(1)=w\Big(\frac{1}{2}\Big),\\ \mathcal{I}^{\frac{2}{3}}z(t)|_{t=0}=\mathcal{D}^{\frac{1}{3}}(t)|_{t=0}=0,\ z(1)=z\Big(\frac{1}{2}\Big),\end{cases}\tag{4.2}$$

where

$$\theta(t,w,z)=\sqrt[4]{w(t)+z(t)}\text{ and }\phi(t,w,z)=\sqrt[3]{w(t)+z(t)}.$$

Then

$$\theta^0=\lim_{(w,z)\to(0,0)}\frac{\theta(t,w,z)}{w+z}=\infty,$$

and similarly $\phi^0 = \infty$. Also by simple computation, one can obtain that

$$\theta^\infty = 0 = \phi^\infty.$$

By Theorem 3.6, the Toppled system (4.2) has at least one positive solution.

Example 4.3. Consider the boundary value problem

$$\begin{cases} \mathcal{D}^{\frac{7}{3}} w(t) + [w(t) + z(t)]^4 = 0,\ \mathcal{D}^{\frac{5}{2}} z(t) + [w(t) + z(t)]^5 = 0,\ t \in (0,1), \\ \mathcal{I}^{\frac{2}{3}} w(t)|_{t=0} = \mathcal{D}^{\frac{1}{3}} w(t)|_{t=0} = 0,\ w(1) = w\left(\frac{1}{4}\right), \\ \mathcal{I}^{\frac{1}{2}} z(t)|_{t=0} = \mathcal{D}^{\frac{1}{2}} z(t)|_{t=0} = 0,\ z(1) = z\left(\frac{1}{3}\right). \end{cases} \tag{4.3}$$

By simple calculation we obtain that $\theta^0 = \phi^0 = 0$ and $\theta_\infty = \phi_\infty = \infty$. Thus by Theorem 3.8, the Toppled system (4.3) has at least one positive solution.

Example 4.4. Consider the toppled system of nonlocal boundary value problem given by

$$\begin{cases} \mathcal{D}^{\frac{9}{4}} w(t) + \dfrac{(1+t)[w(t)+z(t)]}{4\widehat{G}_1} = 0,\ t \in (0,1), \\ \mathcal{D}^{\frac{7}{3}} z(t) + \dfrac{(t^4+2t^2+1)[w(t)+z(t)]}{2(t^2+1)\widehat{G}_2} = 0,\ t \in (0,1), \\ \mathcal{I}^{\frac{3}{4}} w(t)|_{t=0} = \mathcal{D}^{\frac{1}{4}} w(t)|_{t=0} = 0,\ w(1) = w\left(\frac{1}{2}\right), \\ \mathcal{I}^{\frac{2}{3}} z(t)|_{t=0} = \mathcal{D}^{\frac{1}{3}} z(t)|_{t=0} = 0,\ z(1) = z\left(\frac{1}{2}\right). \end{cases} \tag{4.4}$$

By simple computation, we can obtain that $\theta_0 = \phi_0 = \infty$ and $\theta_\infty = \phi_\infty = \infty$. Also, for all $(t, w, z) \in J \times J \times J$, we have

$$\theta(t,w,z) \le \frac{(t+1)}{4\widehat{G}_1} = \frac{\widehat{G}_1^{-1}}{2},$$

$$\phi(t,w,z) \le \frac{(t^4+2t^2+1)^2}{4(t^2+1)\widehat{G}_2} = \frac{\widehat{G}_2^{-1}}{2}.$$

Moreover $a = \frac{1}{2},\ b = \frac{1}{2}, r = 1$. Hence by Theorems 3.9 and 3.10, the Toppled system (4.4) has at least two positive solutions (w_1, z_1) and (w_2, z_2) which satisfy

$$0 < \|(w_1, z_1)\| < 1 < \|(w_2, z_2)\|.$$

Example 4.5. To justify Theorem 3.13 and 3.14, we give the following example.

$$\begin{cases} \mathcal{D}^{\frac{5}{2}} w(t) + \dfrac{(w^2(t)+z^2(t))(20+w(t)+z(t))}{w(t)+z(t)+1} = 0,\ t \in (0,1), \\ \mathcal{D}^{\frac{5}{2}} z(t) + \dfrac{(w(t)+z^2(t))(20+w^2(t)+z^2(t)}{w(t)+z(t)+1} = 0,\ t \in (0,1), \\ \mathcal{D}^{\frac{1}{2}} w(t)|_{t=0} = \mathcal{D}^{\frac{3}{2}} w(t)|_{t=0} = 0,\ w(1) = w\left(\frac{1}{5}\right), \\ \mathcal{D}^{\frac{1}{3}} z(t)|_{t=0} = \mathcal{D}^{\frac{4}{3}} z(t)|_{t=0} = 0,\ z(1) = z\left(\frac{1}{4}\right). \end{cases} \tag{4.5}$$

Since $(C_1) - (C_3)$ hold and also

$$\begin{aligned} \theta^0 = \phi^0 = 22,\quad \theta^\infty = \phi^\infty &= 44, \\ 22\|(w,z)\| < \|\theta(t, w(t), z(t))\| &< 44\|(w,z)\|, \end{aligned}$$

$$\begin{aligned}22\|(w,z)\| < \|\theta(t,w(t),z(t))\| &< 44\|(w,z)\|,\\ \|\phi(t,w(t),z(t))\| < 44\|(w,z)\| &< \|(w,z)\|\widehat{G}_1^{-1},\end{aligned}$$

where $\widehat{G}_1 \approx 0.6196$ and $\widehat{G}_2 \approx 0.5196$.

(A_1) Now $\theta(t,w(t),z(t)) < \|(w,z)\|\widehat{G}_1^{-1} \approx 3.1413\|(w,z)\|$ implies that
$\theta(t,w(t),z(t)) < 44\|(w,z)\| \approx 4.6413\|(w,z)\|$ and
$\phi(t,w(t),z(t)) < 44\|(w,z)\| \approx 3.8022\|(w,z)\|$.

Since all the conditions of Theorem 3.13 are satisfied, the Toppled system (4.5) has no positive solution.

(A_2) Also

$$\begin{aligned}\theta(t,w(t),z(t)) > 22\|(w,z)\| &> \|(w,z)\|\left(\gamma_1^2\int_{\frac{1}{4}}^{\frac{3}{4}}\mathcal{H}_1(1,s)ds\right)^{-1}\\ &\approx 71.072415361\|(w,z)\|\end{aligned}$$

and

$$\begin{aligned}\phi(t,w(t),z(t)) > 22\|(w,z)\| &> \|(w,z)\|\left(\gamma_2^2\int_{\frac{1}{4}}^{\frac{3}{4}}\mathcal{H}_2(s,s)ds\right)^{-1}\\ &\approx 84.680\|(w,z)\|.\end{aligned}$$

Thus by Theorem 3.14, the Toppled system (4.5) has no solution.

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