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On the New Wirtinger Type Inequalities

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Abstract

The aim of this paper to establish some generalized and refinement of Wirtinger type inequality.

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1. Introduction

The classical Wirtinger inequality is given by

$$\int_a^b (f(x))^2 dx \leq \int_a^b (f'(x))^2 dx \tag{1.1}$$

for any $f \in C^1([a,b])$ satisfying $f(a) = f(b) = 0$ in [6]. Then, Beesack extended the inequality (1.1) and proved that

$$\int_a^b (f(x))^4 dx \leq \frac{4}{3}\int_a^b (f'(x))^4 dx$$

for any $f \in C^2([a,b])$ satisfying $f(a) = f(b) = 0$ in [5].
One of the most impressive issues in inequality theory is integral inequalities involving a function and its derivative. Wirtinger inequality in this area of the theorems has been a particular attraction due to close coupling to linear differential equations and differential geometry. Wirtinger's inequality compares the integral of a square of a function with that of a square of its first derivative. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Wirtinger's inequality and its generalizations, see [1]-[11]. The purpose of this paper is to establish some generalized and refinement of Wirtinger type inequalities.

2. Main Results

Now, we present the main results:

Theorem 2.1. *Let $f \in C^1([a,b])$ with $f(a) = f(b) = 0$ and $f' \in L_2[a,b]$, then, we have the following inequality*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^2}{6}\int_a^b [f'(x)]^2 dx.$$

Proof. From the hypotheses, we have

$$[f(x)]^2 \leq \left[\int_a^x f'(t)dt\right]^2, \quad \text{with } f(a) = 0$$

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$$[f(x)]^2 \leq \left[\int_x^b f'(t)dt\right]^2, \quad \text{with } f(b)=0.$$

By using the Cauchy-Shwartz inequality, we have

$$[f(x)]^2 \leq (x-a)\int_a^x \left[f'(t)\right]^2 dt \tag{2.1}$$

and

$$[f(x)]^2 \leq (b-x)\int_x^b \left[f'(t)\right]^2 dt. \tag{2.2}$$

By integrating both sides of the inequality (2.1) from a to $a\lambda+(1-\lambda)b$ for $\lambda \in [0,1]$, we get

$$\int_a^{a\lambda+(1-\lambda)b} [f(x)]^2 dx \leq \int_a^{a\lambda+(1-\lambda)b}\int_a^x (x-a)\left[f'(t)\right]^2 dtdx.$$

Then using Fubini's theorem it follows that

$$\int_a^{a\lambda+(1-\lambda)b} [f(x)]^2 dx \leq \frac{1}{2}\int_a^{a\lambda+(1-\lambda)b} \left[(1-\lambda)^2(b-a)^2-(t-a)^2\right]\left[f'(t)\right]^2 dt.$$

By the change of variable $t=au+(1-u)b$,on the right hand sides integrals, we get

$$\begin{aligned}&\int_a^{a\lambda+(1-\lambda)b} [f(x)]^2 dx \\ \leq\ & \frac{(b-a)^3}{2}\int_\lambda^1 \left[(1-\lambda)^2-(1-u)^2\right]\left[f'(au+(1-u)b)\right]^2 du.\end{aligned} \tag{2.3}$$

Similarly, by integrating both sides of the inequality (2.2) from $a\lambda+(1-\lambda)b$ to b for $\lambda \in [0,1]$, we get

$$\int_{a\lambda+(1-\lambda)b}^b [f(x)]^2 dx \leq \int_{a\lambda+(1-\lambda)b}^b\int_x^b (b-x)\left[f'(t)\right]^2 dtdx.$$

Then using Fubini's theorem it follows that

$$\int_{a\lambda+(1-\lambda)b}^b [f(x)]^2 dx \leq \frac{1}{2}\int_{a\lambda+(1-\lambda)b}^b \left[\lambda^2(b-a)^2-(b-t)^2\right]\left[f'(t)\right]^2 dt.$$

By the change of variable $t=au+(1-u)b$,on the right hand sides integrals, we get

$$\int_{a\lambda+(1-\lambda)b}^b [f(x)]^2 dx \leq \frac{(b-a)^3}{2}\int_0^\lambda \left[\lambda^2-u^2\right]\left[f'(au+(1-u)b))\right]^2 du. \tag{2.4}$$

Adding (2.3) and (2.4), it follows that

$$\begin{aligned}\int_a^b [f(x)]^2 dx \ &\leq\ \frac{(b-a)^3}{2}\int_\lambda^1 \left[(1-\lambda)^2-(1-u)^2\right]\left[f'(au+(1-u)b)\right]^2 du \\ &\quad +\frac{(b-a)^3}{2}\int_0^\lambda \left[\lambda^2-u^2\right]\left|f'(au+(1-u)b))\right|^2 du.\end{aligned}$$

By integrating both sides of the inequality from 0 to 1 with respect to λ, we get

$$\begin{aligned}\int_a^b [f(x)]^2 dx \ &\leq\ \frac{(b-a)^3}{2}\int_0^1\int_\lambda^1 \left[(1-\lambda)^2-(1-u)^2\right]\left[f'(au+(1-u)b)\right]^2 dud\lambda \\ &\quad +\frac{(b-a)^3}{2}\int_0^1\int_0^\lambda \left[\lambda^2-u^2\right]\left[f'(au+(1-u)b))\right]^2 dud\lambda.\end{aligned}$$

By using change order of the integrals, we have

$$\begin{aligned}\int_a^b [f(x)]^2\,dx &\le \frac{(b-a)^3}{2}\int_0^1\int_0^u \left[(1-\lambda)^2-(1-u)^2\right]\left[f'(au+(1-u)b)\right]^2 d\lambda du\\ &+\frac{(b-a)^3}{2}\int_0^1\int_u^1 \left[\lambda^2-u^2\right]\left[f'(au+(1-u)b))\right]^2 d\lambda du\\ &= \frac{(b-a)^3}{6}\int_0^1 \left[f'(au+(1-u)b))\right]^2 du,\end{aligned}$$

which is the desired inequality. □

Theorem 2.2. *Let $f\in C^1([a,b])$ with $f(a)=f(b)=0$, $p>1$, and $f'\in L_p[a,b]$, then, we have the following inequality*

$$\int_a^b |f(x)|^p\,dx \le \frac{(b-a)^{p-1}}{2^{p-1}p}\int_a^b \left|f'(x)\right|^p dx. \tag{2.5}$$

Proof. From the hypotheses, we have

$$|f(x)|^p \le \left(\int_a^x \left|f'(t)\right| dt\right)^p, \quad \text{with } f(a)=0$$

$$|f(x)|^p \le \left(\int_x^b \left|f'(t)\right| dt\right)^p, \quad \text{with } f(b)=0.$$

and hence from Hölder's inequality with indices p and $\frac{p}{p-1}$, it follows that

$$|f(x)|^p \le (x-a)^{p-1}\int_a^x \left|f'(t)\right|^p dt \tag{2.6}$$

and

$$|f(x)|^p \le (b-x)^{p-1}\int_x^b \left|f'(t)\right|^p dt. \tag{2.7}$$

By integrating both sides of the inequality (2.6) from a to $a\lambda+(1-\lambda)b$ for $\lambda\in[0,1]$, we get

$$\begin{aligned}&\int_a^{a\lambda+(1-\lambda)b} |f(x)|^p\,dx \qquad (2.8)\\ \le\ &\int_a^{a\lambda+(1-\lambda)b} (x-a)^{p-1}\int_a^x \left|f'(t)\right|^p dtdx\\ =\ &\frac{(x-a)^p}{p}\left|f'(x)\right|^p\Big|_a^{a\lambda+(1-\lambda)b} - \frac{1}{p}\int_a^{a\lambda+(1-\lambda)b} (x-a)^p\left|f'(x)\right|^p dx\\ \le\ &\frac{(1-\lambda)^p(b-a)^p}{p}\left|f'(a\lambda+(1-\lambda)b)\right|^p\end{aligned}$$

Similarly, by integrating both sides of the inequality (2.7) from $a\lambda+(1-\lambda)b$ to b for $\lambda\in[0,1]$, we get

$$\begin{aligned}&\int_{a\lambda+(1-\lambda)b}^{b} |f(x)|^p\,dx \qquad (2.9)\\ \le\ &\int_{a\lambda+(1-\lambda)b}^{b} (b-x)^{p-1}\int_x^b \left|f'(t)\right|^p dtdx\\ =\ &\frac{(b-x)^p}{p}\left|f'(x)\right|^p\Big|_{a\lambda+(1-\lambda)b}^{b} - \frac{1}{p}\int_{a\lambda+(1-\lambda)b}^{b} (b-x)^p\left|f'(x)\right|^p dx\\ \le\ &\frac{\lambda^p(b-a)^p}{p}\left|f'(a\lambda+(1-\lambda)b)\right|^p.\end{aligned}$$

Adding (2.8) and (2.9), it follows that

$$\int_a^b |f(x)|^p\,dx \le \frac{(b-a)^p}{p}\left[(1-\lambda)^p+\lambda^p\right]\left|f'(a\lambda+(1-\lambda)b)\right|^p .$$

By integrating both sides of the inequality from 0 to 1 with respect to λ, we get

$$\int_a^b |f(x)|^p\,dx \le \frac{(b-a)^p}{p}\int_0^1 \left[(1-\lambda)^p+\lambda^p\right]\left|f'(a\lambda+(1-\lambda)b)\right|^p d\lambda .$$

It is not difficult to reveal that the function

$$h(u)=(1-\lambda)^p+\lambda^p$$

for all $\lambda\in[0,1]$ attains its maximum $\frac{1}{2^{p-1}}$ at the point $\lambda=\frac{1}{2}\in[0,1]$ and using the change of variable $u=a\lambda+(1-\lambda)b$, which is the same as (2.5). This completes the proof. □

Theorem 2.3. *Let $f,g\in C^1([a,b])$ with $f(a)=f(b)=0$, $g(a)=g(b)=0$, and $f',g'\in L_2[a,b]$, then, we have the following inequality*

$$\int_a^b |f(x)|\,|g(x)|\,dx \le \frac{(b-a)^2}{8}\int_a^b \left[\left|f'(x)\right|^2+\left|g'(x)\right|^2\right]dx. \tag{2.10}$$

Proof. From the hypotheses, we have

$$f(x)=\frac{1}{2}\left(\int_a^x f'(t)dt-\int_x^b f'(t)dt\right) \tag{2.11}$$

and

$$g(x)=\frac{1}{2}\left(\int_a^x g'(t)dt-\int_x^b g'(t)dt\right). \tag{2.12}$$

Using the properties of modulus we have

$$|f(x)|\le\frac{1}{2}\left(\int_a^b \left|f'(t)\right|dt\right), \tag{2.13}$$

$$|g(x)|\le\frac{1}{2}\left(\int_a^b \left|g'(t)\right|dt\right). \tag{2.14}$$

Multiplying the left sides and right sides of (2.13) and (2.14) and then integrating both sides of the inequality from a to b with respect to x , we get

$$\int_a^b |f(x)|\,|g(x)|\,dx \le \frac{(b-a)}{4}\left(\int_a^b \left|f'(t)\right|dt\right)\left(\int_a^b \left|g'(t)\right|dt\right).$$

By using the Cauchy-Shwartz inequality, and then using elemantary inequality $\sqrt{mn}\le\frac{1}{2}(m+n)$, $m,n\ge 0$, we have

$$\begin{aligned}\int_a^b |f(x)|\,|g(x)|\,dx &\le \frac{(b-a)^2}{4}\left(\int_a^b \left|f'(t)\right|^2 dt\right)^{\frac{1}{2}}\left(\int_a^b \left|g'(t)\right|^2 dt\right)^{\frac{1}{2}}\\ &\le \frac{(b-a)^2}{8}\int_a^b\left[\left|f'(x)\right|^2+\left|g'(x)\right|^2\right]dx\end{aligned}$$

which is the desired inequality. □

Remark 2.4. *By taking $f=g$ and $f'=g'$ in Theorem 2.3, we have*

$$\int_a^b [f(x)]^2\,dx \le \frac{(b-a)^2}{4}\int_a^b \left[f'(x)\right]^2 dx.$$

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