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Basic properties of certain class of non normal operators

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Abstract

Some properties of k-quasi-M-hyponormal are established in this paper. The ascent and an extension of the well-known Fuglede Putnam's Theorem for such operators as well as other related results are also presented, which complete some results given in [7, 12].

Keywords: k-quasi-M-hyponormal operator ; Bishop's property; Dunford's property; polaroid operator.
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1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex separable Hilbert space, and let $B(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$. Denote by $N(T)$ and $R(T)$ respectively, for the null space and the range of an operator T in $B(\mathcal{H})$. Operators $T, S \in B(\mathcal{H})$ are said to be intertwined by an operator $X \in B(\mathcal{H})$ if $TX = XS$. The familiar Fuglede-Putnam's Theorem asserts that if $X \in B(\mathcal{H})$ intertwines two normal operators $T, S \in B(\mathcal{H})$, then X intertwines their adjoints T^* and S^* too. Several extensions of this result for other classes of non normal operators have been studied by other authors, see [3],[5],[8] and [9]. An operator $T \in B(\mathcal{H})$ is said to be dominant if $R(T-\lambda) \subset R(T-\lambda)^*$ for all $\lambda \in \mathbb{C}$, [10], M-hyponormal if there exists $M > 0$ such that $M(T-\lambda)^*(T-\lambda) \geq (T-\lambda)(T-\lambda)^*$ for all $\lambda \in \mathbb{C}$, [2]. A 1-hyponormal operator is hyponormal. The operator $T \in B(\mathcal{H})$ is said to be k-quasi-M-hyponormal for a positive integer k, if there exists $M > 0$ such that

$$T^{*k}(M(T-\lambda)^*(T-\lambda))T^k \geq T^{*k}(T-\lambda)(T-\lambda)^*T^k$$

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for all $\lambda \in \mathbb{C}$, [7, 12, 13].
This definition is equivalent to

$$\left\|\sqrt{M}(T-\lambda)T^k x\right\| \geq \left\|(T-\lambda)^* T^k x\right\|$$

for all $x \in \mathcal{H}$. If $k = 1$, T is said to be quasi-M-hyponormal. Clearly,

$$M\text{-hyponormal} \subset \text{quasi-}M\text{-hyponormal} \subset k\text{-quasi-}M\text{-hyponormal}$$

Example 1.1. *[13] The matrix* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *on* $\mathbb{C}^2$ *is a* 2*-quasi-*M*-hyponormal but not* M*-hyponormal.*

Properties of this class of operators have been presented in [7, 12]. In this paper, we add some complement results. We give the ascent of $T - \lambda$ where T is an element of such class for every complex scalar λ. We present an extension of the well-known Fuglede-Putnam's Theorem for k-quasi-M-hyponormal operators. The SVEP, Bishop's and Dunford's properties are also established.

2. Main Results

Theorem 2.1. *Let* $T \in B(\mathcal{H})$ *be a* k*-quasi-*M*-hyponormal operator. If* T *has dense range, then* T *is* M*-hyponormal.*

Proof. Let $x \in \mathcal{H}$. Since $\overline{R(T)} = \mathcal{H}$, there exists a sequence $(x_n)_n$ in $\mathcal{H}$ such that $x = \lim\limits_{n\to\infty} Tx_n$. By continuity of T, we get

$$\lim_{n\to\infty} T^k x_n = \lim_{n\to\infty} T^{k-1} T x_n = T^{k-1} x$$

Since T is k-quasi-M-hyponormal,

$$\left\|\sqrt{M}(T-\lambda)T^k x_n\right\| \geq \left\|(T-\lambda)^* T^k x_n\right\|$$

for all $\lambda \in \mathbb{C}$. Thus,

$$\begin{aligned}
\left\|\sqrt{M}(T-\lambda)T^{k-1}x\right\| &= \left\|\sqrt{M}\lim_{n\to\infty}(T-\lambda)T^k x_n\right\| \\
&= \lim_{n\to\infty}\left\|\sqrt{M}(T-\lambda)T^k x_n\right\| \\
&\geq \lim_{n\to\infty}\left\|(T-\lambda)^* T^k x_n\right\| \\
&= \left\|\lim_{n\to\infty}(T-\lambda)^* T^k x_n\right\| \\
&= \left\|(T-\lambda)^* T^{k-1} x\right\|
\end{aligned}$$

Hence, T is $(k-1)$-quasi-M-hyponormal. Since T has dense range, T is $(k-2)$-quasi-M-hyponormal. By iteration, T is M-hyponormal. □

Corollary 2.1. *Let* T *be a nonzero* k*-quasi-*M*-hyponormal operator, but not an* M*-hyponormal. Then* T *admits at least, a non trivial closed invariant subspace.*

Proof. Suppose that T has no non trivial closed invariant subspace. Since $T \neq 0$, $N(T) \neq \mathcal{H}$ and $\overline{R(T)} \neq \{0\}$ are closed invariant subspaces for T. Thus, necessarily, $N(T) = \{0\}$ and $\overline{R(T)} = \mathcal{H}$. By Theorem 2.1, T is M-hyponormal operator, which contradicts the hypothesis. □

Definition 2.1. *[1] For $T \in B(\mathcal{H})$, the smallest integer m such that $N(T^m) = N(T^{m+1})$ is said to be the ascent (length of the null chain) of T, and is denoted by $\alpha(T)$. If such integer does not exist, we shall write $\alpha(T) = \infty$.*

Example 2.1. *Since an M-hyponormal operator is dominant, and according to [9, Lemma 2.1], $\alpha(T) = 1$ for an M-hyponormal operator $T \in B(\mathcal{H})$.*

Definition 2.2. *[1] The smallest integer m such that $R(T^m) = R(T^{m+1})$ is said to be the descent (length of the range chain) of T, and is denoted by $\delta(T)$. If no such integer exists, we set $\delta(T) = \infty$.*

According to [1], $\alpha(T) = \delta(T)$ whenever $\alpha(T)$ and $\delta(T)$ are both finite.

In [13], F. Zuo and H. Zuo showed that k-quasi-M-hyponormal operators have finite ascent. Now, we give the value of this ascent for all complex scalar λ.

Theorem 2.2. *Let T be a k-quasi-M-hyponormal operator. Then :*

(1) $N(T^k) = N(T^{k+1})$

(2) $N((T-\lambda)^2) = N(T-\lambda)$*, for all* $\lambda \in \mathbb{C}, \lambda \neq 0$.

Or equivalent, $\alpha(T) = k$ *and* $\alpha(T-\lambda) = 1$,

Proof. (1). It is enough to show that $N(T)^{k+1} \subset N(T)^k$ since clearly $N((T)^k) \subset N(T)^{k+1}$. Let x be in $N(T^{k+1})$. Then $T^{k+1}x = 0$. Since T is k-quasi-M-hyponormal, there exists $M > 0$ such that

$$0 = \left\| \sqrt{M} T^{k+1} x \right\| \geq \left\| T^* T^k x \right\|$$

So, $x \in N(T^*T^k)$. Thus, for all $z \in \mathcal{H}$

$$\left\langle T^* T^k x, z \right\rangle = 0$$

i.e.,

$$\left\langle T^k x, Tz \right\rangle = 0$$

for all $z \in H$. Therefore, $T^k x \in R(T)^\perp$. Since $R\left(T^k\right) \subset R(T)$,

$$T^k x \in R\left(T^k\right)^\perp \cap R\left(T^k\right) = \{0\}$$

and so $x \in N(T^k)$.

(2). Let $x \in N((T-\lambda)^2)$. Since $N(T-\lambda) \subseteq N(T-\lambda)^*$ by [13, Lemma 2.2], $N(T-\lambda)$ reduces $(T-\lambda)$. Hence, according to the decomposition

$$\mathcal{H} = (N(T-\lambda))^\perp \oplus N(T-\lambda)$$

we can write $x = x_1 + x_2$, where $x_1 \in (N(T-\lambda))^\perp$ and $x_2 \in (N(T-\lambda))$. It follows that

$$(T-\lambda)^2 x = 0 = (T-\lambda)^2 x_1 = (T-\lambda)((T-\lambda)x_1)$$

Thus, $(T-\lambda)x_1 \in N(T-\lambda)$ and $(T-\lambda)x_1 \in (N(T-\lambda))^\perp$. Therefore, $(T-\lambda)x_1 = 0$, and then $x_1 \in N(T-\lambda)$. So $x_1 = 0$. Finally, $x = x_2 \in N(T-\lambda)$. □

Definition 2.3. *[1] For an operator $T \in B(\mathcal{H})$ and $x \in \mathcal{H}$, the local resolvent set of T at x denoted by $\rho_T(x)$, is defined to consist of complex elements z_0 such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathcal{H}$, for which $(T-z)f(z) = x$. The set $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ is called the local spectrum of T at x.*

Definition 2.4. *[1] For every subset F of $\mathbb{C}$, we define the local spectral subspace of T by $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$.*

Definition 2.5. *[1] An operator $T \in B(\mathcal{H})$ is said to have Dunford's property (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$.*

Definition 2.6. *[1] An operator $T \in B(\mathcal{H})$ is said to be polaroid, if every isolated point of the spectrum $\sigma(T)$ of T is a pole of the resolvent of T, or equivalent, if $\lambda \in iso\sigma(T)$, then $\alpha(T-\lambda)$ and $\delta(T-\lambda)$ are finite.*

Definition 2.7. *[1] $T \in B(\mathcal{H})$ is said to have Bishop's property (β) if for each open subset $U \subset \mathbb{C}$ and every sequence of analytic functions $f_n \colon U \to \mathcal{H}$ for which $(T-\lambda)f_n(\lambda) \to 0$ as $n \to \infty$ locally uniformly on each compact subset of U, $f_n(\lambda) \to 0$ as $n \to \infty$ again locally uniformly on U.*

Definition 2.8. *[1] An operator T in $B(\mathcal{H})$ is said to have the single valued extension property , briefly SVEP at a complex number α, if for each open neighborhood V of α, the operator $(T-\lambda)$ is one-to-one for all $\lambda \in V$.*

If furthermore, T has SVEP at every $\alpha \in \mathbb{C}$, then T is said to have SVEP.

According to [1],

$$\text{Bishop's property } (\beta) \ \Rightarrow \ \text{Dunford's property } (C) \Rightarrow \text{SVEP} \tag{1}$$

F. Zuo and S. Mecheri in [12] proved that k-quasi-M-hyponormal operators have Bishop's property (β). Using this result, we present in the sequel, an extension of the Fuglede-Putnam's Theorem for such type of operators. We've then

Proposition 2.1. *The Fuglede-Putnam's Theorem holds for k-quasi-M-hyponormal operators T and S^* in $B(\mathcal{H})$.*

Proof. Operators T and S^* are reduced by their eigenspaces according to [13, Theorem 5], polaroid and having Bishop's property by [12]. Thus, our result holds by [6, Theorem 2.4]. □

Lemma 2.1. *[11] Let T in $B(\mathcal{H})$ and S in $B(\mathcal{K})$. Then, the following assertions are equivalent :*

(1) *The pair (T, S) satisfies the Fuglede-Putnam's Theorem.*

(2) *If $TX = XS$ for some X in $B(\mathcal{K}, \mathcal{H})$, then $\overline{R(X)}$ reduces T, $(N(X))^\perp$ reduces S, and the restrictions $T\big|\overline{R(X)}$, $S\,|(N(X))^\perp$ are unitarily equivalent normal operators.*

Corollary 2.2. *Let $T \in B(\mathcal{H})$ be a pure k -quasi-M-hyponormal operator, and let $S^* \in B(\mathcal{H})$ be k-quasi-M-hyponormal. Then, equation $TX = XS$ implies $X = 0$.*

Proof. Equations $TX = XS$ and $T^*X = XS^*$ hold by the previous Proposition. Hence, restrictions $T\left|\overline{R(X)}\right.$, $S\left|(N(X))^\perp\right.$ are unitarily equivalent normal operators by Lemma 2.1. Since T is pure, $X = 0$ necessarily. □

Definition 2.9. *An operator $T \in B(\mathcal{H})$ is said to be bounded below if there exists $c > 0$ such that $\|x\| \leq c\,\|Tx\|$ for all $x \in \mathcal{H}$.*

Note that such operator is one-to-one. We've then

Proposition 2.2. *Let $T \in B(\mathcal{H})$ be a k-quasi-M-hyponormal operator, and let $S \in B(\mathcal{H})$ be such that the pair (T, S) satisfies the Fuglede-Putnam's Theorem. If $X \in B(\mathcal{H})$ intertwines T and S, then :*

- (i) *If X is one-to-one, then S has SVEP.*
- (ii) *If X is an isometry, then S has Dunford's property (C).*
- (iii) *If X is bounded below, then S has Bishop's property (β).*

Proof. Since T has Bishop's property (β) by [7], T has SVEP and Dunford's property (C) by (1). Thus, assertions (ii), (i) and (iii) hold by [4, Theorem 2.8]. □

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