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GEOMETRY OF SEMI-INVARIANT COISOTROPIC SUBMANIFOLDS IN GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study semi-invariant coisotropic submanifolds of a golden semi-Riemannian manifold. We give some necessary and sufficient conditions on integrability of distributions on semi-invariant coisotropic submanifolds of a golden semi-Riemannian manifold. We obtain some geometric results for such submanifolds. Moreover, we give an example.

1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of significant issues of differential geometry. Lightlike submanifolds of semi-Riemannian manifolds were presented by Duggal-Bejancu in [6] and Kupeli in [17]. Şahin and Güneş studied integrability of distributions of CR lightlike submanifolds in [21]. Bahadır and Kılıç introduced lightlike submanifolds of a semi-Riemannian product manifold with quarter symmetric non-metric connection in [4]. The lightlike submanifolds have been studied by many authors in various spaces for example [1, 2, 3, 7, 8, 9, 16].

As a generalization of almost complex and almost contact structures Yano introduced the notion of an $f-$structure which is a (1,1)-tensor field of constant rank on $\tilde{M}$ and satisfies the equality $f^3 + f = 0$ [23]. In its turn, it has been generalized by Goldberg and Yano in [13], who defined a polynomial structure of degree d which is a (1,1)-tensor field f of constant rank on M and satisfies the equation $Q(f) = f_d + a_d f_{d-1} + ... + a_2 f + a_1 I = 0$, where a_1, a_2, ..., a_d are real numbers and I is the identity tensor of type (1,1). As particular cases of polynomial structures Hretcanu and Crasmareanu defined the Golden structure. Being inspired by the Golden proportion the notion of Golden Riemannian manifold $\tilde{M}$ was defined in [5, 14] by a tensor field on $\tilde{M}$ satisfying $\tilde{P}^2 - \tilde{P} - I = 0$. They studied properties of Golden Riemannian manifolds. Moreover, they studied invariant submanifolds of a Riemannian manifold endowed with a golden structure in [14] and they showed

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that a Golden structure induced on every invariant submanifold a Golden structure, too, in [15]. The integrability of golden structures was investigated in [12]. In [18], Özkan studied the complete and horizontal lifts of the golden structure in the tangent bundle. Golden maps between golden Riemannian manifolds were presented by Şahin and Akyol in [22]. Totally umbilical semi-invariant submanifolds of golden Riemannian manifolds were studied in [11] by Erdoğan and Yıldırım. Poyraz and Yaşar introduced lightlike hypersurfaces and lightlike submanifolds of a golden semi-Riemannian manifold in [19] and [20], respectively. Erdoğan worked transversal lightlike submanifolds of metallic semi-Riemannian manifolds in [10].

In this paper, we study semi-invariant coisotropic submanifolds of a golden semi-Riemannian manifold. We give some necessary and sufficient conditions on integrability of distributions on semi-invariant coisotropic submanifolds of a golden semi-Riemannian manifold. We obtain some geometric results for such submanifolds. Moreover, we give an example.

2. Preliminaries

Let $\tilde{M}$ be a differentiable manifold. If a tensor field $\tilde{P}$ of type (1,1) satisfies the following equation, then $\tilde{P}$ is called a golden structure on $\tilde{M}$

$$\tilde{P}^2 = \tilde{P} + I. \tag{2.1}$$

A golden semi-Riemannian structure on $\tilde{M}$ is a pair $(\tilde{g}, \tilde{P})$ with

$$\tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, \tilde{P}Y). \tag{2.2}$$

Then $(\tilde{M}, \tilde{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [18].

Let $(\tilde{M}, \tilde{g}, \tilde{P})$ be a golden semi-Riemannian manifold, then equation (2.2) is equivalent with

$$\tilde{g}(\tilde{P}X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y) + \tilde{g}(X, Y) \tag{2.3}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold with index q, such that $m, n \geq 1$, $1 \leq q \leq m+n+1$ and (M, g) be an $m-$dimensional submanifold of $\tilde{M}$, where g is the induced metric of $\tilde{g}$ on M. If $\tilde{g}$ is degenerate on the tangent bundle TM of M then M is named a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup \left\{ u \in T_x\tilde{M} : \tilde{g}(u, v) = 0, \forall v \in T_xM, x \in M \right\} \tag{2.4}$$

is a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthonormal distributions. Then, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution, named radical distribution, on M of rank $r > 0$ then the submanifold M of $\tilde{M}$ is named an $r-$lightlike submanifold.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S(TM) \perp Rad(TM) \tag{2.5}$$

and $S\left(TM^{\perp}\right)$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to

TM in $T\tilde{M}_{|_M}$ and $Rad(TM)$ in $S\left(TM^{\perp}\right)^{\perp}$, respectively. Thus we have

$$tr\,(TM) = ltr\,(TM)\perp S\left(TM^{\perp}\right), \tag{2.6}$$

$$T\tilde{M}\mid_M = TM \oplus tr\,(TM) = \{Rad(TM) \oplus ltr\,(TM)\}\perp S\,(TM)\perp S(TM^{\perp}). \tag{2.7}$$

Theorem 2.1. *Let* $(M, g, S(TM), S\left(TM^{\perp}\right))$ *be an* $r-$*lightlike submanifold of a semi-Riemannian manifold* $(\tilde{M}, \tilde{g})$. *Suppose* U *is a coordinate neighbourhood of* M *and* $\{\xi_i\}$, $i \in \{1, .., r\}$ *is a basis of* $\Gamma(Rad\,(TM))_{|_U}$. *Then, there exist a complementary vector subbundle* $ltr\,(TM)$ *of* $Rad(TM)$ *in* $S\left(TM^{\perp}\right)^{\perp}$ *and a basis* $\{N_i\}$, $i \in \{1, .., r\}$ *of* $\Gamma(ltr\,(TM))_{|_U}$ *such that*

$$\tilde{g}\,(N_i, \xi_j) = \delta_{ij}, \quad \tilde{g}\,(N_i, N_j) = 0 \tag{2.8}$$

for any $i, j \in \{1, .., r\}$.

We say that a submanifold $(M, g, S(TM), S\left(TM^{\perp}\right))$ of $\tilde{M}$ is
Case 1: $r-$lightlike if $r < min\,\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S\,(\,TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S\,(TM) = \{0\} = S\left(TM^{\perp}\right)$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. Then, according to the decomposition (2.7), the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$
$$\tilde{\nabla}_X V = -A_V X + \nabla^t_X V \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr\,(TM))$, where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr\,(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr\,(TM)$, respectively. Using the projectoins $L : tr\,(TM) \to ltr\,(TM)$ and $S : tr\,(TM) \to S\left(TM^{\perp}\right)$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.11}$$
$$\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.12}$$
$$\tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W) \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr\,(TM))$ and $W \in \Gamma(S\left(TM^{\perp}\right))$, where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\nabla^l_X N, D^l(X, W) \in \Gamma(ltr\,(TM))$, $\nabla^s_X W, D^s(X, N) \in \Gamma(S\left(TM^{\perp}\right))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$. Then, using (2.11)-(2.13) and $\tilde{\nabla}$ metric connection, we derive

$$g(h^s(X, Y), W) + g(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.14}$$
$$g(D^s(X, N), W) = g(A_W X, N). \tag{2.15}$$

Let J be a projection of TM on $S(TM)$. From (2.5) we have

$$\nabla_X JY = \nabla^*_X JY + h^*(X, JY), \tag{2.16}$$
$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X JY, A^*_\xi X\}$ and $\{h^*(X, JY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S\,(TM))$ and $\Gamma(Rad\,(TM))$, respectively.

By using above equations, we obtain

$$\tilde{g}(h^l(X,JY),\xi) = g(A^*_\xi X, JY), \tag{2.18}$$

$$\tilde{g}(h^*(X,JY),N) = g(A_N X, JY), \tag{2.19}$$

$$\tilde{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.20}$$

Generally, the induced connection ∇ on M is not metric connection. Since $\tilde{\nabla}$ is a metric connection, from (2.11) we derive

$$(\nabla_X g)(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y). \tag{2.21}$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$.

3. Semi-invariant Coisotropic Submanifolds of Semi-Riemannian Golden Manifolds

Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, for any $X \in \Gamma(TM)$, we have

$$\tilde{P}X = PX + wX, \tag{3.1}$$

where PX and wX are the tangential and transversal components of $\tilde{P}X$, respectively. Similarly, for any $V \in \Gamma(tr(TM))$, we have

$$\tilde{P}V = BV + CV, \tag{3.2}$$

where BV and CV are the tangential and transversal components of $\tilde{P}X$, respectively.

Definition 3.1. Let $(M, g, S(TM))$ be a lightlike submanifold of golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. If $\tilde{P}(Rad(TM)) \subset S(TM)$, $\tilde{P}(ltr(TM)) \subset S(TM)$ and $\tilde{P}(S(TM^\perp)) \subset S(TM)$ then we call that M is a semi-invariant lightlike submanifold.

If we set $D_1 = \tilde{P}(Rad(TM))$, $D_2 = \tilde{P}(ltr(TM))$ and $D_3 = \tilde{P}(S(TM^\perp))$ then we have

$$S(TM) = D_0 \perp \{D_1 \oplus D_2\} \perp D_3. \tag{3.3}$$

Thus we derive

$$TM = D_0 \perp \{D_1 \oplus D_2\} \perp D_3 \perp Rad(TM), \tag{3.4}$$

$$T\tilde{M} = D_0 \perp \{D_1 \oplus D_2\} \perp D_3 \perp S(TM^\perp) \perp \{Rad(TM) \oplus ltr(TM)\}. \tag{3.5}$$

According to this definition we can write

$$D = D_0 \perp D_1 \perp Rad(TM), \tag{3.6}$$

and

$$D^\perp = D_2 \perp D_3. \tag{3.7}$$

Thus we have

$$TM = D \oplus D^\perp. \tag{3.8}$$

If M is a semi-invariant coisotropic submanifold, we know that $S(TM^{\perp}) = \{0\}$. Then we have

$$S(TM) = \{D_1 \oplus D_2\} \perp D_0, \tag{3.9}$$

$$TM = \{D_1 \oplus D_2\} \perp D_0 \perp Rad(TM), \tag{3.10}$$

$$T\tilde{M} = \{D_1 \oplus D_2\} \perp D_0 \perp \{Rad(TM) \oplus ltr(TM)\}, \tag{3.11}$$

$$TM = D \oplus D_2. \tag{3.12}$$

Proposition 3.2. The distribution D_0 are D are invariant distributions with respect to $\tilde{P}$.

Example 3.3. Let $(\tilde{M} = \mathbb{R}^5_2, \tilde{g})$ be a 5-dimensional semi-Euclidean space with signature $(+,+,-,-,+)$ and $(x_1, x_2, x_3, x_4, x_5)$ be the standard coordinate system of $\mathbb{R}^5_2$. If we define a mapping $\tilde{P}$ by $\tilde{P}(x_1, x_2, x_3, x_4, x_5) = (\phi x_1, \phi x_2, \phi x_3, (1-\phi)x_4, (1-\phi)x_5)$ then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^5_2$. Let M be a submanifold of $\tilde{M}$ given by

$$\begin{aligned}
x_1 &= u_1 + \phi u_2 - \frac{\phi}{2(2+\phi)} u_3 + \sqrt{2}\arctan u_4, x_2 = u_1 + \phi u_2 + \frac{\phi}{2(2+\phi)} u_3, \\
x_3 &= \sqrt{2}u_1 + \sqrt{2}\phi u_2 - \frac{\sqrt{2}\phi}{2(2+\phi)} u_3 + \arctan u_4, x_4 = \phi u_1 - u_2 + \frac{1}{2(2+\phi)} u_3, \\
x_5 &= \phi u_1 - u_2 - \frac{1}{2(2+\phi)} u_3,
\end{aligned}$$

where u_i, $1 \le i \le 4$, are real parameters. Thus $TM = span\{U_1, U_2, U_3, U_4\}$ where

$$\begin{aligned}
U_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_4} + \phi\frac{\partial}{\partial x_5}, \\
U_2 &= \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \sqrt{2}\phi\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}, \\
U_3 &= \frac{1}{2(2+\phi)}(-\phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} - \sqrt{2}\phi\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}), \\
U_4 &= \frac{\sqrt{2}}{1+u_4^2}\frac{\partial}{\partial x_1} + \frac{1}{1+u_4^2}\frac{\partial}{\partial x_3}.
\end{aligned}$$

It is easy to check that M is a $1-$lightlike submanifold and U_1 is a degenerate vector. Then we have $Rad(TM) = span\{U_1\}$ and $S(TM) = span\{U_2, U_3, U_4\}$. By direct calculations we get

$$ltr(TM) = span\{N = -\frac{1}{2(2+\phi)}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_4} - \phi\frac{\partial}{\partial x_5})\}.$$

Furthermore, we can write $D_0 = span\{U_4\}$, $D_1 = span\{U_2\}$, $D_2 = span\{U_3\}$. Thus M is a semi-invariant coisotropic submanifold of $\tilde{M}$.

Theorem 3.4. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the distribution D is integrable iff*

$$h^l(\tilde{P}X, \tilde{P}Y) = h^l(\tilde{P}X, Y) + h^l(X, Y) \tag{3.13}$$

for any $X, Y \in \Gamma(D)$.

Proof. For any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$ the distribution D is integrable iff

$$g([X,Y], \tilde{P}\xi) = 0.$$

Then, from (2.2), (2.3) and (2.11) we obtain

$$g(\left[\tilde{P}X, Y\right], \tilde{P}\xi) = \tilde{g}(h^l(\tilde{P}X, \tilde{P}Y) - h^l(\tilde{P}X, Y) - h^l(X,Y), \xi) \tag{3.14}$$

which completes the proof. □

Theorem 3.5. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then $Rad(TM)$ is integrable iff*
(i) $g(h^*(\xi, \tilde{P}\xi'), N) = g(h^*(\xi', \tilde{P}\xi), N)$,
(ii) $\tilde{g}(h^l(\xi, \tilde{P}\xi'), \xi_1) = \tilde{g}(h^l(\xi', \tilde{P}\xi), \xi_1)$,
(iii) $g(A^*_{\xi}\xi', X) = g(A^*_{\xi'}\xi, X)$
for any $X \in \Gamma(D_0)$, $\xi, \xi', \xi_1 \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$.

Proof. $Rad(TM)$ is integrable iff

$$g([\xi, \xi'], \tilde{P}N) = g([\xi, \xi'], \tilde{P}\xi_1) = g([\xi, \xi'], X) = 0$$

for any $X \in \Gamma(D_0)$, $\xi, \xi', \xi_1 \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$. Then, from (2.2), (2.11), (2.16) and (2.17) we obtain

$$\begin{aligned} g([\xi, \xi'], \tilde{P}N) &= \tilde{g}(\tilde{\nabla}_{\xi}\xi' - \tilde{\nabla}_{\xi'}\xi, \tilde{P}N) = \tilde{g}(\tilde{\nabla}_{\xi}\tilde{P}\xi' - \tilde{\nabla}_{\xi'}\tilde{P}\xi, N) \\ &= \tilde{g}(h^*(\xi, \tilde{P}\xi') - h^*(\xi', \tilde{P}\xi), N), \end{aligned} \tag{3.15}$$

$$\begin{aligned} g([\xi, \xi'], \tilde{P}\xi_1) &= \tilde{g}(\tilde{\nabla}_{\xi}\xi' - \tilde{\nabla}_{\xi'}\xi, \tilde{P}\xi_1) = g(\tilde{\nabla}_{\xi}\tilde{P}\xi' - \tilde{\nabla}_{\xi'}\tilde{P}\xi, \xi_1) \\ &= \tilde{g}(h^l(\xi, \tilde{P}\xi') - h^l(\xi', \tilde{P}\xi), \xi_1), \end{aligned} \tag{3.16}$$

$$g([\xi, \xi'], X) = \tilde{g}(\tilde{\nabla}_{\xi}\xi' - \tilde{\nabla}_{\xi'}\xi, X) = g(A^*_{\xi}\xi' - A^*_{\xi'}\xi, X) = 0. \tag{3.17}$$

This completes the proof. □

Theorem 3.6. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, $\tilde{P}Rad(TM)$ is integrable iff*
(i) $\tilde{g}(h^l(\tilde{P}\xi_1, \xi_2), \xi) = \tilde{g}(h^l(\tilde{P}\xi_2, \xi_1), \xi)$,
(ii) $g(A_N\tilde{P}\xi_1, \tilde{P}\xi_2) = g(A_N\tilde{P}\xi_2, \tilde{P}\xi_1)$,
(iii) $g(A^*_{\xi_1}\tilde{P}\xi_2, \tilde{P}X) = g(A^*_{\xi_2}\tilde{P}\xi_1, \tilde{P}X)$
for any $X \in \Gamma(D_0)$, $\xi_1, \xi_2, \xi \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$.

Proof. $\tilde{P}Rad(TM)$ is integrable iff

$$g(\left[\tilde{P}\xi_1, \tilde{P}\xi_2\right], \tilde{P}\xi) = \tilde{g}(\left[\tilde{P}\xi_1, \tilde{P}\xi_2\right], N) = g(\left[\tilde{P}\xi_1, \tilde{P}\xi_2\right], X) = 0,$$

for any $X \in \Gamma(D_0)$, $\xi_1, \xi_2, \xi \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$. Since $\tilde{\nabla}$ is a metric connection, from (2.2), (2.3), (2.11), (2.12) and (2.17) we get

$$\begin{aligned} g(\left[\tilde{P}\xi_1, \tilde{P}\xi_2\right], \tilde{P}\xi) &= \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\tilde{P}\xi_2 - \tilde{\nabla}_{\tilde{P}\xi_2}\tilde{P}\xi_1, \tilde{P}\xi) \\ &= \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\tilde{P}\xi_2, \tilde{P}\xi) - \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_2}\tilde{P}\xi_1, \tilde{P}\xi) \\ &= \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\tilde{P}\xi_2, \xi) + \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\xi_2, \xi) \\ &\quad -\tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_2}\tilde{P}\xi_1, \xi) - \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_2}\xi_1, \xi) \\ &= \tilde{g}(h^l(\tilde{P}\xi_1, \tilde{P}\xi_2), \xi) + \tilde{g}(h^l(\tilde{P}\xi_1, \xi_2), \xi) \\ &\quad -\tilde{g}(h^l(\tilde{P}\xi_2, \tilde{P}\xi_1), \xi) - \tilde{g}(h^l(\tilde{P}\xi_2, \xi_1), \xi) \\ &= \tilde{g}(h^l(\tilde{P}\xi_1, \xi_2), \xi) - \tilde{g}(h^l(\tilde{P}\xi_2, \xi_1), \xi), \end{aligned} \tag{3.18}$$

$$\begin{aligned} \tilde{g}(\left[\tilde{P}\xi_1, \tilde{P}\xi_2\right], N) &= \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\tilde{P}\xi_2 - \tilde{\nabla}_{\tilde{P}\xi_2}\tilde{P}\xi_1, N) \\ &= \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\tilde{P}\xi_2, N) - \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_2}\tilde{P}\xi_1, N) \\ &= -\tilde{g}(\tilde{P}\xi_2, \tilde{\nabla}_{\tilde{P}\xi_1}N) + \tilde{g}(\tilde{P}\xi_1, \tilde{\nabla}_{\tilde{P}\xi_2}N) \\ &= g(A_N\tilde{P}\xi_1, \tilde{P}\xi_2) - g(A_N\tilde{P}\xi_2, \tilde{P}\xi_1), \end{aligned} \tag{3.19}$$

$$\begin{aligned} \tilde{g}(\left[\tilde{P}\xi_1, \tilde{P}\xi_2\right], X) &= \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\tilde{P}\xi_2 - \tilde{\nabla}_{\tilde{P}\xi_2}\tilde{P}\xi_1, X) \\ &= \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_1}\xi_2, \tilde{P}X) - \tilde{g}(\tilde{\nabla}_{\tilde{P}\xi_2}\xi_1, \tilde{P}X) \\ &= g(A^*_{\xi_1}\tilde{P}\xi_2, \tilde{P}X) - g(A^*_{\xi_2}\tilde{P}\xi_1, \tilde{P}X). \end{aligned} \tag{3.20}$$

Thus the proof is completed. □

Theorem 3.7. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, each leaf of radical distribution is totally geodesic iff*

(i) $A^*_{\xi_2}\xi_1 \in \Gamma(D_1)$,
(ii) $g(h^*(\xi_1, \tilde{P}\xi_2), N) = 0$,
for any $\xi, \xi_1, \xi_2 \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$.

Proof. Radical distribution is totally geodesic iff

$$g(\nabla_{\xi_1}\xi_2, \tilde{P}\xi) = g(\nabla_{\xi_1}\xi_2, X) = g(\nabla_{\xi_1}\xi_2, \tilde{P}N) = 0,$$

for any $X \in \Gamma(D_0)$, $\xi, \xi_1, \xi_2 \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$. Using (2.2), (2.11), (2.16) and (2.17), we have

$$g(\nabla_{\xi_1}\xi_2, \tilde{P}\xi) = -g(A^*_{\xi_2}\xi_1, \tilde{P}\xi), \tag{3.21}$$

$$g(\nabla_{\xi_1}\xi_2, X) = -g(A^*_{\xi_2}\xi_1, X), \tag{3.22}$$

$$g(\nabla_{\xi_1}\xi_2, \tilde{P}N) = g(h^*(\xi_1, \tilde{P}\xi_2), N). \tag{3.23}$$

From (3.21)-(3.23) the proof is completed. □

Definition 3.8. A semi-invariant submanifold M of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ is said to be D-totally geodesic (resp. D_2-totally geodesic) if its the second fundamental form h satisfies $h(X, Y) = 0$ (resp. $h(Z, W) = 0$), for any $X, Y \in \Gamma(D)$, $(Z, W \in \Gamma(D_2))$.

Theorem 3.9. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is D-totally geodesic submanifold iff for any $X \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$, $A^*_\xi X$ has no component in $\Gamma(D_0 \perp D_2)$.*

Proof. Since $\tilde{\nabla}$ is a metric connection, from (2.9) and (2.17) we obtain

$$\tilde{g}(h(X, \tilde{P}Y), \xi) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, \xi) = -\tilde{g}(\tilde{\nabla}_X \xi, \tilde{P}Y) = \tilde{g}(A^*_\xi X, \tilde{P}Y) \tag{3.24}$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$. Thus using (3.24) we derive $h(X, \tilde{P}Y) = 0$ iff $A^*_\xi X$ has no component in $\Gamma(D_0 \perp D_2)$. □

Theorem 3.10. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is D_2-totally geodesic submanifold iff $A^*_\xi Y$ has no component in $\Gamma(D_1)$, for any $Y \in \Gamma(D_2)$, $\xi \in \Gamma(Rad(TM))$.*

Proof. Since $\tilde{\nabla}$ is a metric connection, from (2.9) and (2.17) we derive

$$\tilde{g}(h(Y, Z), \xi) = \tilde{g}(\tilde{\nabla}_Y Z, \xi) = -\tilde{g}(\tilde{\nabla}_Y \xi, Z) = \tilde{g}(A^*_\xi Y, Z) \tag{3.25}$$

for any $Y, Z \in \Gamma(D_2)$, $\xi \in \Gamma(Rad(TM))$. Thus from the equations (3.25) we conclude $h(Y, Z) = 0$ iff $A^*_\xi Y$ has no component in $\Gamma(D_1)$. □

Definition 3.11. Let M be a proper semi-invariant r-lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. M is said to be mixed-geodesic submanifold if the second fundamental form of M satisfies $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D_2)$.

Theorem 3.12. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the following statements are equivalent:*

i) M is mixed geodesic,
ii) $A_V X$ has no component in D_2,
iii) $A^*_\xi X$ has no component in D_1,
iv) $\nabla^*_Y \tilde{P}\xi \in \Gamma(D_1)$
for any $X \in \Gamma(D)$, $Y \in \Gamma(D_2)$, $\xi \in \Gamma(Rad(TM))$, $V \in \Gamma(tr(TM))$.

Proof. M is mixed geodesic iff

$$g(h(X, Y), \xi) = 0,$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D_2)$, $\xi \in \Gamma(Rad(TM))$. Choosing $Y \in \Gamma(D_2)$, there is a vector field $V \in \Gamma(tr(TM))$ such that $Y = \tilde{P}V$. Using (2.2), (2.9) and (2.10) we have

$$\begin{aligned} \tilde{g}(h(X, Y), \xi) &= \tilde{g}(\tilde{\nabla}_X Y, \xi) = \tilde{g}(\tilde{\nabla}_X \tilde{P}V, \xi) \\ &= \tilde{g}(\tilde{\nabla}_X V, \tilde{P}\xi) = -g(A_V X, \tilde{P}\xi). \end{aligned} \tag{3.26}$$

Thus we derive $(i) \Longleftrightarrow (ii)$. Since $\tilde{\nabla}$ is a metric connection, from (2.9) and (2.17) we get

$$\tilde{g}(h(X, Y), \xi) = g(Y, A^*_\xi X). \tag{3.27}$$

Hence we obtain $(i) \Longleftrightarrow (iii)$. Using (2.2) and (2.9) and the fact that $\tilde{\nabla}$ is a metric connection, we derive

$$\begin{aligned} \tilde{g}(h(\tilde{P}X,Y),\xi) &= \tilde{g}(h(Y,\tilde{P}X),\xi) = \tilde{g}(\tilde{\nabla}_Y\tilde{P}X,\xi) = -\tilde{g}(\tilde{P}X,\tilde{\nabla}_Y\xi) \\ &= -\tilde{g}(X,\tilde{\nabla}_Y\tilde{P}\xi) = -g(X,\nabla^*_Y\tilde{P}\xi). \end{aligned} \tag{3.28}$$

Thus we derive $(i) \Longleftrightarrow (iv)$. □

Theorem 3.13. *Let $(M, g, S(TM))$ be a semi-invariant coisotropic submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D_2 is parallel on M iff $A_V X \in \Gamma(D_2)$ for any $X \in \Gamma(D_2)$, $V \in \Gamma(tr(TM))$.*

Proof. For any $X, Y \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $Z \in \Gamma(D_0)$, The distribution D_2 is parallel on M iff

$$\tilde{g}(\nabla_X Y, N) = g(\nabla_X Y, \tilde{P}N) = g(\nabla_X Y, Z) = 0.$$

Choosing $Y \in \Gamma(D_2)$, there is a vector field $V \in \Gamma(tr(TM))$ such that $Y = \tilde{P}V$ and from (2.2), (2.3), (2.9) and (2.10) we have

$$\tilde{g}(\nabla_X Y, N) = \tilde{g}(\tilde{\nabla}_X\tilde{P}V, N) = \tilde{g}(\tilde{\nabla}_X V, \tilde{P}N) = -g(A_V X, \tilde{P}N), \tag{3.29}$$

$$\begin{aligned} g(\nabla_X Y, \tilde{P}N) &= \tilde{g}(\tilde{\nabla}_X\tilde{P}V, \tilde{P}N) = \tilde{g}(\tilde{\nabla}_X V, \tilde{P}N) + \tilde{g}(\tilde{\nabla}_X V, N) \\ &= -g(A_V X, \tilde{P}N) - g(A_V X, N), \end{aligned} \tag{3.30}$$

$$g(\nabla_X Y, Z) = \tilde{g}(\tilde{\nabla}_X\tilde{P}V, Z) = \tilde{g}(\tilde{\nabla}_X V, \tilde{P}Z) = -g(A_V X, \tilde{P}Z). \tag{3.31}$$

From (3.29) $A_V X$ has no component D_1, for any $X \in \Gamma(D_2)$. Thus (3.30) and (3.31) $A_V X$ has no component $Rad(TM)$ and D_0, which completes the proof. □

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