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Research Article

Mathematics Teachers' Views on Mathematical Thinking

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Abstract

The aim of this research is to examine the opinions of mathematics teachers about mathematical thinking. The study examined teachers' knowledge level and views regarding mathematical thinking within the scope of a class titled "Development of Mathematical Thinking Skills in Children" in educational sciences institute of a university. The study was conducted in a pattern of descriptive case study following a qualitative paradigm. Descriptive case studies are definitive. Participants considered mathematical thinking as a process in which they related real life and mathematics in preeducation. In posteducation process, however, the participants were able to define mathematical thinking with several different viewpoints and examples and explain the process with examples.

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Introduction

Skills which are expected in today's world such as problem-solving, reasoning and modeling require a high level of mathematical thinking (Suzuki, 1998). Mathematical thinking allows individuals to grasp the importance of using information and skills as well as learn individually and independently (Isoda & Katagiri, 2012). Mathematical thinking begins with an individual's perception of objects around them and effort to make sense of the relationship among those objects (Tall, 1995). When mathematical thinking is mentioned, a mathematical situation may come to mind, and the effective use of mathematical rules and procedures to achieve a certain outcome may come. However, mathematical thinking is the application of mathematical processes, either openly or not, in solving problems (Henderson, 2002). If the solution of a problem requires high-level thinking skills such as privatization, generalization, estimating, producing a hypothesis, controlling the accuracy of the hypothesis, mathematical thinking will be realized (Yeşildere & Türnüklü, 2007). From this viewpoint, mathematical thinking is defined as the union of complex processes such as

guessing, induction, deduction, description, generalization, modeling, verification, etc. (Liu & Niess, 2006).

Although the importance of student knowledge is accepted in constructing mathematics learning activities, studies reveal that teachers and teacher candidates struggle greatly in discovering and interpreting students' thoughts (Crespo, 2000, 2003; Kazemi & Franke, 2004; Moyer & Milewicz, 2002; Steinberg, et al.; Wallach & Even, 2005). Empson and Junk (2004) claimed that teachers following a student-centered teaching program knew about non-standard strategies developed by students in multistage operations; however, they did not know about rare strategies. A similar study conducted in secondary school level by Baş, Erbaş and Çetinkaya (2011) observed that three teachers who were teaching 9th grade fell behind in guessing the strategies which students can use in algebra and algebraic thinking structures which were behind these strategies. Wallach and Even (2005) asked an experienced teacher who taught 4th grade to interpret his/her two students' comments and actions in the process of solving a problem and found that there were contradictions between students' utterances and actions and what the teacher heard and interpreted.

Numerous projects applied with the purpose of supporting mathematics teachers' professional development show that teachers who consider their students' mathematical thinking are more successful in creating student-centered learning environments (Carpenter, Fennema, et al., 1989; Cobb, et al., 1990, Cobb, et al., 1991; Fennema, et al., 1996; Franke & Kazemi, 2001; Franke, et al., 2001). Researchers formed professional development programs which support teachers and teacher candidates in examining and interpreting students' thinking styles. Teaching to the Big Ideas (TBI) which is one of those programs gathered 36 teachers together every two weeks for four years and discussed both mathematics notions and analyzed how their students can understand these notions and where they might have problems through the example cases in their classes. Schifter (1998) observed the fractions classes of two teachers who attended this project and identified that these teachers were successful in listening to their students, interpreting and analyzing different thinking styles. Besides, these teachers also improved in creating atmospheres where students can review and enhance their ideas. Similarly, Kazami and Franke (2004) asked ten teachers to apply the same mathematics problems in their classes. Then, the teachers got together and discussed the answers given by students, described strategies used by the students and made a comparison among them. The teachers who participated in the study initially assessed the

strategies of the students as successful and unsuccessful; but as the project advanced, they needed to develop methods to understand students' strategies. At the end of the project, teachers managed to focus on how students reasoned while solving the problem and notice the difference in the details.

As part of mathematics teaching classes in the education faculties of our country, primary school mathematics teacher candidates study students' mathematical thinking styles, possible difficulties they may have, common mistakes, misconceptions, and its reasons, the relationship between mathematics and life (National Higher Education Institution, 2018). However, we have limited knowledge of how well the teacher candidates acquire this information and skills. Several recent studies reveal the importance of teachers being aware of their students' mathematical thinking (Cooper, 2009; Crespo, 2000; Even & Tirosh, 2008; Fraivillig, et al., 1999; McLeman & Cavell 2009; Moss, 2009; Philipp, 2008). Then, a teacher should be aware of the students' mathematical thinking and develop their teaching within the frame of these thinking (Olkun & Toluk, 2004). In this context, examining teachers' present state on mathematical thinking and contributing to them is deemed important. Although there are many pieces of research in the literature about mathematical thinking, few focus on teachers' perceptions (Ersoy & Güner, 2014; Koparan, Güven & Karataş, 2014; Palinussa, 2013; Toheri & Winarso, 2017; Yorulmaz, et al., 2017) and this study motivated by this point of view. Moreover, researchers had also observed that although the lesson plans prepared by teacher candidates involved concepts such as "mathematical thinking" when they were asked to point at the "skills to be declared", they were unable to give satisfactory answers to the researchers' question of "What is mathematical thinking?". This means that they didn't know in depth what "mathematical thinking" actually refers to. The purpose of the present study is to examine mathematics teachers' views on mathematical thinking. The aim of this research is to examine the opinions of mathematics teachers about mathematical thinking. The answers are sought in the research problem statement "is how the math teachers' opinions on mathematical thinking?" has been identified as.

Method

The study examined teachers' knowledge level and views regarding mathematical thinking within the scope of a class titled "Development of Mathematical Thinking Skills in

Children" in educational sciences institute of a university. From this viewpoint, 7 female, 3 male teachers who actively continued teaching and were thought to be open for improvement (considering their status of doing a master's degree) were chosen to participate in the study via convenient sampling which is one of the purposeful sampling methods. The study was conducted within the scope of teachers' "Development of Mathematical Thinking Skills in Children" class (the aim was to examine the change in teachers' mathematical thinking caused by this class). At the beginning of the study, participants' opinions of mathematical thinking were taken through a structured interview approach. Data gathered were analyzed via content analysis and then divided into categories, sub-categories, and codes. In the lecture conducted by one of the researchers, the sections in the literature about mathematical thinking are presented. Then, teachers' opinions of mathematical thinking were taken via a structured interview approach. These data were also analyzed via content analysis and divided into categories, sub-categories, and codes. Considering the aim of the study, it can be said that the study was conducted in a pattern of descriptive case study following a qualitative paradigm. Descriptive case studies are definitive. That is, it is conducted for the researcher to describe a certain subject (Subaşı & Okumuş, 2017). Participants examined the data gathered in order to ensure the construct validity of the study. Also, as much detail about the study as possible was shared and findings were revealed clearly and orderly (Yin, 2003).

Process and Teaching of "Development of Mathematical Thinking Skills in Children" Class

The class titled "Development of Mathematical Thinking Skills in Children" which is an MA class was conducted in 14 weeks in total. The aim of the class is for students to be able to perceive the development and importance of mathematical thinking skill in children. Also, within the scope of the class, topics of mathematical thinking structure and development, types of mathematical thinking, characteristics of mathematical thinking, the importance of mathematical thinking and being aware of it; differences in students' perception, association and application skills according to age; methods and techniques used to acquire mathematical thinking skill were dealt with and related national and international articles were examined. During the classes, following week's topics were shared with the participants and they were asked to do necessary preparations and readings. The class was conducted in an environment where brainstorming and discussion were involved.

Findings

Findings gathered in the postgraduate class "Development of Mathematical Thinking in Children" which was taught by the second researcher were reported as "Preeducation findings" and "Posteducation findings"

Preeducation Findings

At the beginning of the study, structured interviews were conducted in order to determine the participants' views and the level of knowledge regarding mathematical thinking. The interviews were transcribed and then content analysis was conducted through repeated readings done by the researchers at different times. Match percentage between the two researchers was found 88%. Findings gathered form preeducation data are presented in Table1 .

Table 1. Preeducation findings

Preeducation findings	Addition on present knowledge	Addition on present knowledge (K_1, K_2, K_3, K_7) Reasoning (K_4, K_6, K_7, K_9, K_{10}) Analyzing(K_4, K_6)
	Availability for improvement	Innate ability (K_2, K_5) Independent (K_8)
	Relationship with daily life (K_2, K_5, K_6)	

When the pretraining findings are examined in detail; participants explained mathematical thinking under these categories: Addition on present knowledge, availability for improvement and relationships with daily life. K_3 defined mathematical thinking as "*The process of learning new information based on what we know earlier.*" K_4 explained mathematical thinking saying *"…Reaching new information by following the logical rules along with the ideas obtained from the results.*". K_6 explained it as *"Access to new information by analyzing the information available ... "*

Participants think of mathematical thinking as an innate, independent process as mathematics in life and course within mathematics. K_2 explained it saying: "*It is an innate ability. I think it is the ability to integrate the topics learned in mathematics, creating new theories and concepts in mathematics through observing situations in daily life…*"

Three of the participants explained mathematical thinking as the relationship between daily life and mathematics. One of these participants was K_5; *"I think that mathematical thinking; is to use mathematics to overcome the difficulties encountered in life."*.

The participants had a one-dimensional approach in defining mathematical thinking and put emphasis only on mathematical understanding is a thinking process. Also, they stated that being aware of this process had a great influence on constructing their own learning. K_1 explains it saying: "*With the help of mathematical thinking, we do not have difficulty in relating subjects we constructed by ourselves with the next subject. But sometimes it can be very difficult to make a connection between concepts in eccentric learning where we learn by deactivating mathematical thinking*.".

Posteducation Findings

Structured interviews were conducted again at the end of postgraduate class process. Interviews were transcribed, and their content analysis was conducted through repeated readings done by researchers at different times. Match percentage between the two researchers was 86%. Findings gathered through posteducation data are presented in Table2.

Table 2. Posteducation findings

Posteducation findings	Functional thinking	Reflection (K_1) Rule (K_4)
	Thinking via analogies	Integers (K_1, K_4, K_5, K_6, K_7) The idea of order (K_2) Solving equations (K_3, K_4)
	Inductive/Deductive thinking	Summing up to a certain point(K_1, K_2, K_3, K_6) Quadrangles (K_1, K_3, K_9, K_{10}) Number of a subset (K_4, K_7) Paper folding (K_5)
	Thinking via manipulatives	Solving problems with the help of counting stamps (K_1, K_8) Modular arithmetic (K_7)
	Abstract thinking	Abstract thinking Reflection (K_4) Functions (K_8) Effort of concretization Division(K_5) Cubic expansion(K_2) Counting stamps(K_3)
	Thinking via modeling Integrative thinking	Area of a triangle (K_1, K_8) Pythagorean relation- Exponential number (K_8)
	Developmental thinking Thinking from simple to complex	Construction of numbers (K_8) Fractions (K_1, K_3) Addition (K_4, K_5)

According to findings gathered via posteducation interviews, the participants tried to explain mathematical thinking under nine titles with examples. They tried to state those titles themselves and preferred to exemplify. K_4, who was one of the participants, tried to explain mathematical thinking saying: "*If we can awake that thought in student, actually a basic mathematical thinking skill is then acquired. For example, let's say we buy goods for x Liras and we will sell it for y Liras. If a student can see the profit and loss as input and output, it is the point where they gain functional thinking skill. This actually requires mathematical thinking.*" K_4 continued to elaborate saying: "*…for example if we can directly draw the shape without following the critical steps while finding a reflection symmetry of an object, this is mathematical thinking*."

K_2 explained mathematical thinking saying: "*Let's think about applying the subject of digits. Let's say red straws represent unit digits, yellow straws represent tens digit and green straws represent hundreds digit. So, when you show two straws, the student/learner will think about number 2, and when you show them three yellow straws, they will think about the number thirty. With this analogical approach, students will have acquired mathematical thinking.*"

K_8, who was one of the participants, explained mathematical thinking by saying: "*…We sometimes have an approach from specific to general. For instance, if you fold an A4 paper in two, it becomes two separate pieces. If you fold again, there are four separate areas. You can have eight different areas by folding it three times. It can continue like this and the individual can think that after n. folding, there will be 2n areas. Inductive thinking like this is also mathematical thinking.*" K_8 who tried to explain mathematical thinking through different approaches continued and added another viewpoint: "*… for example, operations with negative and positive integers can be explained by using counting stamps. I mean, mathematical thinking can be activated by using objects that can be used instead of mathematical symbols, that is, manipulatives…*" K_6 explained mathematical thinking saying: "*Let's look at the topic of slope for example. If you cut an A4 paper 6 cm horizontally and 8 cm vertically, the slope can be observed. This is actually transferring the way of thinking in our mind into a model and it is a way of transferring mathematical thinking into real life*." K_7 explained it saying: "*In order for mathematical thinking to form a meaningful system within itself, you need to consider and assess the structures as a whole. I mean, if you do not consider fractional exponents while teaching fractions, this will be a problem while teaching radical expressions. So, mathematical thinking is the process of integrated thinking.*" K_7 continued: "Topics which are considered in the process happen according to sub-titles of mentally and physically developmental features. For example, we learn to count numbers first and we do

not learn complex numbers right after that. There is an order, like, integers and rational numbers because we construct via mathematical thinking. Then, mathematical thinking is a developmental process at the same time."

K_3 described mathematical thinking by saying: *"…it is the ability to think in a more complex way based on the simple one, I think. It is an individual's effort to simplify a problem and then solve it in the desired way*."

In the light of all these statements, we concluded that participants explained mathematical thinking in two dimensions: mathematical thinking regarding mathematical methods and mathematics-oriented mathematical thinking. Mathematical thinking regarding mathematical methods can be described as thinking of helpful methods of mathematical thinking. In this context, methods provided by the participants are defined as functional thinking, thinking via analogies, inductive/deductive thinking, thinking with the help of manipulatives, abstract thinking, thinking via modeling, integrative thinking, developmental thinking and thinking from simple to complex. Mathematics-oriented mathematical thinking is topics of construction of numbers, the concept of function, mental creation of geometric conversions stated by the participants. This finding is supported by this statement of participant K_4 : "*Mathematical thinking is all of the processes of defining strategies to reach the solution of a problem and application and monitoring of the solution.*"

Discussion and Conclusions

Participants considered mathematical thinking as a process in which they related real life and mathematics in preeducation. Besides, participants could not exactly explain the mathematical thinking system in the examples they provided. Also, they were not aware of mathematical thinking regarding mathematical methods. In addition, participants considered mathematical thinking as an individual process and gave examples in this regard. In addition, participants; they evaluated mathematical thinking individually and did not exemplify for their students' learning. It is seen that they consider mathematical thinking as a skill that needs to be developed, but they do not have detailed knowledge about the way how this thinking skill work.

In posteducation process, however, the participants were able to define mathematical thinking with several different viewpoints and examples and explain the process with examples. Besides, the participants considered mathematical thinking with pedagogical

approach; thus, enriched their definitions and examples in this context. In posteducation, the participants emphasized that being able to understand mathematical thinking may influence students' thinking and accordingly, their learning. In addition to this, they also give detailed explanations to improve mathematical thinking in their students.

While the current situation of teachers in MA education was as examined, the suggestion to conduct a similar study with in-service teachers and teacher candidates seems compulsory. Additionally, prospective primary school and mathematics teachers' awareness need to be raised by allowing them to experience examples of mathematical thinking in both mathematics field courses and mathematics education courses. In addition, a detailed examination of prospective teachers' knowledge about mathematical skills is among the suggestions of the research.

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