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Double Laplace Decomposition Method and Exact Solutions of Hirota, Schrödinger and Complex mKdV Equations

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Abstract

In this paper, a powerful method, named as the double Laplace decomposition method, is used to obtain exact solutions of nonlinear partial differential equations subject to initial conditions. We especially interested in Hirota, Schrödinger and complex modified KdV equations with their initial conditions. The double Laplace deceomposition method is applied to these equations. We then gain complex-valued solutions, yield the given initial conditions. Moreover, we give some nonlinear partial equations to demonstrate that this method effective, useful, and powerful tool for getting real-valued functions.

Keywords: *Double Laplace transform, decomposition method, exact solution, Hirota equation, Schrödinger equation, complex mKdV equation.*
2010 Mathematics Subject Classification: *35C07, 35C09*

1. Introduction

In modern science and engineering, a great deal of scientific events and engineering problems can be modelled by linear or nonlinear partial differential equations (LPDEs, NLPDEs). In nature, N/LPDEs may not be considered without being exposed to any forces and some conditions. Scientists mostly focus on N/LPDEs which are subject to initial conditions. It is therefore important to gain the solutions of such N/LPDEs. For this aim, several analytical and numerical methods have been established until now. The perturbation method [1]-[4], the homotopy perturbation method [4]-[5], the Adomian decomposition method [7]-[11], the modified decomposition method [7], [12]-[15], the Laplace decomposition method [7], [15], [16]-[18], the double Laplace decomposition method [20]-[25], and others. Among these methods, we utilize the double Laplace decomposition method, combines the double Laplace transform and Adomian decomposition method to find solutions for NLPDEs with initial values.
The Laplace transform has attracted a great deal of attention and many applications in modern science and engineering. This transform is mostly used for one variable function, $f(x)$. For a function of two variables, $f(x,t)$, the double Laplace transform is more convenient and suitable. There are numerous applications for the Laplace transform, but there are insufficient work on the double Laplace transform. In the literature, we see some applications. In 2011, some significant theorems on two dimensional Laplace transform are proposed by Aghilli and Moghaddam[19], and they applied the suggested method to nonhomogeneous parabolic partial differential equations. In 2012, Elzaki [22] combined double Laplace transform and modified variational iteration method, and solved nonlinear convolution partial differential equations by the proposed method. Eltayeb and Kilicman [23] used the double Laplace transform to solve some differential equations and integro-differential equations in 2013. Debnath [24] paid his attention to the properties and convolution theorem for the double Laplace transform in 2016. Dhunde and Waghmare [25] applied double Laplace transform technique in order to solve partial integro-differential equations. In these applications, it is clearly seen that the double Laplace decomposition method is powerful one to obtain solutions of real-valued functions.
Here we give some information about Hirota, Schrödinger, and complex mKdv equations, hence this work mainly focuses on these NLPDEs. The well-known Hirota equation [26] is given by

$$iu_t+u_{xx}+2|u|^2u+iau_{xxx}+6ia|u|^2u_x=0, \tag{1.1}$$

where $u(x,t)$ is the complex amplitude of slowly changing optical field, the subscripts t and x represent the temporal and spatial partial derivatives, respectively, and α is a small parameter. The equation (1.1) describes the propagation of femtosecond soliton pulse in the single mode fibers. u_{xx}, $|u|^2u$, u_{xxx}, and $|u|^2u_x$ demonstrate the group velocity dispersion, self phase modulation, third order dispersion, and self

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steepening, respectively [27]. Hirota equation plays significant role in modern science and therfore is of many applications in the literature, see [28]-[32].
The Schrödinger equation, another famous mathematical and physical equation, is derived from the equation (1.1). For $\alpha = 0$, the equation (1.1) gives the Scrödinger equation [33] as follows:

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{1.2}$$

Here $u(x,t)$ is a complex function of x and t. The equation (1.2) defines the propagation of pulses in single mode fibers in the condition of ignoring fiber loss. It also characterizes the evolution of the evelope of modulated nonlinear wave groups. And also, it is noticed in nonlinear wave propagation in dispersive and inhomogeneous media. Furthermore, it has significant roles in several areas of physics including water waves, nonlinear optics, plasma physics, quantum mechanics, and so on, see [34], [35]. Because of its importance in these areas, there are large number of works on obtainin the exact and approximate analytical solutions to Schrödinger equation, such as [36],[37].
In addition to Schrödinger equation, removing the terms of group velocity dispersion and self phase modulation from the equation (1.1) grants the complex modified KdV equation(shortly, cmKdV). The equation reads

$$u_t + \alpha u_{xxx} + 6\alpha |u|^2 u_x = 0, \tag{1.3}$$

which covers the dynamics for the amplitude of wave packet [38]. Here $u(x,t)$ is a complex-valued function of x and t. The cmKdV equation (1.3) is the theoretical model for propagation of the nametic optical fibers [39]. It also has applications in the propagation of transverse magnetic waves and few-cycle optical pulses [40]. Our main intent is to demonstrate that the double Laplace decomposition method is impressive, efficient, and fruitful for solving NLPDEs subject to the initial conditions. Therefore, we utulize this method to obtain the solutions of Hirota equation, Schrödinger equation, and complex mKdV equation, whose solutions are complex-valued functions. The application of this method to these equations indicate that the double Laplace decomposition method is impressive tool in order to get solutions for complex-valued functions. To exemplify usefullness of this method for real-valued functions, we aslo put forward some applications.
This work is prepared as follows. In section 2, we give some informations about double Laplace transform. We then highlights the double Laplace decomposition method in section 3. We obtain the solutions of Hirota equation, Schrödinger equation, complex mKdV equation, and two more equations subject to initial conditions in section 4. Finally, we give some conclusions in section 5.

2. Some Notes On Double Laplace Transform

Let us consider $f(x,t)$, a function of two varibale x and t. The double Laplace transform of $f(x,t)$ is defined by the following double integral:

$$L_x L_t[f(x,t)] = F(p,s) = \int_0^\infty \int_0^\infty e^{-px-st} f(x,t)dtdx, \tag{2.1}$$

whenever this integral exists. Here $x,t \geq 0$ and p, s are complex numbers [41].
Let α and β be sufficiently large constants. The inverse double Laplace transform $L_x^{-1}L_t^{-1}[F(p,s)] = f(x,t)$ is defined by

$$f(x,t) = L_x^{-1}L_t^{-1}[F(p,s)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}dp \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}F(p,s)ds \tag{2.2}$$

where $F(p,s)$ must be an analytic function for all p and s in the region defined by the inequalities $Rep \leq c$ and $Res \leq d$.

Definition 2.1. *A function $f(x,t)$ is said to be of exponential order $a > 0$ and $b > 0$ on $0 \leq x < \infty, 0 \leq t < \infty$, if there exists a positive constant K such that $|f(x,y)| \leq Ke^{ax+by}$.*

Theorem 2.2. *If a function $f(x,t)$, continous in $(0,X)$ and $(0,T)$, is of exponential order $exp(ax+bt)$, then the double Laplace transform of $f(x,t)$ exists whenever $Rep > a$ and $Req > b$.*

Proof. The proof of this theorem is given in [42]. □

Because of this fact that all functions are supposed to be of exponential order in this paper.

Definition 2.3. *Let $f(x,t)$ and $g(x,t)$ be continous functions for $x,t \leq 0$ and of exponential order. Then, the double convolution of the functions $f(x,t)$ and $g(x,t)$ is defined by*

$$f(x,t) **g(x,t) = \int_0^t \int_0^x f(x-\eta, t-\zeta)g(\eta,\zeta)d\eta d\zeta. \tag{2.3}$$

Theorem 2.4. *Suppose that $f(x,t)$ and $g(x,t)$ have double Laplace transforms say, $F(p,s)$ and $G(p,s)$, respectively. The double Laplace transform of the convolution of $f(x,t)$ and $g(x,t)$ is*

$$L_x L_t[f(x,t) **g(x,t)] = F(p,s)G(p,s). \tag{2.4}$$

Proof. Firstly, the double Laplace transform is applied to the convolution $f(x,t) **g(x,t)$. Then, we have

$$L_x L_t[f(x,t) **g(x,t)] = \int_0^\infty \int_0^\infty e^{-px-st}(f(x,t) **g(x,t))dxdt. \tag{2.5}$$

By the definition of the convolution, we get

$$L_xL_t[f(x,t)**g(x,t)] = \int_0^\infty \int_0^\infty e^{-px-st}(\int_0^t \int_0^x f(x-\eta,t-\zeta)g(\eta,\zeta)d\eta d\zeta)dxdt. \tag{2.6}$$

For simplicity, we use the notations $\xi = x-\eta$ and $\mu = t-\zeta$. After that, the integral turns into

$$L_xL_t[f(x,t)**g(x,t)] = (\int_0^\infty \int_0^\infty e^{-p\eta-s\zeta} f(\eta,\zeta)d\eta d\zeta)(\int_0^\infty \int_0^\infty e^{-p\xi-\mu} f(\xi,\mu)d\xi d\mu), \tag{2.7}$$

where $\eta,\zeta,\xi,and\mu \geq 0$. From this equality, it is easily seen that

$$L_xL_t[f(x,t)**g(x,t)] = F(p,s)G(p,s). \tag{2.8}$$

□

Theorem 2.5. *Let $f(x,t)$ and $g(x,t)$ be continuous functions defined for $x,t \geq 0$ and having double Laplace transforms, $F(p,s)$, and $G(p,s)$, respectively. If $F(p,s) = G(p,s)$, then $f(x,t) = g(x,t)$.*

Proof. Let α and β be sufficiently large constants. Then, $f(x,t)$ can be written as

$$f(x,t) = L_x^{-1}L_t^{-1}[F(p,s)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}dp \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}F(p,s)ds. \tag{2.9}$$

Putting the given condition that $F(p,s) = G(p,s)$ into equation (2.9) yields

$$f(x,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}dp \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}G(p,s)ds = L_x^{-1}L_t^{-1}[G(p,s)] = g(x,t). \tag{2.10}$$

□

Here we give some fundamental properties of double Laplace transform and inverse double Laplace transform.
Let a, b, and c be constants.

1. $L_xL_t[c] = \frac{c}{ps}$.

2. $L_xL_t[e^{ax+bt}] = \frac{1}{(p-a)(s-b)}$.

3. $L_xL_t[e^{i(ax+bt)})] = \frac{(ps-ab)+i(as+bp)}{(p^2+a^2)(s^2+b^2)}$.

4. $L_xL_t[cos(ax+bt)] = \frac{ps-ab}{(p^2+a^2)(s^2+b^2)}$.

5. $L_xL_t[sin(ax+bt)] = \frac{as+bp}{(p^2+a^2)(s^2+b^2)}$.

6.$L_xL_t[x^m t^n] = \frac{m!n!}{p^{m+1}s^{n+1}}$ where m and n are positive integers.

7.If $f(x,t) = h(x)g(t)$, then $L_xL_t[f(x,t)] = L_xL_t[h(x)]L_xL_t[g(t)]$.

8.$L_xL_t[e^{ax+bt}f(x,t)] = F[p-a,s-b]$.

9.$L_xL_t[f(ax,bt)] = \frac{1}{ab}F[\frac{p}{a},\frac{s}{b}]$.

10.$L_xL_t[.]$ and $L_x^{-1}L_t^{-1}[.]$ are linear transformations, that is,

$$L_xL_t[c_1f_1(x,t)+c_2f_2(x,t)] = c_1L_xL_t[f_1(x,t)]+c_2L_xL_t[f_2(x,t)], \tag{2.11}$$

and

$$L_x^{-1}L_t^{-1}[c_1F_1(p,s)+c_2F_2(p,s)] = c_1L_x^{-1}L_t^{-1}[F_1(p,s)]+c_2L_x^{-1}L_t^{-1}[F_2(p,s)], \tag{2.12}$$

where c_1 and c_2 are arbitrary constants.
Now we introduce the general formulas for the double Laplace transform of a function $f(x,t)$ with any integer order partial derivatives w.r.t x and t as follows:

$$L_xL_t[\frac{\partial^n f(x,t)}{\partial x^n}] = p^nF(p,s) - \sum_{i=0}^{n-1} p^{n-1-i}L_t[\frac{\partial^i f(0,t)}{\partial x^i}], \tag{2.13}$$

and

$$L_xL_t[\frac{\partial^m f(x,t)}{\partial t^m}] = s^mF(p,s) - \sum_{j=0}^{m-1} s^{m-1-j}L_x[\frac{\partial^j f(x,0)}{\partial t^j}]. \tag{2.14}$$

For the first and second order partial derivatives, we have

$$L_xL_t[\frac{\partial f(x,t)}{\partial x}] = pF(p,s)-F(0,s), \quad L_xL_t[\frac{\partial^2 f(x,t)}{\partial x^2}] = p^2F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x}, \tag{2.15}$$

$$L_xL_t[\frac{\partial f(x,t)}{\partial t}] = sF(p,s)-F(p,0), \quad L_xL_t[\frac{\partial^2 f(x,t)}{\partial t^2}] = s^2F(p,s)-sF(p,0)-\frac{\partial F(p,0)}{\partial t}. \tag{2.16}$$

3. Outline of The Method

This method is described as in the following manner. Let us consider the nonlinear nonhomogeneous partial differential equation in operator form

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=h(x,t) \tag{3.1}$$

with initial conditions $u(0,t)=f(t)$ and $u_x(0,t)=g(t)$. Here L is a second order partial differential operator with respect to x, R is a remaining linear operator, N represents a general nonlinear differantial operator, and $h(x,t)$ is a source term.
At the beginning of this method, the double Laplace transform is applied to both sides of the equation (3.1). Then we have

$$L_xL_t[Lu(x,t)+Ru(x,t)+Nu(x,t)]=L_xL_t[h(x,t)]. \tag{3.2}$$

Using the linearity and the differentiation properties of the double Laplace transform yields

$$U(p,s)=\frac{F(s)}{p}+\frac{G(s)}{p^2}+\frac{1}{p^2}L_xL_t[h(x,t)]-\frac{1}{p^2}[L_xL_t[Ru(x,t)]+L_xL_t[Nu(x,t)]] \tag{3.3}$$

where $U(p,s)$, $F(s)$, and $G(s)$ represents the double Laplace transforms of $u(x,t)$, $f(t)$, and $g(t)$, respectively.
After this step, we use the following decomposition series

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\dots \tag{3.4}$$

for the linear terms. And also, the infinite series defined by

$$N(u(x,t))=\sum_{n=0}^{\infty}A_n(u(x,t)), \tag{3.5}$$

is used for the nonlinear terms. Here A_n represents the Adomian polynomials, described by

$$A_n=\frac{1}{n!}\frac{d^n}{d\alpha^n}\left[N(\sum_{i=0}^{\infty}\alpha^i u_i)\right]_{\alpha=0},n=0,1,2,\dots \tag{3.6}$$

From this definition, we get the first terms as below:

$$A_0=N(u_0),\quad A_1=u_1N'(u_0),\quad A_2=u_2N'(u_0)+\frac{1}{2!}u_1^2N''(u_0). \tag{3.7}$$

Now we substitute (3.4) and (3.5) into the equation (3.3), and afterwards we get

$$L_xL_t[\sum_{n=0}^{\infty}u_n(x,t)]=\frac{F(s)}{p}+\frac{G(s)}{p^2}+\frac{1}{p^2}L_xL_t[h(x,t)]-\frac{1}{p^2}[L_xL_t[R[\sum_{n=0}^{\infty}u_n(x,t)]]+L_xL_t[\sum_{n=0}^{\infty}A_n]]. \tag{3.8}$$

The inverse double Laplace transform is applied to both sides of the equation (3.8), and by the linearity of the inverse transform, we obtain

$$\sum_{n=0}^{\infty}u_n(x,t)=f(t)+xg(t)+L_x^{-1}L_t^{-1}[\frac{1}{p^2}L_xL_t[h(x,t)]]-L_x^{-1}L_t^{-1}[\frac{1}{p^2}[L_xL_t[R[\sum_{n=0}^{\infty}u_n(x,t)]]+L_xL_t[\sum_{n=0}^{\infty}A_n]]]. \tag{3.9}$$

Comparing both sides of the equation (3.9) yields the following equalities:

$$u_0(x,t)=f(t)+xg(t)+L_x^{-1}L_t^{-1}[\frac{1}{p^2}L_xL_t[h(x,t)]], \tag{3.10}$$

$$u_1(x,t)=-L_x^{-1}L_t^{-1}[\frac{1}{p^2}[L_xL_t[R[u_0(x,t)]+L_xL_t[A_0]]], \tag{3.11}$$

$$u_2(x,t)=-L_x^{-1}L_t^{-1}[\frac{1}{p^2}[L_xL_t[R[u_1(x,t)]+L_xL_t[A_1]]]. \tag{3.12}$$

The general form of the recursive relation is given by

$$u_{n+1}=-L_x^{-1}L_t^{-1}[\frac{1}{p^2}[L_xL_t[R[u_n(x,t)]+L_xL_t[A_n]]],\quad n\geq 0. \tag{3.13}$$

Obtaining the components u_0, u_1, u_2, ... from the above recursive relation and putting them into the expansion (3.4) provide us with the solution $u(x,t)$.

4. Applications of The Method

4.1. Solving Hirota Equation

We consider the nonhomogeneous Hirota equation given by

$$iu_t + u_{xx} + 2|u|^2 u + i\alpha u_{xxx} + 6i\alpha|u|^2 u_x = xe^{it} + 6i\alpha x^2 e^{it} + 2x^3 e^{it}, \tag{4.1}$$

with the initial conditions $u(0,t) = 0$ and $u_x(0,t) = e^{it}$. The equation (4.1) can be written as

$$u_{xx} = xe^{it} + 6i\alpha x^2 e^{it} + 2x^3 e^{it} - iu_t - 2|u|^2 u - i\alpha u_{xxx} - 6i\alpha|u|^2 u_x. \tag{4.2}$$

We first apply the double Laplace transform to both sides of the equation (4.2). By the properties of the double Laplace transform we have

$$L_x L_t[u(x,t)] = U(p,s) = xe^{it} + \frac{1}{p^2} L_x L_t[xe^{it} + 6i\alpha x^2 e^{it} + 2x^3 e^{it}] - \frac{1}{p^2}[L_x L_t[iu_t + i\alpha u_{xxx}] + L_x L_t[2|u|^2 u + 6i\alpha|u|^2 u_x]]. \tag{4.3}$$

Here we use the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{4.4}$$

for the linear terms, and

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(u(x,t)), \tag{4.5}$$

for the nonlinear terms. Putting these into the equation (4.3) gives

$$L_x L_t[\sum_{n=0}^{\infty} u_n(x,t)] = \frac{1}{p^2(s-i)} + \frac{1}{p^2} L_x L_t[xe^{it} + 6i\alpha x^2 e^{it} + 2x^3 e^{it}] - \frac{1}{p^2}[L_x L_t[R[\sum_{n=0}^{\infty} u_n(x,t)]] + L_x L_t[\sum_{n=0}^{\infty} A_n]], \tag{4.6}$$

where $R[u] = iu_t + i\alpha u_{xxx}$ and $A_n[u] = 2|u|^2 u + 6i\alpha|u|^2 u_x$. Then, taking the inverse double Laplace transform of the equation (4.6) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = xe^{it} + L_x^{-1} L_t^{-1}[\frac{1}{p^2} L_x L_t[xe^{it} + 6i\alpha x^2 e^{it} + 2x^3 e^{it}]] - L_x^{-1} L_t^{-1}[\frac{1}{p^2}[L_x L_t[R[\sum_{n=0}^{\infty} u_n(x,t)]] + L_x L_t[\sum_{n=0}^{\infty} A_n]]]. \tag{4.7}$$

From the equation (4.7), we obtain the recursive relation:

$$u_0(x,t) = xe^{it} + L_x^{-1} L_t^{-1}[\frac{1}{p^2} L_x L_t[xe^{it} + 6i\alpha x^2 e^{it} + 2x^3 e^{it}]], \tag{4.8}$$

$$u_1(x,t) = -L_x^{-1} L_t^{-1}[\frac{1}{p^2}[L_x L_t[iu_{0t} + iu_{0xxx}] + L_x L_t[A_0]]], \tag{4.9}$$

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$$u_{n+1} = -L_x^{-1} L_t^{-1}[\frac{1}{p^2}[L_x L_t[R[u_n(x,t)] + L_x L_t[A_n]]], \quad n \geq 0. \tag{4.10}$$

Eventually, we obtain

$$u_0(x,t) = xe^{it} - \frac{1}{6} e^{it} x^3 + \frac{1}{2} i\alpha e^{3it} x^4 + \frac{1}{10} e^{3it} x^5, \tag{4.11}$$

$$u_1(x,t) = \frac{1}{2} ie^{it} x^2 + 2ae^{3it} x^3 + \frac{1}{6} e^{it} x^3 - \frac{1}{2} iae^{3it} x^4 - \frac{1}{2} ie^{3it} x^4 - \frac{1}{120} e^{it} x^5 - \frac{1}{10} e^{3it} x^5 + \frac{13}{60} iae^{3it} x^6 + \frac{13}{420} e^{3it} x^7 + \frac{3}{7} a^2 e^{5it} x^7 \tag{4.12}$$

$$- \tfrac{1}{48} iae^{3it} x^8 - \tfrac{9}{70} iae^{5it} x^8 - \tfrac{1}{432} e^{3it} x^9 - \tfrac{1}{120} e^{5it} x^9 - \tfrac{1}{9} a^2 e^{5it} x^9 + \tfrac{iae^{3it}}{1080} x^{10} + \tfrac{7}{225} iae^{5it} x^{10} + \tfrac{3}{20} ia^3 e^{7it} x^{10} + \tfrac{1}{132} a^2 e^{5it} x^{11}$$

$$+ \tfrac{3}{44} a^2 e^{7it} x^{11} + \tfrac{e^{3it}}{11880} x^{11} + \tfrac{1}{550} e^{5it} x^{11} - \tfrac{1}{495} iae^{5it} x^{12} - \tfrac{21iae^{7it}}{2200} x^{12} - \tfrac{1}{48} ia^3 e^{7it} x^{12} - \tfrac{e^{7it}}{2600} x^{13} - \tfrac{e^{5it}}{9360} x^{13} - \tfrac{1}{52} a^4 e^{9it} x^{13} - \tfrac{29a^2 e^{7it}}{3120} x^{13}$$

$$+ \tfrac{11}{910} ia^3 e^{9it} x^{14} + \tfrac{23iae^{7it}}{18200} x^{14} + \tfrac{19a^2 e^{9it}}{7000} x^{15} + \tfrac{e^{7it}}{21000} x^{15} - \tfrac{iae^{9it}}{4000} x^{16} - \tfrac{e^{9it}}{136000} x^{17}$$

From u_0 and u_1, we get the noise terms as $-\frac{1}{6} e^{it} x^3$, $\frac{1}{2} i\alpha e^{3it} x^4$, and $\frac{1}{10} e^{3it} x^5$. Deleting these terms from the first component u_0 gives the desired solution:

$$u(x,t) = xe^{it}. \tag{4.13}$$

4.2. Solving Schrödinger Equation

We consider the following nonhomogeneous Schrödinger equation

$$iu_t + u_{xx} + 2|u|^2u = 2it^2 - 2x^2t - 2ix^6t^6, \tag{4.14}$$

with the initial conditions $u(0,t) = 0$ and $u_x(0,t) = 0$. We can rewrite the equation (4.14) as follows:

$$u_{xx} = 2it^2 - 2x^2t - 2ix^6t^6 - iu_t - 2|u|^2u. \tag{4.15}$$

Applying the double Laplace transform to both sides of the equation (4.15) and using the properties of the double Laplace transform gives

$$L_xL_t[u(x,t)] = U(p,s) = \frac{1}{p^2}L_xL_t[2it^2 - 2x^2t - 2ix^6t^6] - \frac{1}{p^2}[L_xL_t[iu_t] + L_xL_t[2|u|^2u]]. \tag{4.16}$$

Here the expansions

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{4.17}$$

and

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(u(x,t)), \tag{4.18}$$

are used for the linear and nonlinear terms, respectively. We put these expansions into the equation (4.16). Then we get

$$L_xL_t[\sum_{n=0}^{\infty} u_n(x,t)] = \frac{1}{p^2}L_xL_t[2it^2 - 2x^2t - 2ix^6t^6] - \frac{1}{p^2}[L_xL_t[R[\sum_{n=0}^{\infty} u_n(x,t)]] + L_xL_t[\sum_{n=0}^{\infty} A_n]], \tag{4.19}$$

where $R[u] = iu_t$ and $A_n[u] = 2|u|^2u$. Applying the inverse double Laplace transform ot both sides of the equation (4.19) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = L_x^{-1}L_t^{-1}[\frac{1}{p^2}L_xL_t[2it^2 - 2x^2t - 2ix^6t^6] - L_x^{-1}L_t^{-1}[\frac{1}{p^2}[L_xL_t[R[\sum_{n=0}^{\infty} u_n(x,t)]] + L_xL_t[\sum_{n=0}^{\infty} A_n]]]. \tag{4.20}$$

If we compared the both sides of the equation (4.20), then we get the recursive relation as below:

$$u_0(x,t) = L_x^{-1}L_t^{-1}[\frac{1}{p^2}L_xL_t[2it^2 - 2x^2t - 2ix^6t^6]], \tag{4.21}$$

$$u_1(x,t) = -L_x^{-1}L_t^{-1}[\frac{1}{p^2}[L_xL_t[iu_{0t}] + L_xL_t[A_0]]], \tag{4.22}$$

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$$u_{n+1} = -L_x^{-1}L_t^{-1}[\frac{1}{p^2}[L_xL_t[R[u_n(x,t)] + L_xL_t[A_n]]], \quad n \geq 0. \tag{4.23}$$

We therefore obtain

$$u_0(x,t) = ix^2t^2 - \frac{1}{6}x^4t - \frac{1}{28}ix^8t^6 \tag{4.24}$$

$$u_1(x,t) = \frac{1}{6}x^4t + \frac{1}{180}ix^6 + \frac{1}{28}ix^8t^6 - \frac{17}{1260}x^{10}t^5 - \frac{1}{792}ix^{12}t^4 + \frac{1}{19656}x^{14}t^3 - \frac{3}{2548}x^{14}t^{10} + \frac{17}{3360}x^{16}t^9 + \frac{1}{51408}ix^{18}t^8 \tag{4.25}$$

$$+\frac{3}{148960}ix^{20}t^{14} - \frac{1}{362208}x^{22}t^{13} - \frac{1}{7134400}ix^{26}t^{18}.$$

Comparing the first two components, u_0 and u_1, gives the noise terms, $-\frac{1}{6}x^4t$ and $-\frac{1}{28}ix^8t^6$. By canceling these terms from the first component u_0, we obtained the desired solution as:

$$u(x,t) = ix^2t^2. \tag{4.26}$$

4.3. Solving Complex mKdV Equation

We consider the nonhomogeneous complex mKdV equation as

$$u_t+\alpha u_{xxx}+6\alpha|u|^2u_x = ixe^{it}+6\alpha x^2xe^{3it}, \tag{4.27}$$

with the initial condition $u(x,0)=x$. In other way, the equation (4.27) is given by

$$u_t = ixe^{it}+6\alpha x^2xe^{3it}-\alpha u_{xxx}-6\alpha|u|^2u_x. \tag{4.28}$$

By applying the double Laplace transform to both sides of the equation (4.28) and using the properties of the double Laplace transform, we get

$$L_xL_t[u(x,t)] = U(p,s) = \frac{1}{s}L_xL_t[ixe^{it}+6\alpha x^2xe^{3it}]-\frac{1}{s}[L_xL_t[\alpha u_{xxx}]+L_xL_t[6\alpha|u|^2u]]. \tag{4.29}$$

We then put the expansions (3.4) and (3.5) into the equation (4.29). We therefore have

$$L_xL_t[\sum_{n=0}^{\infty}u_n(x,t)] = \frac{1}{s}L_xL_t[ixe^{it}+6\alpha x^2xe^{3it}]-\frac{1}{s}[L_xL_t[R[\sum_{n=0}^{\infty}u_n(x,t)]]+L_xL_t[\sum_{n=0}^{\infty}A_n]], \tag{4.30}$$

where $R[u]=\alpha u_{xxx}$ and $A_n[u]=6\alpha|u|^2u]$. By applying the inverse double Laplace transform ot both sides of the equation (4.30), we obtain

$$\sum_{n=0}^{\infty}u_n(x,t) = L_x^{-1}L_t^{-1}[\frac{1}{s}L_xL_t[ixe^{it}+6\alpha x^2xe^{3it}]]-L_x^{-1}L_t^{-1}[\frac{1}{s}[L_xL_t[R[\sum_{n=0}^{\infty}u_n(x,t)]]+L_xL_t[\sum_{n=0}^{\infty}A_n]]]. \tag{4.31}$$

From the above equality, we get the recursive relation as follows:

$$u_0(x,t) = x+L_x^{-1}L_t^{-1}[\frac{1}{s}L_xL_t[ixe^{it}+6\alpha x^2xe^{3it}]], \tag{4.32}$$

$$u_1(x,t) = -L_x^{-1}L_t^{-1}[\frac{1}{s}[L_xL_t[R[u_0]]+L_xL_t[A_0]]], \tag{4.33}$$

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$$u_{n+1} = -L_x^{-1}L_t^{-1}[\frac{1}{s}[L_xL_t[R[u_n(x,t)]+L_xL_t[A_n]]], \quad n\geq 0. \tag{4.34}$$

We hence attain

$$u_0(x,t) = xe^{it}+2i\alpha x^2-2i\alpha x^2e^{3it} \tag{4.35}$$

$$u_1(x,t) = -96a^4e^{3it}x^5+48a^4e^{6it}x^5-\frac{32}{3}a^4e^{9it}x^5+96ia^4tx^5+\frac{176a^4x^5}{3}-120ia^3e^{it}x^4-\frac{120}{7}ia^3e^{7it}x^4+60ia^3e^{4it}x^4+ \tag{4.36}$$

$$\tfrac{540}{7}ia^3x^4-24a^2e^{2it}x^3+\tfrac{48}{5}a^2e^{5it}x^3+\tfrac{72a^2x^3}{5}+2iae^{3it}x^2-2iax^2.$$

From the first two components, u_0 and u_1,we observe the noise terms as $2i\alpha x^2$ and $2i\alpha x^2e^{3it}$. Removing these terms from the first component u_0 provides us with the solution

$$u(x,t) = xe^{it}. \tag{4.37}$$

4.4. Examples

Here we solve two nonhomogeneous nonlinear partial differential equations subject to the initial conditions by the double Laplace decomposition method.

4.4.1. Example 1

For the first example, we consider the following equation

$$u_{tt}-\alpha^2u_{xx}+\beta u-\gamma u^2 = t\alpha^2sin(x)+t\beta sin(x)-t^2\gamma sin(x)^2, \tag{4.38}$$

with the initial values $u(x,0)=0$ and $u_t(x,0)=sin(x)$. First thing is to get u_{tt} alone in the left side and put the other terms into the right side. We hence have

$$u_{tt} = t\alpha^2sin(x)+t\beta sin(x)-t^2\gamma sin(x)^2+\alpha^2u_{xx}-\beta u+\gamma u^2. \tag{4.39}$$

Applying the double Laplace tranform to both sides of the equation (4.39) gives

$$L_xL_t[u(x,t)] = U(p,s) = \frac{1}{s^2}L_xL_t[sin(x)]+\frac{1}{s^2}L_xL_t[t\alpha^2sin(x)+t\beta sin(x)-t^2\gamma sin(x)^2]-\frac{1}{s^2}[L_xL_t[\alpha^2u_{xx}-\beta u]+L_xL_t[\gamma u^2]]. \tag{4.40}$$

By following the same process shown above, we get

$$u_0(x,t)=tsin(x)+L_x^{-1}L_t^{-1}[\frac{1}{s^2}L_xL_t[t\alpha^2sin(x)+t\beta sin(x)-t^2\gamma sin(x)^2]], \tag{4.41}$$

$$u_1(x,t)=-L_x^{-1}L_t^{-1}[\frac{1}{s^2}[L_xL_t[R[u_0]_t]+L_xL_t[A_0]]] \tag{4.42}$$

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$$u_{n+1}=-L_x^{-1}L_t^{-1}[\frac{1}{s^2}[L_xL_t[R[u_n(x,t)]+L_xL_t[A_n]]], \quad n\geq 0. \tag{4.43}$$

From above, we obtain

$$u_0(x,t)=-\frac{1}{24}\gamma t^4+\frac{1}{24}\gamma t^4\cos(2x)+\frac{1}{6}\alpha^2t^3\sin(x)+\frac{1}{6}\beta t^3\sin(x)+t\sin(x), \tag{4.44}$$

$$\begin{aligned}u_1(x,t)=&\frac{1}{34560}\gamma^3t^{10}+\frac{1}{103680}\gamma^3t^{10}\cos(4x)-\frac{1}{25920}\gamma^3t^{10}\cos(2x)+\frac{1}{10368}\alpha^2\gamma^2t^9\sin(3x)-\frac{1}{3456}\alpha^2\gamma^2t^9\sin(x)\\&+\tfrac{1}{10368}\beta\gamma^2t^9\sin(3x)-\tfrac{1}{3456}\beta\gamma^2t^9\sin(x)+\tfrac{1}{4032}\alpha^4\gamma t^8+\tfrac{1}{2016}\alpha^2\beta\gamma t^8+\tfrac{1}{4032}\beta^2\gamma t^8-\tfrac{1}{4032}\alpha^4\gamma t^8\cos(2x)-\tfrac{1}{2016}\alpha^2\beta\gamma t^8\cos(2x)\\&-\tfrac{1}{4032}\beta^2\gamma t^8\cos(2x)-\tfrac{1}{336}\gamma^2t^7\sin(x)+\tfrac{1}{1008}\gamma^2t^7\sin(3x)+\tfrac{1}{180}\alpha^2\gamma t^6+\tfrac{1}{144}\beta\gamma t^6-\tfrac{1}{90}\alpha^2\gamma t^6\cos(2x)-\tfrac{1}{144}\beta\gamma t^6\cos(2x)\\&-\tfrac{1}{120}\alpha^4t^5\sin(x)-\tfrac{1}{60}\alpha^2\beta t^5\sin(x)-\tfrac{1}{120}\beta^2t^5\sin(x)+\tfrac{1}{24}\gamma t^4-\tfrac{1}{24}\gamma t^4\cos(2x)-\tfrac{1}{6}\alpha^2t^3\sin(x)-\tfrac{1}{6}\beta t^3\sin(x).\end{aligned} \tag{4.45}$$

Here the noise terms are $\frac{1}{24}\gamma t^4$, $\frac{1}{24}\gamma t^4\cos(2x)$, $\frac{1}{6}\alpha^2t^3\sin(x)$, and $\frac{1}{6}\beta t^3\sin(x)$. If we remove the noise terms from u_0, then we obtain the solution

$$u(x,t)=tsin(x). \tag{4.46}$$

4.4.2. Example 2

For the first example, we consider the following equation

$$u_{tt}-u_{xxx}u_t-u_{xxx}=-2sin(x), \tag{4.47}$$

with the initial values $u(x,0)=cos(x)$ and $u_t(x,0)=1$. Firstly, let us consider the equation (4.47) as follows:

$$u_{tt}=-2sin(x)+u_{xxx}u_t+u_{xxx}. \tag{4.48}$$

Then, we apply the double Laplace tranform to both sides of the equation (4.48) yields

$$L_xL_t[u(x,t)]=U(p,s)=cos(x)+t+\frac{1}{s^2}L_xL_t[-2sin(x)]-\frac{1}{s^2}[L_xL_t[u_{xxx}]+L_xL_t[u_{xxx}u_t]]. \tag{4.49}$$

In the same manner above, we have

$$u_0(x,t)=cos(x)+t+\frac{1}{s^2}L_xL_t[-2sin(x)], \tag{4.50}$$

$$u_1(x,t)=-L_x^{-1}L_t^{-1}[\frac{1}{s^2}[L_xL_t[R[u_0]_t]+L_xL_t[A_0]]] \tag{4.51}$$

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$$u_{n+1}=-L_x^{-1}L_t^{-1}[\frac{1}{s^2}[L_xL_t[R[u_n(x,t)]+L_xL_t[A_n]]], \quad n\geq 0. \tag{4.52}$$

where $R[u_n]=u_{nxxx}$ and $A_n[u]=u_{nxxx}u_{nt}$. From this recursive relation, we obtain

$$u_0(x,t)=cos(x)+t-t^2sin(x), \tag{4.53}$$

$$u_1(x,t)=t^2sin(x)-\frac{1}{6}t^3+\frac{1}{6}t^3cos(2x)+\frac{1}{6}t^4cos(x)-\frac{1}{20}t^5sin(2x). \tag{4.54}$$

It is easily seen that $t^2sin(x)$ is the noise term. We take away the noise term from u_0 to attain the solution as

$$u(x,t)=cos(x)+t. \tag{4.55}$$

5. Conclusion

In this present paper, we focus on the double Laplace decomposition method. We take the advantage of this method in order to obtain the exact solutions of some significant NLPDEs, namely Hirota, Schrödinger, cmKdV, and two more equations with the initial conditions. It is clearly demonstrated that this method is really convenient, appropriate, advantageous, and sufficient to acquire the exact solutions of NLPDEs subject to the given initial conditions. It is also seen that this method is simple and direct. Moreover, we quickly obtain the exact solution with the help of the noise terms. The best part of this method is that there is no need for linearization of nonlinear terms thanks to the Adomian polynomials compared to other methods. We eventually state that this method is indeed trustworthy and applicable to almost all NLPDEs subject to the initial conditions.

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