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On Numerical Approach to The Rate of Convergence and Data Dependence Results for a New Iterative Scheme

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Abstract

The aim of this paper is to obtain results of the strong convergence, rate of convergence and data dependence for a new three step iterative scheme using contraction mappings and to give examples for the rate of convergence and data dependence results. After these numerical approachs, it can be seen that the new iterative scheme has a better rate of convergence with respect to the other iterative schemes in the literature. The results obtained in this paper may be interpreted as a refinement and improvement of the previously known results.

Keywords: *A new iterative scheme, Strong convergence, Data Dependence, Contraction mappings.*
2010 Mathematics Subject Classification: *47H10, 47H09.*

1. Introduction

Fixed point theory is an effective and useful tool for solving problems encountered in various kinds of science from mathematics to economy. The applicability of this theory to other fields has attracted the attention of many researchers and it has therefore become a new field of study with a wide range of literature. The fixed point of a mapping is the invariant point under this mapping that is:
Let B be a Banach space, and C be a nonempty, closed, convex subset of B. Let A be a mapping from a set C to itself. An element x in C is said to be a fixed point of A if $Ax = x$.
In general terms, the aim of fixed point theory is to determine the appropriate conditions to be put on the mapping or on the set where the mapping is defined in order to obtain this fixed point. Once these conditions have been determined, a number of fixed point theorems have been obtained for various types of mappings classes by defining the algorithms called iteration in order to reach the fixed point of the mappings (see [1]-[8]).
In 2016, Ullah and Arshad [9] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = Ty_n \\ y_n = T\left((1-\alpha_n)z_n + \alpha_n Tz_n\right) \\ z_n = T\left((1-\beta_n)x_n + \beta_n Tx_n\right) \end{cases} \tag{1.1}$$

Very recently, Ertürk et. al [10] introduced a new iterative scheme as follows:

$$\begin{cases} u_0 \in C, \\ u_{n+1} = Tv_n \\ v_n = T\left(T(w_n)\right) \\ w_n = T\left((1-\alpha_n)u_n + \alpha_n Tu_n\right) \end{cases} \tag{1.2}$$

where $(\alpha_n)_{n=1}^{\infty} \in [0,1]$. They proved that this method has a better convergence rate than Ishikawa [2], Mann [3], Noor [4], CR [6], Picard [7], Picard-S [11], Thakur et al. [12], Vatan twostep [13], Abbas and Nazir [14], Normal-S [15], Modified-SP [16], S* [17] and (1.1) iterative schemes.
Now, we have the following problem:
Is it possible to define an iterative scheme whose convergence rate is faster than the iterative scheme (1.2) ?

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We give an affirmative answer for this question by defining the following iterative scheme:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = Ty_n \\ y_n = T\left(\frac{(1-\alpha_n)}{k} z_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Tz_n\right) \\ z_n = T\left(T\left(x_n\right)\right) \end{cases} \tag{1.3}$$

where $(\alpha_n)_{n=1}^{\infty} \in [0,1]$ and $k \in \mathbb{N}$.
In this paper, we show that iterative scheme (1.3) can be used approximate fixed point of contraction mappings. Further, we proved that there is equivalency between iterative scheme (1.3) and iterative scheme (1.2). Moreover, we show that iterative scheme (1.3) has a better convergence rate than iterative scheme (1.2) and in order to demonstrate the efficiency of iterative scheme (1.3), we give a numerical example. Finally, we show that a data dependence result can be obtain for contraction mappings using iterative scheme (1.3).

2. Preliminaries and Basic Results

Lemma 2.1 ([18]). *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real sequences satisfying the following condition:*

$a_{n+1} \leq (1-\mu_n)a_n + b_n,$

where $\mu_n \in (0,1)$ for all $n \geq n_0$, $\sum_{n=1}^{\infty} \mu_n = \infty$ and $\frac{b_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([19]). *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative real sequence and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ satisfying the following condition:*

$a_{n+1} \leq (1-\mu_n)a_n + \mu_n \eta_n,$

where $\mu_n \in (0,1)$ such that $\sum_{n=1}^{\infty} \mu_n = \infty$ and $\eta_n \geq 0$. Then the following inequality holds:

$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \eta_n.$

Definition 2.3 ([9]). *Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called contraction if there exists $\delta \in (0,1)$ such that*

$$\|Tx - Ty\| \leq \delta \|x - y\| \tag{2.1}$$

for all $x, y \in C$.

Definition 2.4 ([20]). *Let $\{c_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ are two iterative schemes converging to the same fixed point p_* of a mapping T. We say that $\{c_n\}_{n=0}^{\infty}$ converges faster than $\{d_n\}_{n=0}^{\infty}$ to p_* if*

$$\lim_{n\to\infty} \frac{\|c_n - p_*\|}{\|d_n - p_*\|} = 0. \tag{2.2}$$

Definition 2.5 ([19]). *Let T, $S : C \to C$ be two operators. We say that S is an approximate operator of T if for all $x \in C$ and for a fixed $\varepsilon > 0$ if $\|Tx - Sx\| \leq \varepsilon$.*

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}_{n=1}^{\infty}$ be iterative sequence generated by (1.3) with real sequence such that $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$ satisfying $\sum_{k=1}^{n} \alpha_k = \infty$. Then, $\{x_n\}_{n=1}^{\infty}$ converges to the unique fixed point p_* of T.*

Proof. Banach-Contraction Principle guarantees the existence and uniqueness of fixed point p_* of T. From (1.3) and (2.1), we have

$$\begin{aligned} \|z_n - p_*\| &= \|T(Tx_n) - p_*\| \leq \delta \|Tx_n - p_*\| \\ &\leq \delta^2 \|x_n - p_*\| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|y_n - p_*\| &= \left\| T\left(\frac{(1-\alpha_n)}{k} z_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Tz_n\right) - Tp_* \right\| \\ &\leq \delta \left\| \left(\frac{(1-\alpha_n)}{k} z_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Tz_n\right) - p_* \right\| \\ &\leq \delta \frac{(1-\alpha_n)}{k} \|z_n - p_*\| + \delta^2 \left(1 - \frac{(1-\alpha_n)}{k}\right) \|z_n - p_*\| \\ &= \delta \left[\frac{(1-\alpha_n)}{k} + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right)\right] \|z_n - p_*\| \end{aligned} \tag{3.2}$$

and

$$\|x_{n+1}-p_*\| \le \delta \|y_n - p_*\| \tag{3.3}$$

Substituting (3.1) in (3.2) and (3.2) in (3.3) respectively, we have

$$\|x_{n+1}-p_*\| \le \delta^4 \left[\frac{(1-\alpha_n)}{k} + \delta\left(1-\frac{(1-\alpha_n)}{k}\right)\right] \|x_n - p_*\| \tag{3.4}$$

Since

$$\left[\frac{(1-\alpha_n)}{k} + \delta\left(1-\frac{(1-\alpha_n)}{k}\right)\right] = \delta\left(1+\frac{(1-\alpha_n)(1-\delta)}{k\delta}\right)$$

we have

$$\begin{aligned}\|x_{n+1}-p_*\| &\le \delta^5\left(1+\frac{(1-\alpha_n)(1-\delta)}{k\delta}\right)\|x_n-p_*\| \\ &\le \delta\left(1+\frac{(1-\alpha_n)(1-\delta)}{k\delta}\right)\|x_n-p_*\|.\end{aligned}$$

Repeating this proocess n-times, we obtain

$$\begin{aligned}\|x_n-p_*\| &\le \delta\left(1+\frac{(1-\alpha_{n-1})(1-\delta)}{k\delta}\right)\|x_{n-1}-p_*\| \\ \|x_{n-1}-p_*\| &\le \delta\left(1+\frac{(1-\alpha_{n-2})(1-\delta)}{k\delta}\right)\|x_{n-2}-p_*\| \\ &\vdots \\ \|x_1-p_*\| &\le \delta\left(1+\frac{(1-\alpha_0)(1-\delta)}{k\delta}\right)\|x_0-p_*\|.\end{aligned}$$

Hence

$$\|x_{n+1}-p_*\| \le \delta^{n+1}\prod_{i=0}^{n}\left(1+\frac{(1-\alpha_i)(1-\delta)}{k\delta}\right)\|x_0-p_*\|. \tag{3.5}$$

Since

$$\left(1+\frac{(1-\alpha_n)(1-\delta)}{k\delta}\right) \le \frac{1}{\delta}\left(1-\frac{\alpha_n(1-\delta)}{k}\right)$$

we obtain

$$\begin{aligned}\|x_{n+1}-p_*\| &\le \delta\prod_{i=0}^{n}\left(1-\frac{\alpha_i(1-\delta)}{k}\right)\|x_0-p_*\| \\ &\le \delta\prod_{i=0}^{n} e^{\frac{-(1-\delta)}{k}\alpha_i}\|x_0-p_*\| \\ &= \delta\frac{1}{e^{\frac{(1-\delta)}{k}\sum_{i=1}^{n}\alpha_i}}\|x_0-p_*\|.\end{aligned}$$

Taking the limit in both sides of the above inequality, it can be seen that $x_n \to p_*$ as $n \to \infty$. □

Theorem 3.2. *Let X, C and T with a fixed point p_* be the same as in Theorem 3.1. Let $\{u_n\}_{n=1}^{\infty}$ is defined by iterative scheme (1.2) for $u_0 \in C$ and $\{x_n\}_{n=1}^{\infty}$ is defined by (1.3) for $x_0 \in C$ with a real sequence $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$ satisfying $\sum_{k=1}^{n}\alpha_k = \infty$. Then the following assertions are equivalent:*

i) The new iterative scheme (1.3) converges to p_*.
ii) The iterative scheme (1.2) converges to p_*.

Proof. We will show that $(i) \Rightarrow (ii)$, that is, if the iterative scheme (1.3) converges, then the iterative scheme(1.2) does too. Now, by using these two iterative schemes we obtain

$$\begin{aligned}\|z_n-w_n\| &= \|T(T(x_n)) - T((1-\alpha_n)u_n+\alpha_n Tu_n)\| \\ &\le \delta\|T(x_n)-((1-\alpha_n)u_n+\alpha_n Tu_n)\| \\ &\le \delta(1-\alpha_n)\|Tx_n-u_n\|+\alpha_n\delta^2\|x_n-u_n\| \\ &\le \delta[1-\alpha_n(1-\delta)]\|x_n-u_n\|+\delta(1-\alpha_n)\|x_n-Tx_n\|,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\|y_n - v_n\| &\leq \left\| T\left(\frac{(1-\alpha_n)}{k} z_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) T z_n\right) - T\left(T\left(w_n\right)\right)\right\| \\ &\leq \delta \frac{(1-\alpha_n)}{k} \|z_n - T w_n\| + \delta^2 \left(1 - \frac{(1-\alpha_n)}{k}\right) \|z_n - w_n\| \\ &\leq \delta^2 \|z_n - w_n\| + \delta \frac{(1-\alpha_n)}{k} \|z_n - T z_n\|\end{aligned} \tag{3.7}$$

and

$$\|x_{n+1} - u_{n+1}\| \leq \delta \|y_n - v_n\|. \tag{3.8}$$

Substituting (3.6) in (3.7) and (3.7) in (3.8) respectively, we obtain

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| \leq\; & [1 - \alpha_n(1-\delta)] \|x_n - u_n\| \\ & + \delta^4 (1-\alpha_n) \|x_n - T x_n\| \\ & + \delta^2 \frac{(1-\alpha_n)}{k} \|z_n - T z_n\|\end{aligned}$$

Let

$$\begin{aligned}\mu_n &= \alpha_n(1-\delta) \in (0,1) \\ a_n &= \|x_n - u_n\|, \\ b_n &= \delta^4 (1-\alpha_n) \|x_n - T x_n\| \\ &\quad + \delta^2 \frac{(1-\alpha_n)}{k} \|z_n - T z_n\|\end{aligned}$$

Furthermore, using $T p_* = p_*$ and $\|x_n - p_*\| \to 0$, we have

$$\begin{aligned}\|x_n - T x_n\| &\leq \|x_n - p_*\| + \delta \|x_n - p_*\| \\ &= (1+\delta) \|x_n - p_*\|.\end{aligned}$$

Then, $\|x_n - T x_n\| \to 0$. Similarly,

$$\begin{aligned}\|z_n - T z_n\| &\leq (1+\delta) \|z_n - p_*\| \\ &= (1+\delta) \|T\left(T\left(x_n\right)\right) - p_*\| \\ &\leq (1+\delta)\delta \|T x_n - p_*\| \\ &\leq (1+\delta)\delta^2 \|x_n - p_*\|,\end{aligned}$$

Because of these results, we obtain $b_n \to 0$. By applying Lemma 2.1, we have $a_n = \|x_n - u_n\| \to 0$ as $n \to \infty$. Consequently,

$$\|x_{n+1} - u_{n+1}\| \to 0 \text{ as } n \to \infty.$$

The other part of this equivalency can be prove similar way. Hence we omit it. □

Theorem 3.3. *Let X, C and T with a fixed point p_* be the same as in Theorem 3.1. Let $\{\alpha_n\}$ be real sequence in $[0,1]$ satisfying (i) $\alpha_1 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N}$. For given $u_1 = x_1 \in C$, consider the iterative sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ defined by (1.3) and (1.2) respectively. Then $\{x_n\}_{n=1}^{\infty}$ converges to p_* faster than $\{u_n\}_{n=1}^{\infty}$ does.*

Proof. From (3.5) in Theorem 3.1 we have the following inequality

$$\|x_{n+1} - p_*\| \leq \delta^{n+1} \prod_{i=0}^{n} \left(1 + \frac{(1-\alpha_i)(1-\delta)}{k\delta}\right) \|x_0 - p_*\|. \tag{3.9}$$

and also

$$\|u_{n+1} - p_*\| \leq \|u_0 - p_*\| \prod_{i=0}^{n} [1 - \alpha_i(1-\delta)]. \tag{3.10}$$

Applying assumption (i) to (3.9) and (3.10) respectively, we obtain

$$\begin{aligned}\|x_{n+1} - p_*\| &\leq \delta^{n+1} \prod_{i=0}^{n} \left(1 + \frac{(1-\alpha_1)(1-\delta)}{k\delta}\right) \|x_0 - p_*\| \\ &= \delta^{n+1} \|x_0 - p_*\| \left(1 + \frac{(1-\alpha_1)(1-\delta)}{k\delta}\right)^{n+1},\end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \|u_{n+1}-p_*\| &\leq \|u_0-p_*\|\prod_{i=0}^{n}[1-\alpha_1(1-\delta)] \\ &\leq \|u_0-p_*\|[1-\alpha_1(1-\delta)]^{n+1} \end{aligned} \tag{3.12}$$

Define,

$$a_n=\delta^{n+1}\|x_0-p_*\|\left(1+\frac{(1-\alpha_1)(1-\delta)}{k\delta}\right)^{n+1}$$

$$b_n=\|u_0-p_*\|[1-\alpha_1(1-\delta)]^{n+1},$$

and

$$\begin{aligned} \psi_n &= \frac{a_n}{b_n}=\frac{\delta^{n+1}\|x_0-p_*\|\left(1+\frac{(1-\alpha_1)(1-\delta)}{k\delta}\right)^{n+1}}{\|u_0-p_*\|[1-\alpha_1(1-\delta)]^{n+1},} \\ &= \delta^{n+1}\left(\frac{1+\frac{(1-\alpha_1)(1-\delta)}{k\delta}}{1-\alpha_1(1-\delta)}\right)^{n+1} \\ &= \left[\delta\left(\frac{1+\frac{(1-\alpha_1)(1-\delta)}{k\delta}}{1-\alpha_1(1-\delta)}\right)\right]^{n+1} \end{aligned}$$

Since $k\in\mathbb{N}$, $\delta\in(0,1)$ and $\alpha_1\leq 1$ we have

$$\begin{aligned} \delta\left(1+\frac{(1-\alpha_1)(1-\delta)}{k\delta}\right) &= \delta+\frac{(1-\alpha_1)(1-\delta)}{k} \\ &< \delta+(1-\alpha_1)(1-\delta) \\ &= 1-\alpha_1(1-\delta) \end{aligned}$$

That is $\psi_n<1$. Therefore $\lim_{n\to\infty}\psi_n=0$. From Definition 2.4, we obtain that $\{x_n\}_{n=1}^{\infty}$converges faster than $\{u_n\}_{n=1}^{\infty}$. □

In order to support the analytical proof of Theorem 3.3 and to demonstrate the efficiency of iterative scheme (1.3), we give a numerical example:

Example 3.4. *Let $X=\mathbb{R}$ and $C=[0,1)$. Let $T:C\to C$ be a mapping defined by $T(x)=\frac{1}{2}\cos x^2-\frac{1}{2}\sin x^2$ for all $x\in C$. T is a contraction with the contractivity factor $\delta\in[0.66,1)$. Also unique fixed point of this mapping is $p_*=0,40952291290289$. Choose $\alpha_n=\frac{1}{4}$ and $k=100$ with the initial value $x_0=0.99$. The following table shows that the new iterative scheme (1.3) converges faster than iterative scheme (1.2):*

Table 1: Comparison rate of convergence between iterative scheme (1.3) and iterative scheme (1.2) with initial value $x_0=0.99$.

Iteration Steps	Iterative Scheme (1.3)	Iterative Scheme (1.2)
1	0,99000000000000	0,99000000000000
2	0,40046287233997	0,42600277956979
3	0,40972848713874	0,41004313957591
⋮	⋮	⋮
9	0,40952291290292	0,40952291290339
10	0,40952291290289	0,40952291290291
11	0,40952291290289	0,40952291290289
⋮	⋮	⋮

Table 1 shows that new iterative scheme reaches to the fixed point at the 10^{th} step while iterative scheme (1.2) reaches at the 11^{th} step. The following figure is graphical presentation of the above result:

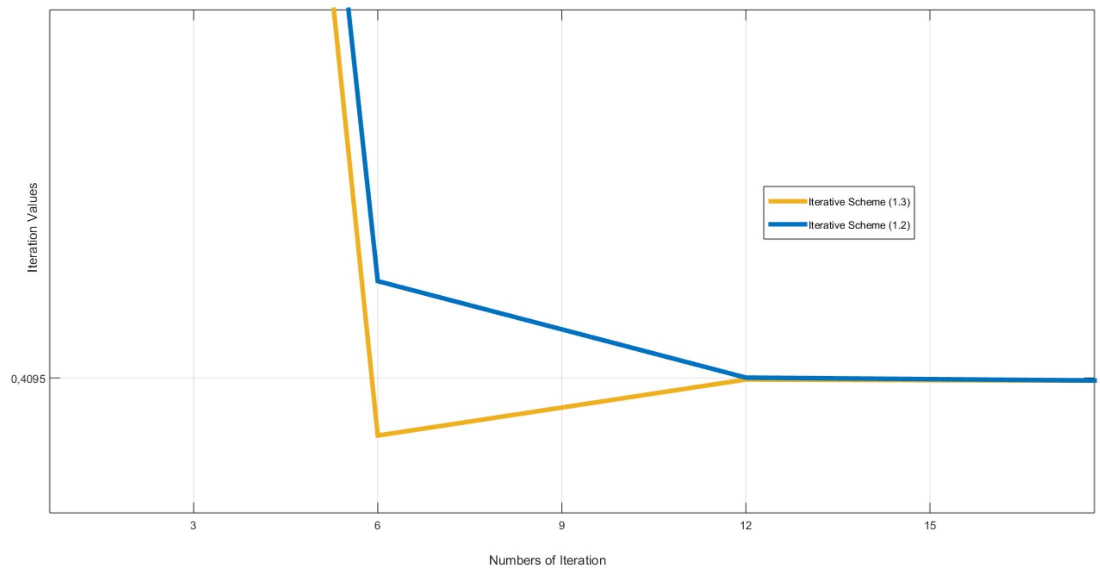

Figure 3.1: Graph of Iterative Scheme (1.3) and Iterative Scheme (1.2)

Theorem 3.5. *Let S be an approximate operator of T. Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1.3) for T and define an iterative sequence $\{u_n\}_{n=1}^{\infty}$ as follows:*

$$\begin{cases} u_1 \in C, \\ u_{n+1} = Sv_n \\ v_n = S\left(\frac{(1-\alpha_n)}{k} w_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Sw_n\right) \\ w_n = S\left(S(u_n)\right) \end{cases} \tag{3.13}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ be real sequence in $[0,1]$ satisfying $\frac{1}{2} \leq \frac{\alpha_n}{k}$ for all $n \in \mathbb{N}$. If $Tp_* = p_*$ and $Sx_* = x_*$ such that $u_n \to x_*$ as $n \to \infty$, then we have

$$\|p_* - x_*\| \leq \frac{10\varepsilon}{1-\delta},$$

where $\varepsilon > 0$ is a fixed number.

Proof. From (1.3) and (3.13), we have

$$\begin{aligned} \|z_n - w_n\| &= \|T(T(x_n)) - S(S(u_n))\| \\ &\leq \|T(T(x_n)) - T(S(u_n))\| + \|T(S(u_n)) - S(S(u_n))\| \\ &\leq \delta^2 \|x_n - u_n\| + (\delta+1)\varepsilon \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
\|y_n - v_n\| &= \Bigg\| T\left(\frac{(1-\alpha_n)}{k} z_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) T z_n\right) \\
&\quad - S\left(\frac{(1-\alpha_n)}{k} w_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) S w_n\right)\Bigg\| \\
&\leq \Bigg\| T\left(\frac{(1-\alpha_n)}{k} z_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) T z_n\right) \\
&\quad - T\left(\frac{(1-\alpha_n)}{k} w_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) S w_n\right)\Bigg\| \\
&\quad + \Bigg\| T\left(\frac{(1-\alpha_n)}{k} w_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) S w_n\right) \\
&\quad - S\left(\frac{(1-\alpha_n)}{k} w_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) S w_n\right)\Bigg\| \\
&\leq \delta \Bigg\| \frac{(1-\alpha_n)}{k} z_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) T z_n \\
&\quad - \left(\frac{(1-\alpha_n)}{k} w_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) S w_n\right)\Bigg\| + \varepsilon \\
&\leq \delta \frac{(1-\alpha_n)}{k} \|z_n - w_n\| \\
&\quad + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right) \|T z_n - S w_n\| + \varepsilon \\
&\leq \delta \frac{(1-\alpha_n)}{k} \|z_n - w_n\| + \delta^2 \left(1 - \frac{(1-\alpha_n)}{k}\right) \|z_n - w_n\| \\
&\quad + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right) \varepsilon + \varepsilon \\
&= \delta \left[\frac{(1-\alpha_n)}{k} + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right)\right] \|z_n - w_n\| \\
&\quad + \left[\delta\left(1 - \frac{(1-\alpha_n)}{k}\right) + 1\right] \varepsilon
\end{aligned} \tag{3.15}$$

Substituting (3.14) in (3.15), we obtain

$$\begin{aligned}
\|y_n - v_n\| &\leq \delta^3 \left[\frac{(1-\alpha_n)}{k} + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right)\right] \|x_n - u_n\| \\
&\quad + \delta(\delta+1) \left[\frac{(1-\alpha_n)}{k} + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right)\right] \varepsilon \\
&\quad + \left[\delta\left(1 - \frac{(1-\alpha_n)}{k}\right) + 1\right] \varepsilon
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \|T y_n - S v_n\| \\
&\leq \delta \|y_n - v_n\| + \varepsilon.
\end{aligned} \tag{3.17}$$

Substituting (3.16) in (3.17) we obtain

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leq \delta^4 \left[\frac{(1-\alpha_n)}{k} + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right)\right] \|x_n - u_n\| \\
&\quad + \delta^2(\delta+1) \left[\frac{(1-\alpha_n)}{k} + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right)\right] \varepsilon \\
&\quad + \delta\left[\delta\left(1 - \frac{(1-\alpha_n)}{k}\right) + 1\right] \varepsilon + \varepsilon
\end{aligned}$$

Since $\delta \in (0,1)$ $k \in \mathbb{N}$ and $\alpha_n \in [0,1]$ for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leq \left(1 - \frac{\alpha_n (1-\delta)}{k}\right) \|x_n - u_n\| \\
&\quad + \delta^2(\delta+1) \left[\frac{(1-\alpha_n)}{k} + \delta\left(1 - \frac{(1-\alpha_n)}{k}\right)\right] \varepsilon \\
&\quad + \delta\left[\delta\left(1 - \frac{(1-\alpha_n)}{k}\right) + 1\right] \varepsilon + \varepsilon
\end{aligned} \tag{3.18}$$

Also we have

$$\delta^2(\delta+1)\left[\frac{(1-\alpha_n)}{k}+\delta\left(1-\frac{(1-\alpha_n)}{k}\right)\right]\varepsilon<2\varepsilon,$$

and

$$\delta\left[\delta\left(1-\frac{(1-\alpha_n)}{k}\right)+1\right]\varepsilon\leq 2\varepsilon.$$

Moreover, from hypothesis, we obtain

$$1-\frac{\alpha_n}{k}\leq\frac{\alpha_n}{k}.$$

Hence, from (3.18) and the above inequalities, we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq \left(1-\frac{\alpha_n(1-\delta)}{k}\right)\|x_n-u_n\|+5\varepsilon\\ &\leq \left(1-\frac{\alpha_n(1-\delta)}{k}\right)\|x_n-u_n\|+\frac{\alpha_n(1-\delta)}{k}\frac{10\varepsilon}{(1-\delta)}\end{aligned}$$

Denote that,

$$\begin{aligned}a_n &= \|x_n-u_n\|,\\ \mu_n &= \frac{\alpha_n}{k}(1-\delta)\in(0,1),\\ \eta_n &= \frac{10\varepsilon}{(1-\delta)}.\end{aligned}$$

It follows from Lemma 2.2 that,

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}\|x_n-u_n\|\\ &\leq \limsup_{n\to\infty}\left\{\frac{10\varepsilon}{(1-\delta)}\right\}\\ &= \frac{10\varepsilon}{(1-\delta)}\end{aligned}$$

We know from Theorem 3.1 that $x_n\to p_*$ and using hypotesis, we obtain

$$\|p_*-x_*\|\leq\frac{10\varepsilon}{1-\delta}.$$

□

Example 3.6. *Let $C=[0,1]$ be endowed with usual metric. Define operator $T:C\to C$ by $Tx=\frac{e^x}{10}$. It is easy to check that T satisfies contraction condition with $\delta\in[0.17,1)$ and hence it has a unique fixed point $p_*=0.1118$. Define operator $S:C\to C$ by*

$$Sx=\frac{1}{10}+\frac{x-0.005}{10}-\frac{(x+0.005)^2}{20}-\frac{(x+0.8)^3}{60}+\frac{(x-0.01)^4}{240} \tag{3.19}$$

By utilizing Wolfram Mathematica 9 software package, we get

$$\max_{x\in C}|T-S|=0.168919.$$

Hence, for all $x\in C$ and for a fixed $\varepsilon=0.168919>0$, we have

$$|Tx-Sx|\leq 0.168919.$$

Thus, S is an approximate operator of T in the sense of Definition 2.5. Moreover, from (3.19) $u_*=0,09663289814977$ is the unique fixed point for the operator S in $C=[0,1]$. Hence the distance between two fixed points p_* and u_* is $|p_*-u_*|=0.0152$. If $Su=$

$\frac{1}{10}+\frac{u-0.005}{10}-\frac{(u+0.005)^2}{20}-\frac{(u+0.8)^3}{60}+\frac{(u-0.01)^4}{240}$ *and we put* $\alpha_n=\frac{1}{5}$ *and* $k=200$ *for all* $n\in\mathbb{N}$ *in (3.13), then we obtain*

$$\begin{cases} u_0\in C, \\ u_{n+1}=\frac{1}{10}+\frac{v_n-0.005}{10}-\frac{(v_n+0.005)^2}{20}-\frac{(v_n+0.8)^3}{60}-\frac{(v_n-0.01)^4}{240} \\ v_n=\frac{1}{10}+\frac{\left[\begin{array}{c}\frac{4}{1000}w_n+\frac{996}{10000}+\frac{996(w_n-0.005)}{10000}\\ -\frac{996(w_n+0.005)^2}{20000}-\frac{996(w_n+0.8)^3}{60000}+\frac{996(w_n-0.01)^4}{240000}-0.005\end{array}\right]}{10} \\ -\frac{\left[\begin{array}{c}\frac{4}{1000}w_n+\frac{996}{10000}+\frac{996(w_n-0.005)}{10000}\\ -\frac{996(w_n+0.005)^2}{20000}-\frac{996(w_n+0.8)^3}{60000}+\frac{996(w_n-0.01)^4}{240000}+0.005\end{array}\right]^2}{20} \\ -\frac{\left[\begin{array}{c}\frac{4}{1000}w_n+\frac{996}{10000}+\frac{996(w_n-0.005)}{10000}\\ -\frac{996(w_n+0.005)^2}{20000}-\frac{996(w_n+0.8)^3}{60000}+\frac{996(w_n-0.01)^4}{240000}+0.8\end{array}\right]^3}{60} \\ +\frac{\left[\begin{array}{c}\frac{4}{1000}w_n+\frac{996}{10000}+\frac{996(w_n-0.005)}{10000}\\ -\frac{996(w_n+0.005)^2}{20000}-\frac{996(w_n+0.8)^3}{60000}+\frac{996(w_n-0.01)^4}{240000}-0.01\end{array}\right]^4}{240} \\ w_n=\frac{1}{10}+\frac{\left[\begin{array}{c}\frac{1}{10}+\frac{u_n-0.005}{10}-\frac{(u_n+0.005)^2}{20}\\ -\frac{(u_n+0.8)^3}{60}+\frac{(u_n-0.01)^4}{240}-0.005\end{array}\right]}{10} \\ -\frac{\left[\begin{array}{c}\frac{1}{10}+\frac{u_n-0.005}{10}-\frac{(u_n+0.005)^2}{20}\\ -\frac{(u_n+0.8)^3}{60}+\frac{(u_n-0.01)^4}{240}+0.005\end{array}\right]^2}{20} \\ -\frac{\left[\begin{array}{c}\frac{1}{10}+\frac{u_n-0.005}{10}-\frac{(u_n+0.005)^2}{20}\\ -\frac{(u_n+0.8)^3}{60}+\frac{(u_n-0.01)^4}{240}+0.8\end{array}\right]^3}{60} \\ +\frac{\left[\begin{array}{c}\frac{1}{10}+\frac{u_n-0.005}{10}-\frac{(u_n+0.005)^2}{20}\\ -\frac{(u_n+0.8)^3}{60}+\frac{(u_n-0.01)^4}{240}-0.01\end{array}\right]^4}{240} \end{cases} \tag{3.20}$$

The following table shows that the sequence $\{u_n\}_{n=0}^{\infty}$ *generated by (3.20) converges to the fixed point* $u_*=0,09663289814977$.

Table 2: Convergence test for the iterative scheme (3.20) with initial value $u_0=1$.

Iteration Steps	Iterative Scheme 3.20
1	1
2	0,09663260932081
3	0,09663289814967
4	0,09663289814977

Then, we can find the following estimate,

$$|p_*-x_*|\leq\frac{10\times(0.168919)}{1-0.17}=2.03516.$$

4. Conclusion

In this paper, some fixed point theorems are obtained by defining a new iterative scheme. The obtained results show us, under contraction mappings, the new iterative scheme is faster than the others which are referred to in this paper. Some of these results have been supported by nontrivial examples with the help of programs such as MATLAB and MATHEMATICA. Consequently, iterative scheme (1.3) is the fastest one among three step iterative schemes in current literature.

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