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Surface Area Computation Established by Cubic Spline Functions

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Abstract

The genesis of open surfaces were made of with one dimensional cubic hermite spline functions extending to two directions on previous studies. A calculus algorithm for approximately finding the area of the open rectangular surfaces created before in this study were given. The results of the application examples were provided at a computer software developed.

Keywords: *cubic spline, two dimensional splines, open surface area*
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1. Introduction

In computational applications of applied mathematics, spline functions are significant. Creation of one-dimensional spline functions was given by Graphic Constructor [1]. The cubic spline functions given in [1] study were extended in two directions to form the solution visualision of the Cauchy problem for linear one dimensional t-hyperbolic PDE in [4]. The value of any point on the surface was computed in [5] with help of the partial derivative values defined in the cardinal points that characterize the resulting open surface. This computation will help in this study. Recall of previous constructions and computations will continue until the end of the section.
$a,b,c,d \in \mathbf{R}$ and $\Omega = [a,b] \times [c,d]$, consider the rectangle on tOx plane as Ω region. For the

$a = t_0 < t_1 < \cdots < t_i < \cdots < t_{m-1} = b, \quad m \geq 1$ and

$c = x_0 < x_1 < \cdots < x_j < \cdots < x_{n-1} = d, \quad n \geq 1$

$i = 0,1,\cdots,m-1, \quad j = 0,1,\cdots,n-1$

Ω region divided into $(n-1)\times(m-1)$ subregions that

$$\omega_{i,j} = \{(t,x) : t_i \leq t \leq t_{i+1},\ x_j \leq x \leq x_{j+1}\} \tag{1.1}$$

$i = 0,1,\cdots,m-2,\ j = 0,1,\cdots,n-2.$

For any $\omega_{i,j}$ subregion have cardinal points as ω_{t_i,x_j}, ω_{t_{i+1},x_j}, $\omega_{t_{i+1},x_{j+1}}$, $\omega_{t_i,x_{j+1}}$. The cardinal points of each $\omega_{i,j}$ subregion defines a grid Ω_{grd}. Been presented a function $\lambda : \Omega_{grd} \to \mathbf{R}$ on the grid extended on the Ω region in[4]. At that rate,

$$\mathrm{U} = \{u_{(0,0)},\ u_{(0,1)},\cdots,\ u_{(0,n-1)},\ u_{(1,0)},\cdots,\ u_{(m-1,n-1)}\},$$

$$\mathrm{G}_T = \{g_{T\,(0,0)},\ g_{T\,(0,1)},\cdots,\ g_{T\,(0,n-1)},\ g_{T\,(1,0)},\cdots,\ g_{T\,(m-1,n-1)}\},$$

$$\mathrm{G}_X = \{g_{X\,(0,0)},\ g_{X\,(0,1)},\cdots,\ g_{X\,(0,n-1)},\ g_{X\,(1,0)},\cdots,\ g_{X\,(m-1,n-1)}\},$$

$$u_{(i,j)} \in \mathbf{R},\ g_{T\,(i,j)} \in \mathbf{R},\ g_{X\,(i,j)} \in \mathbf{R},$$

$$\lambda(t_i,x_j) = u_{(i,j)},\ \lambda_t'(t_i,x_j) = g_{T\,(i,j)},\ \lambda_x'(t_i,x_j) = g_{X\,(i,j)},$$

$$f : \Omega \to \mathbf{R},\ f(t_i,x_j) = u_{(i,j)},\ \lambda(t_i,x_j) = f(t_i,x_j)$$

$$i = 0,1,\cdots,m-1,\ j = 0,1,\cdots,n-1.$$

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Computation of $f:\Omega\to\mathbf{R},\ f(t,x)$ differentiable real functions were given in [4, 5].

$S(t,x_0),\ S(t,x_1),\ S(t,x_2),\dots,\ S(t,x_{n-1}),\qquad t_0\le t\le t_{n-1}$

$H(t_0,x),\ H(t_1,x),\ H(t_2,x),\dots,\ H(t_{m-1},x),\qquad x_0\le x\le x_{m-1}$

$S(t,x_j),\ j=0,1,\cdots,n-1,\ t_0\le t\le t_{m-1}$ describe direction of t spline functions and $H(t_i,x),\ i=0,1,\cdots,m-1,\ x_0\le x\le x_{n-1}$ describe direction of x spline functions [5]. Let been considered that

$$T=\{t_i\,|\,t_i<t_{i+1},\, i=0,1,\cdots,m-1,\, m\ge 1\},$$

$$X=\{x_j\,|\,x_j<x_{j+1},\, j=0,1,\cdots,n-1,\, n\ge 1\}.$$

Superscript $^{(T)}$ and $^{(X)}$ notations will point out that related values on direction of spline functions.
U and G_T datasets provides for $j=0,1,\cdots,n-1$, n pieces

$$U_j^{(T)}=\{u_i^{(T)j}=u_{(i,j)}\,|\,i=0,1,\cdots,m-1\},$$

$$G_j^{(T)}=\{g_i^{(T)j}=g_{T_{(i,j)}}\,|\,i=0,1,\cdots,m-1\}$$

vectors for each $S\left(t,x_j\right)$ spline functions direction of t.
Furthermore,U with G_X dataset provides for $i=0,1,\cdots,m-1$, m pieces

$$U_i^{(X)}=\{u_j^{(X)i}=u_{(i,j)}\,|\,j=0,1,\cdots,n-1\},$$

$$G_i^{(X)}=\{g_j^{(X)i}=g_{X_{(i,j)}}\,|\,j=0,1,\cdots,n-1\}$$

vectors for each $H(t_i,x)$ spline functions direction of x. Obvious that $u_i^{(T)j}=u_j^{(X)i}$. Hereby, the $m+n$ pieces one-dimensional spline functions be calculated. Differentiable real-valued function *cubicSPL* was submitted in detail in [1, 4, 5]. In this instance, one dimensional computations are

$$s(t)=CubicSPL(T,U_r^{(T)},G_r^{(T)},t),\ t\in[t_0,t_{m-1}],\ r\in\{j\,|\,j=0,1,\cdots,n-1\},$$

$$h(x)=CubicSPL(X,U_p^{(X)},G_p^{(X)},x),\ x\in[x_0,x_{n-1}],\ p\in\{i\,|\,i=0,1,\cdots,m-1\}.$$

Let been that $\tau\in(t_p,\,t_{p+1})$ and $\xi\in(x_r,\,x_{r+1})$, the value of (τ,ξ) of the surface can be computed in two different layouts. First layout is

$$u_j^{(X)aux}=s_j(\tau)=CubicSPL(T,U_j^{(T)},G_j^{(T)},\tau),\quad j=0,1\cdots,n-1,$$

$$g_j^{(X)aux}=g_j^{(X)p}\,\frac{t_{p+1}-\tau}{t_{p+1}-t_p}+g_j^{(X)p+1}\,\frac{|t_p-\tau|}{t_{p+1}-t_p},\quad j=0,1\cdots,n-1,$$

$$h(\xi)=CubicSPL(X,U_{aux}^{(X)},G_{aux}^{(X)},\xi).\tag{1.2}$$

Second layout is

$$u_i^{(T)aux}=h_i(\xi)=CubicSPL(X,U_i^{(X)},G_i^{(X)},\xi),\quad i=0,1\cdots,m-1,$$

$$g_i^{(T)aux}=g_i^{(T)r}\,\frac{x_{r+1}-\xi}{x_{r+1}-x_r}+g_i^{(T)r+1}\,\frac{|x_r-\xi|}{x_{r+1}-x_r},\quad i=0,1\cdots,m-1,$$

$$s(\tau)=CubicSPL(T,U_{aux}^{(T)},G_{aux}^{(T)},\tau).\tag{1.3}$$

$h(\xi)$ at equation(1.2) and $s(\tau)$ at equation(1.3) offers identical approximations to $f(\tau,\,\xi)$.

2. Open Surface Area

This section will focus on surface area of the surface defined in the previous section. Let been $p\in\{i\,|\,i=0,1,\cdots,m-2\}$, $r\in\{j\,|\,i=0,1,\cdots,n-2\}$, when taken into account the particular one $\omega_{p,r}$ subregions that belong to Ω(1.1), the $v,\widehat{v},w,\widehat{w}\in\mathbf{R}^3$ vectors' formation that depending on the values representing by the cardinal points are

$$v=\begin{pmatrix} t_{p+1}-t_p \\ 0 \\ u_{p+1}^{(T)r}-u_p^{(T)r}\end{pmatrix},\qquad \widehat{v}=\begin{pmatrix} 0 \\ x_{r+1}-x_r \\ u_{r+1}^{(X)p}-u_r^{(X)p}\end{pmatrix},$$

$$w=\begin{pmatrix} t_p-t_{p+1} \\ 0 \\ u_p^{(T)r+1}-u_{p+1}^{(T)r+1}\end{pmatrix},\qquad \widehat{w}=\begin{pmatrix} 0 \\ x_r-x_{r+1} \\ u_r^{(X)p+1}-u_{r+1}^{(X)p+1}\end{pmatrix}.$$

$\mathscr{E}_\Omega$ is represent the surface over establish by $f:\Omega\to\mathbf{R}$ function on Ω region. $\mathscr{E}_{\omega_{p,r}}$ is the surface piece that relation with $\omega_{p,r}$. $\|\cdot\|$ will be denote the Euclidean norm of a vector and $\langle\cdot,\cdot\rangle$ will be denote vector inner product. $q=v-\widehat{v}$, $\|v\|,\|\widehat{v}\|,\|q\|$ are sides lengths of a triangle. Vertices of triangle are junction with surface at $f(t_p,x_r)$, $f(t_{p+1},x_r)$, $f(t_p,x_{r+1})$. $-q=w-\widehat{w}$, $\|w\|,\|\widehat{w}\|,\|q\|$ are sides lengths of other triangle, it's vertices are adjacent at surface at $f(t_{p+1},x_{r+1})$, $f(t_p,x_{r+1})$, $f(t_{p+1},x_r)$.

$$\Pi^{(1)}(\mathscr{E}_{\omega_{p,r}})\approx\frac{1}{2}\left(\|v\|\,\|\widehat{v}\|\sqrt{1-\left(\frac{\langle v,\widehat{v}\rangle}{\|v\|\,\|\widehat{v}\|}\right)^2}+\|w\|\,\|\widehat{w}\|\sqrt{1-\left(\frac{\langle w,\widehat{w}\rangle}{\|w\|\,\|\widehat{w}\|}\right)^2}\right)$$

gives rudimentary approximation to area calculation of $\mathscr{E}_{\omega_{p,r}}$ which about $\omega_{p,r}$.
Another approach can be made as follows,
Let been $v^{(1)}$, $v^{(2)}$, $v^{(3)}$, $v^{(4)}\in\mathbf{R}^3$, these vectors' generation are that connected to the values represented by the cardinal points as

$$t_\mu=\frac{t_{p+1}+t_p}{2},\ x_\mu=\frac{x_{r+1}+x_r}{2},\ u_\mu=f\left(t_\mu,x_\mu\right),$$

$$v^{(1)}=\begin{pmatrix}-t_\mu\\ x_\mu\\ u_p^{(T)r+1}-u_\mu\end{pmatrix},\qquad v^{(2)}=\begin{pmatrix}t_\mu\\ x_\mu\\ u_{p+1}^{(T)r+1}-u_\mu\end{pmatrix},$$

$$v^{(3)}=\begin{pmatrix}t_\mu\\ -x_\mu\\ u_{p+1}^{(T)r}-u_\mu\end{pmatrix},\qquad v^{(4)}=\begin{pmatrix}-t_\mu\\ -x_\mu\\ u_p^{(T)r}-u_\mu\end{pmatrix}.$$

Area of $\mathscr{E}_{\omega_{p,r}}$ related with $\omega_{p,r}$ is as follows

$$\begin{aligned}\Pi^{(2)}(\mathscr{E}_{\omega_{p,r}})\approx\frac{1}{2}\Bigg(&\left\|v^{(1)}\right\|\left\|v^{(2)}\right\|\sqrt{1-\left(\frac{\left\langle v^{(1)},v^{(2)}\right\rangle}{\|v^{(1)}\|\,\|v^{(2)}\|}\right)^2}\\ &+\left\|v^{(2)}\right\|\left\|v^{(3)}\right\|\sqrt{1-\left(\frac{\left\langle v^{(2)},v^{(3)}\right\rangle}{\|v^{(2)}\|\,\|v^{(3)}\|}\right)^2}\\ &+\left\|v^{(3)}\right\|\left\|v^{(4)}\right\|\sqrt{1-\left(\frac{\left\langle v^{(3)},v^{(4)}\right\rangle}{\|v^{(3)}\|\,\|v^{(4)}\|}\right)^2}\\ &+\left\|v^{(4)}\right\|\left\|v^{(1)}\right\|\sqrt{1-\left(\frac{\left\langle v^{(4)},v^{(1)}\right\rangle}{\|v^{(4)}\|\,\|v^{(1)}\|}\right)^2}\Bigg).\end{aligned}$$

Thus, overall area of the $\mathscr{E}_\Omega$ surface congruous with region Ω is

$$\Pi(\mathscr{E}_\Omega)\approx\sum_{i=0}^{m-2}\sum_{j=0}^{n-2}\Pi^{(2)}(\mathscr{E}_{\omega_{i,j}}).$$

3. Splitting to segments

For the purpose of better approximation, each subregions has to split to more $\widehat{\omega}$ segments by step widths given as

$$L^{(T)},L^{(X)}\in\mathbf{Z},\quad L^{(T)}\geq 1,\ L^{(X)}\geq 1,\quad \eta^{(T)}=\frac{t_{i+1}-t_i}{L^{(T)}},\quad \eta^{(X)}=\frac{x_{j+1}-x_j}{L^{(X)}}.$$

$$\left.\begin{aligned}&t_k=t_i+\eta^{(T)}k,\\ &x_l=x_j+\eta^{(X)}l,\\ &u_k^{(T)l}=f(t_k,x_l),\end{aligned}\right\}\begin{aligned}&k=0,1,\cdots,L^{(T)}-1,\\ &l=0,1,\cdots,L^{(X)}-1.\end{aligned}$$

$$\Pi^{(2)}(\mathscr{E}_{\omega_{i,j}})\approx\sum_{k=0}^{L^{(T)}-2}\sum_{l=0}^{L^{(X)}-2}\Pi^{(2)}(\mathscr{E}_{\widehat{\omega}_{k,l}}).$$

Example 3.1. *Considering that*

$T=[1,2,4,5,6]$, $X=[0.5,2,3,5]$,

$$U=\left\{\begin{array}{rrrrr} 1, & -1, & 2, & 1, & 3 \\ -1, & 1, & 2, & 2, & 1 \\ 2, & 1, & -2, & 3, & 2 \\ 4, & 3, & 2, & 2, & 4 \end{array}\right\},\ G_T=\left\{\begin{array}{rrrrr} -2, & 0, & 0, & 0, & 0 \\ 1, & 1.5, & -0.4, & -1, & 0 \\ 1, & 0, & 0, & 0.5, & -0.5 \\ 1.5, & 0.5, & 0.2, & 0.4, & -1 \end{array}\right\}$$

and $G_X=\left\{\begin{array}{rrrrr} -1.3 & 1.4 & 0.9 & 0.7 & -0.1 \\ 0 & 1.2 & 0 & 0.8 & 0 \\ 0 & 0.3 & 0 & 0.1 & 0.1 \\ -0.5 & 0 & 1.3 & -0.4 & -0.3 \end{array}\right\}$ *datums are represent an open surface, table (1) is comprised results for several selected* $L^{(T)}$ *and* $L^{(X)}$ *values.*

Table 1: Computation results for several selected $L^{(T)}$ and $L^{(X)}$ values of example (3.1).

$L^{(T)}$	$L^{(X)}$	Area	$L^{(T)}$	$L^{(X)}$	Area	$L^{(T)}$	$L^{(X)}$	Area
1	1	46.69228	11	11	50.06450	37	37	50.11714
1	2	47.59803	12	12	50.07358	38	38	50.11741
2	1	48.03892	13	13	50.08069	39	39	50.11766
2	2	48.80278	14	14	50.08636	50	50	50.11949
2	3	49.40648	15	15	50.09094	55	55	50.11999
3	2	49.66742	16	16	50.09471	60	60	50.12036
3	3	49.44832	17	17	50.09784	70	70	50.12089
3	4	49.66095	18	18	50.10047	80	80	50.12123
4	3	49.75763	29	29	50.11389	90	90	50.12146
4	4	49.72240	30	30	50.11444	91	91	50.12148
5	5	49.85829	31	31	50.11494	92	92	50.12150
6	6	49.93540	32	32	50.11539	93	93	50.12152
7	7	49.98317	33	33	50.11581	94	94	50.12153
8	8	50.01477	34	34	50.11619	100	100	50.12163
9	9	50.03674	35	35	50.11653	200	200	50.12216
10	10	50.05263	36	36	50.11685	300	300	50.12226

Example 3.2.

$T=[-1,1]$, $X=[-1,1]$, $U=\left\{\begin{array}{rr} -1, & 1 \\ 1, & -1 \end{array}\right\}$,

$G_T=\left\{\begin{array}{rr} 1, & 1 \\ -1, & -1 \end{array}\right\}$, $G_X=\left\{\begin{array}{rr} 1, & -1 \\ 1, & -1 \end{array}\right\}$,

Given these datums, according to the different selected segmentations quantity, few results are shown in table (2).

Table 2: A few results for chosen different $L^{(T)}$ and $L^{(X)}$ values.

$L^{(T)}$	$L^{(X)}$	Area	$L^{(T)}$	$L^{(X)}$	Area	$L^{(T)}$	$L^{(X)}$	Area
1	1	5.65685	10	10	5.12755	95	95	5.12321
1	2	5.41421	30	30	5.12364	100	100	5.12320
2	1	5.41421	50	50	5.12333	200	200	5.12317
2	2	5.23607	70	70	5.12325	300	300	5.12316

Example 3.3.

$T=[-1,1]$, $X=[-1,1]$, $U=\left\{\begin{array}{rr} -1, & 1 \\ 1, & -1 \end{array}\right\}$,

$G_T=\left\{\begin{array}{rr} -1, & -1 \\ 1, & 1 \end{array}\right\}$, $G_X=\left\{\begin{array}{rr} -1, & 1 \\ -1, & 1 \end{array}\right\}$,

Here handled same segmentations quantity with previous example (3.2), few results are shown in the table (3).

Table 3: A few results for chosen different $L^{(T)}$ and $L^{(X)}$ values.

$L^{(T)}$	$L^{(X)}$	Area	$L^{(T)}$	$L^{(X)}$	Area	$L^{(T)}$	$L^{(X)}$	Area
1	1	5.65685	10	10	6.92092	95	95	6.94262
1	2	5.41421	30	30	6.94042	100	100	6.94264
2	1	5.41421	50	50	6.94198	200	200	6.94281
2	2	6.36003	70	70	6.94241	300	300	6.94284

4. Conclusion

When the vertices points and the middle point of quadrilateral partitioned surface particle is considered, the areas of the four contiguous triangles overlapping by these points were computed. This treat was repeated for all portions.
In synopsis, this exertion is an approach to the

$$\Pi(\mathscr{E}_\Omega) = \int\int_\Omega \sqrt{\left(\frac{\partial f}{\partial t}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2 + 1}\, dt\, dx$$

calculus for the function $f : \Omega \to \mathbf{R}$ defined by cubic spline functions except definition of f as $f(t,x) < 0$.
Example results were provided by a computer console application that evolved in the course of this study. This console application attainable of *http://www.oguzersinan.net.tr/softwares* address.

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