Konuralp Journal of Mathematics, 7 (1) (2019) 122-127

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

D_a-Homothetic Deformation and Ricci Solitons in $(k,\mu)-$ Contact Metric Manifolds

H. G. Nagaraja[1], D. L. Kiran Kumar[1] and D. G. Prakasha[2*]

[1]*Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bengaluru – 560 056, India*
[2]*Department of Mathematics, Karnatak University, Dharwad– 580003, India*
[*]*Corresponding author E-mail: prakashadg@gmail.com*

Abstract

In this paper, we study (k,μ)-contact metric manifold under D_a-homothetic deformation. It is proved that a D_3-homothetic deformed locally symmetric $(1,-4)$-contact metric manifold is a Sasakian manifold and the Ricci soliton is shrinking. Further, ξ^*-projectively flat and h-projectively semisymmetric (k,μ)-contact metric manifolds under D_a-homothetic deformation are studied and obtained interesting results.

Keywords: *D_a-homothetic deformation; Ricci solitons; (k,μ)-contact metric manifolds; projective curvature tensor.*
2010 Mathematics Subject Classification: *53D10; 53D15.*

1. Introduction

In modern mathematics the study of contact manifolds has become a matter of growing interest due to its role in explaining physical phenomena in the context of mathematical physics. In 1995, Blair[1] presented the thought of contact metric manifolds for which the characteristic vector field ξ belongs to the (k,μ)-nullity distribution for some real numbers k and μ. Such manifolds are known as (k,μ)-contact metric manifolds. The class of (k,μ)-contact metric manifolds encases both Sasakian and non-Sasakian structures. A full classification of (k,μ)-contact metric manifolds was given by Boeckx [4]. (k,μ)-contact metric manifolds are invariant under D_a-homothetic transformation. It is noted that the class of space acquired through D_a-homothetic deformation [19] is a contact metric manifold whose curvature satisfying $R(X,Y)\xi = 0$. (k,μ)-contact metric manifolds have been studied widely by several authors such as ([15, 3, 5, 6, 7, 8, 9, 13, 14, 16, 17, 21]) and numerous others.
Ricci soliton, introduced by Hamilton [10] are natural generalizations of the Einstein metrics, and is defined on a Riemannian manifold (M,g). A Ricci soliton (g,V,λ) defined on (M,g) as

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

where L_V denotes the Lie-derivative of Riemannian metric g along a vector field V, λ is a constant and X,Y are arbitrary vector fields on M. A Ricci soliton is said to shrinking or steady or expanding to the extent that λ is negative, zero or positive respectively. Ricci solitons have been studied extensively in the context of contact geometry; we may refer to [11, 20]) and references therein.
The paper is organized as follows: after a short audit of (k,μ) contact metric manifold in section 2, we study D_a-homothetic deformation and Ricci soliton in a (k,μ) contact metric manifolds in section 3.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be contact if it has a global 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ on M. Given a contact 1-form η there always exists a unique vector field ξ such that $(d\eta)(\xi,X) = 0$. Polarization of $d\eta$ on the contact subbundle D (defined by $D = 0$), yields a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\phi^2 X = -X + \eta(X)\xi, \;\; \phi\xi = 0, \;\; g(X,\xi) = \eta(X), \;\; \eta(\xi) = 1, \;\; \eta\circ\phi = 0, \tag{2.1}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

Email addresses: **hgnraj@yahoo.com (H. G. Nagaraja), kirankumar250791@gmail.com (D. L. Kiran Kumar), prakashadg@gmail.com (D. G. Prakasha)**

$$g(X,\phi Y) = d\eta(X,Y), \quad g(X,\phi Y) = -g(Y,\phi X), \tag{2.3}$$

for all vector fields X,Y on M. In a contact metric manifold, we characterize a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ signifies the Lie differentiation. At this point h is symmetric and satisifies $h\phi = -\phi h$. Likewise we have $Tr\cdot h = Tr\cdot\phi h = 0$ and $h\xi = 0$.
Moreover, if ∇ signifies the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi hX. \tag{2.4}$$

For a contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, the (k,μ))-nullity distribution is

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) = \{Z \in T_pM | R(X,Y)Z = \\ k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{2.5}$$

for any $X,Y \in T_pM$ and for some real numbers k and μ, R is the curvature tensor. Hence if the characteristic vector field $\xi \in N(k,\mu)$, then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[(\eta(Y)hX - \eta(X)hY)]. \tag{2.6}$$

Thus, a contact metric manifold satisfying relation (2.6) is known as a (k,μ)-contact metric manifold [1]. On (k,μ)-contact metric manifold, we have $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k,μ)-nullity condition completely determines the curvature of M^{2n+1} [1]. Actually, for a (k,μ)-contact manifold the condition of being Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent. In particular, if $\mu = 0$, then the (k,μ)-nullity distribution $N(k,\mu)$ is reduced to k-nullity distribution $N(k)$ [18]. If $\xi \in N(k)$, then we call contact metric manifold M is an $N(k)$- contact metric manifold [2]. A D_a-homothetic deformation:

$$\phi^* = \phi,\ \xi^* = \frac{1}{a}\xi,\ \eta^* = a\eta,\ g^* = ag + a(a-1)\eta\otimes\eta, \tag{2.7}$$

for a positive real constant a, deforms a contact metric structure into another contact metric structure and preserves Sasakian, K-contact and (k,μ)-contact structures. However the form of the (k,μ) nullity condition is preserved under a D_a-homothetic deformation with

$$k^* = \frac{k + a^2 - 1}{a^2}, \qquad \mu^* = \frac{\mu + 2a - 2}{a}. \tag{2.8}$$

In a (k,μ)-contact metric manifold the following relations hold [12] [1];

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.9}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.10}$$

$$\begin{aligned} (\nabla_X h)Y &= [(1-k)g(X,\phi Y) - g(X,\phi hY)]\xi \\ &- \eta(Y)[(1-k)\phi X + \phi hX] - \mu\eta(X)\phi hY, \end{aligned} \tag{2.11}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.12}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.13}$$

$$\begin{aligned} S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &+ [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \quad n \geq 1, \end{aligned} \tag{2.14}$$

$$r = 2n[2n - 2 + k - n\mu], \tag{2.15}$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci-operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. From (2.4), it follows that

$$(\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.16}$$

Lemma 2.1. *[1] A $(2n+1)$ dimensional contact metric manifold $M(\phi,\xi,\eta,g)$ with ξ belonging to (k,μ)-nullity distribution. Then for any vector fields X,Y,Z*

$$\begin{aligned} &R(X,Y)hZ - hR(X,Y)Z \\ &= [k\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\} \\ &+ \mu(k-1)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}]\xi \\ &+ k[g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY + g(Z,\phi hY)\phi X] - g(Z,\phi hX)\phi Y \\ &+ \eta(Z)\{\eta(X)hY - \eta(Y)hX\}] \\ &+ \mu[(k-1)\eta(Z)\{\eta(Y)X - \eta(X)Y\} + 2g(\phi X,Y)\phi hZ]. \end{aligned} \tag{2.17}$$

3. D-homothetic deformation and Ricci solitons in $(k,\mu)-$ Contact metric manifolds

Throughout this paper the quantities with $*$ signify the quantities in $(M,\phi^*,\xi^*,\eta^*,g^*)$ and quantities without $*$ are from (M,ϕ,ξ,η,g). The connection between the associations ∇ and ∇^* is given by

$$\nabla^*_X Y = \nabla_X Y + (1-a)[\eta(Y)\phi X + \eta(X)\phi Y] + \frac{(1-a)}{a} g(\phi hX, Y)\xi, \tag{3.1}$$

for any vector fields X,Y on M. We now calculate the Riemann curvature tensor R^* of $(M,\phi^*,\xi^*,\eta^*,g^*)$ as follows:

$$R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z. \tag{3.2}$$

Using (2.4), (2.7), (2.9), (2.10), (2.11) and (3.1) in (3.2), we obtain

$$\begin{aligned} R^*(X,Y)Z = {} & R(X,Y)Z + (1-a)[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & + 2\eta(X)\eta(Z)hY - 2\eta(Y)\eta(Z)hX + 2g(\phi Y,X)\phi Z \\ & + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi] + \frac{(1-a)}{a}[2\eta(Y)g(hX,Z)\xi \\ & - 2\eta(X)g(hY,Z)\xi + (1-k)\{\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi\} \\ & + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX] \\ & + (a^2-1)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \end{aligned} \tag{3.3}$$

for any vector fields X,Y,Z on M.
On contracting (3.3), we get the Ricci tensor S^* of D_a-homothetically deformed $(k,\mu)-$ contact metric manifolds as

$$\begin{aligned} S^*(Y,Z) = {} & aS(Y,Z) + (a-1)[\{a^2-2a-k+1\}g(Y,Z) \\ & + \{2na^2+2na+2a-a^2+k-1\}\eta(Y)\eta(Z) + a\{2+\mu\}g(hY,Z)]. \end{aligned} \tag{3.4}$$

First, assume that M is a locally symmetric $(k,\mu)-$ contact metric manifold under D_a-homothetic deformation. Then we have

$$(\nabla^*_X R^*)(Y,Z)W = 0. \tag{3.5}$$

On a suitable contraction of (3.5), we have

$$(\nabla^*_X S^*)(Z,W) = \nabla^*_X S^*(Z,W) - S^*(\nabla^*_X Z, W) - S^*(Z, \nabla^*_X W) = 0. \tag{3.6}$$

Taking $W=\xi^*$ in the above equation yields

$$\nabla^*_X S^*(Z,\xi^*) - S^*(\nabla^*_X Z,\xi^*) - S^*(Z,\nabla^*_X\xi^*) = 0. \tag{3.7}$$

Making use of (2.1), (2.4), (3.1) and (3.4) in (3.7), we obtain

$$a^2 S(Z,\phi X) + aS(Z,\phi hX) + Ag(X,\phi Z) + Bg(\phi hX, Z) = 0, \tag{3.8}$$

where
$A = 2na\{k+a^2-1\} - a(a-1)\{a^2-2a-k+1\} - a(a-1)(k-1)(2+\mu)$
and
$B = -2n\{k+a^2-1\} + (a-1)\{a^2-2a-k+1\} - a^2(a-1)(2+\mu)$.
Replacing X by ϕX in (3.8), we get

$$\begin{aligned} & -a^2S(Z,X) + aS(Z,hX) + Ag(X,Z) \\ & + \{2nka^2 - A\}\eta(X)\eta(Z) + Bg(hX,Z) = 0. \end{aligned} \tag{3.9}$$

Taking $X=Z=e_i$ in (3.9) and summing up with respect to $i, 1\le i\le 2n+1$ and using (2.15) we obtain

$$-2na^2(2n-2-n\mu+k) - 2na(k-1)(2n-2+\mu) + 2nka^2 + 2nA = 0. \tag{3.10}$$

From (3.10), we get

$$\mu = \frac{(3-a)(k+a^2-1)+2(a-1)na}{a(k-n-1)}. \tag{3.11}$$

At this point for $a=3$ and $k=1$, we get $\mu=-4$. From (2.8), this demonstrates that $k^*=1$ and $\mu^*=0$.
Thus we state the following:

Theorem 3.1. *A D_3-homothetic deformed locally symmetric $(1,-4)$-contact metric manifold is a Sasakian manifold.*

Next, let $(M,\phi^*,\xi^*,\eta^*,g^*)$ be a D_a-homothetic deformed (k,μ)-contact metric manifold. A Ricci soliton (g^*,V,λ) is defined on $(M,\phi^*,\xi^*,\eta^*,g^*)$ as

$$(L_V^* g^*)(X,Y)+2S^*(X,Y)+2\lambda g^*(X,Y)=0. \tag{3.12}$$

Where $L_V^* g^*$ denotes the Lie-derivative of Riemannian metric g^* along a vector field V, S^* is the Ricci tensor on $(M,\phi^*,\xi^*,\eta^*,g^*)$. Further, suppose that the potential vector field V is the Reeb vector field ξ^*, i.e., $V=\xi^*$ on $(M,\phi^*,\xi^*,\eta^*,g^*)$. Then from (3.12) we have

$$(L_{\xi^*}^* g^*)(X,Y)+2S^*(X,Y)+2\lambda g^*(X,Y)=0. \tag{3.13}$$

From (2.4) and (3.1) we have

$$\begin{aligned}(L_{\xi^*}^* g^*)(X,Y) &= g^*(\nabla_X^*\xi^*,Y)+g^*(X,\nabla_Y^*\xi^*)\\ &= -2g(\phi hX,Y).\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we find that

$$S^*(X,Y)=g(\phi hX,Y)-\lambda a\{g(X,Y)+(a-1)\eta(X)\eta(Y)\}. \tag{3.15}$$

Replacing X and Y by ξ^* in (3.15) and using (3.4), we get

$$\lambda=\frac{-2n}{a}\{k+a^2-1\}. \tag{3.16}$$

Thus we state the following:

Theorem 3.2. *A Ricci soliton on a D_a-homothetic deformed (k,μ)-contact metric manifold is shrinking.*

For a Sasakian case $k=1$, then from (3.16), we get

$$\lambda=-2na. \tag{3.17}$$

Corollary 3.3. *A Ricci soliton on a D_a-homothetic deformed Sasakian manifold is shrinking.*

By the covariant differentiation of the Ricci tensor S^* with respect to X, we have

$$(\nabla_X^* S^*)(Z,W)=\nabla_X^* S^*(Z,W)-S^*(\nabla_X^* Z,W)-S^*(Z,\nabla_X^* W). \tag{3.18}$$

Replacing W by ξ^* in (3.18), we get

$$(\nabla_X^* S^*)(Z,\xi^*)=\nabla_X^* S^*(Z,\xi^*)-S^*(\nabla_X^* Z,\xi^*)-S^*(Z,\nabla_X^*\xi^*). \tag{3.19}$$

Now, using (3.15), (3.1), (2.7), (2.4) in (3.19), we get

$$\begin{aligned}(\nabla_X^* S^*)(Z,\xi^*) &= \lambda a\{g(\phi X,Z)+g(\phi hX,Z)-\eta(\nabla_X Z)\}\\ &\quad -\lambda a\{\eta(\nabla_X Z)+\frac{(a-1)}{a}g(\phi hX,Z)\}\\ &\quad +S^*(Z,\phi X)+\frac{1}{a}S^*(Z,\phi hX).\end{aligned} \tag{3.20}$$

On simplification and using (2.9), (3.15), we get

$$(\nabla_X^* S^*)(Z,\xi^*)=g(hX,Z)+\frac{(1-k)}{a}\{g(X,Z)-\eta(X)\eta(Z)\}. \tag{3.21}$$

On the other hand, taking the covariant differentiation of (3.4) with respect to X and then using (2.4), (2.7), (2.11), (2.13), (2.14) and (3.1), we obtain

$$\begin{aligned}(\nabla_X^* S^*)(Z,\xi^*) &= -aS(Z,\phi X)-S(Z,\phi hX)\\ &\quad -(a-1)[(a^2-2a-k+1)\{g(\phi X,Z)+\frac{1}{a}g(\phi hX,Z)\}\\ &\quad +a(2+\mu)\{g(hZ,\phi X)+\frac{(k-1)}{a}g(\phi X,Z)\}].\end{aligned} \tag{3.22}$$

Comparing the equations (3.21) and (3.22), then contracting with respect to X and Z, we get

$$k=1. \tag{3.23}$$

Thus we state the following:

Theorem 3.4. *A D_a-homothetic deformed (k,μ) contact metric manifold admitting a Ricci soliton $(g^*,V,\lambda)$is a Sasakian manifold.*

The projective curvature tensor [22] P^* under D_a-homothetic deformation ∇^* is defined by

$$P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n}\{S^*(Y,Z)X - S^*(X,Z)Y\}. \tag{3.24}$$

By interchanging X and Y in (3.24), we have

$$P^*(Y,X)Z = R^*(Y,X)Z - \frac{1}{2n}\{S^*(X,Z)Y - S^*(Y,Z)X\}. \tag{3.25}$$

On adding (3.24) and (3.25) and using the fact that $R(X,Y)Z + R(Y,X)Z = 0$, we get

$$P^*(X,Y)Z + P^*(Y,X)Z = 0. \tag{3.26}$$

From (3.3), (3.24) and first Bianchi identity $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ with respect to ∇, we obtain

$$P^*(X,Y)Z + P^*(Y,Z)X + P^*(Z,X)Y = 0. \tag{3.27}$$

Hence, from (3.26) and (3.27), shows that projective curvature tensor under D_a-homothetic deformation in a (k,μ) contact metric manifold is skew-symmetric and cyclic.
Now using (2.7), (3.3) and (3.15) in (3.24), we obtain

$$\begin{aligned}
P^*(X,Y)Z = {} & R(X,Y)Z + (1-a)[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\
& + 2\eta(X)\eta(Z)hY - 2\eta(Y)\eta(Z)hX + 2g(\phi Y,X)\phi Z \\
& + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi] + \frac{(1-a)}{a}[2\eta(Y)g(hX,Z)\xi \\
& - 2\eta(X)g(hY,Z)\xi + (1-k)\{\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi\} \\
& + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX] \\
& + (a^2-1)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\
& - \frac{1}{2n}[g(\phi hY,Z)X - g(\phi hX,Z)Y - \lambda a\{g(Y,Z)X - g(X,Z)Y\} \\
& - \lambda a(a-1)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}].
\end{aligned} \tag{3.28}$$

Replacing X by ξ^* in (3.28); then using (2.1) and (2.6), we get

$$\begin{aligned}
P^*(X,Y)\xi^* = {} & [\frac{k+a^2-1}{a^2} - \lambda]\{\eta(Y)X - \eta(X)Y\} \\
& + [\frac{\mu+2a-2}{a}]\{\eta(Y)hX - \eta(X)hY\}.
\end{aligned} \tag{3.29}$$

If the vector fields X and Y are orthogonal to ξ, then we have $P^*(X,Y)\xi^* = 0$. That is, $M(\phi^*,\xi^*,\eta^*,g^*)$ is ξ^*-projectively flat. Thus, we can express the accompanying:

Theorem 3.5. *A D_a-homothetically deformed (k,μ)-contact metric manifold is ξ^*-projectively flat, that is, $P^*(X,Y)\xi^* = 0$ if and only if the vector fields X and Y are orthogonal to ξ.*

Let $M(\phi^*,\xi^*,\eta^*,g^*)$ be a $(2n+1)$ dimensional h-projectively semisymmetric (k,μ) contact metric manifold under D_a-homothetic deformation. Then we have

$$P^* \cdot h = 0. \tag{3.30}$$

From (3.30), it follows that

$$P^*(X,Y)hZ - hP^*(X,Y)Z = 0. \tag{3.31}$$

Making use of (2.6) to (2.12), (2.17), (3.4), (3.15) and (3.28) in (3.31), we have

$$\begin{aligned}
& \{k+a-1+\frac{(k-1)(1-a)}{a}\}\{g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi \\
& + g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY + g(\phi hY,Z)\phi X - g(\phi hX,Z)\phi Y\} \\
& + \{\mu(k-1) + \frac{2(k-1)(a-1)}{a}\}\{\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi\} \\
& + \{k+a^2-1+\frac{\lambda a(a-1)}{2n}\}\{\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\} \\
& + (k-1)(2-2a-\mu)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\
& + 2(\mu+2a-2)g(\phi X,Y)\phi hZ + \frac{(k-1)}{2n}\{g(\phi X,Z)Y - g(\phi Y,Z)X\} \\
& + \frac{\lambda a}{2n}\{g(hY,Z)X - g(hX,Z)Y - g(Y,Z)hX - g(X,Z)hY\} \\
& + \frac{1}{2n}\{g(\phi hY,Z)hX - g(\phi hX,Z)hY\} = 0.
\end{aligned} \tag{3.32}$$

Replace X by hX in (3.32) and using the fact that $h^2 = (k-1)\phi^2$, we obtain

$$\begin{aligned}
&\frac{(k+a^2-1)}{a}[(k-1)\{\eta(Y)g(X,Z)\xi - \eta(X)\eta(Y)\eta(Z)\xi\} \\
&+(k-1)g(\phi Y,Z)\phi X + g(\phi hX,Z)\phi hY + +g(\phi hY,Z)\phi hX \\
&+(k-1)g(\phi X,Z)\phi Y] + \{\mu(k-1) + \frac{2(k-1)(a-1)}{a}\}g(hX,Z)\eta(Y)\xi \\
&+\{k+a^2-1+\frac{\lambda a(a-1)}{2n}\}(k-1)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Y)\eta(Z)\xi\} \\
&+(k-1)(\mu+2a-2)\eta(Y)\eta(Z)hX + 2(\mu+2a-2)g(\phi hX,Y)\phi hZ \\
&+\frac{\lambda a}{2n}[g(hY,Z)hX + g(hX,Z)hY + (k-1)g(Y,Z)\{X-\eta(X)\xi\}] \\
&+\frac{(k-1)}{2n}\{g(\phi hX,Z)Y - g(\phi Y,Z)hX\} \\
&+\frac{(k-1)}{2n}\{g(\phi hY,Z)(-X+\eta(X)\xi) + g(\phi X,Z)hY\} = 0.
\end{aligned} \tag{3.33}$$

Replace $Y = \xi^*$ in (3.33) and then taking an inner product with U, we get

$$\begin{aligned}
&\frac{(k+a^2-1)}{a}(k-1)\{g(X,Z)\eta(U) - \eta(X)\eta(Z)\eta(U)\} \\
&+\{\mu(k-1) + \frac{2(k-1)(a-1)}{a}\}g(hX,Z)\eta(U) \\
&+\{k+a^2-1+\frac{\lambda a^2}{2n}\}(k-1)\{\eta(Z)g(X,U) - \eta(X)\eta(Z)\eta(U)\} \\
&+(k-1)(\mu+2a-2)\eta(Z)g(hX,U) + \frac{(k-1)}{2n}g(\phi hX,Z)\eta(U) = 0.
\end{aligned} \tag{3.34}$$

Again taking $Z = \xi^*$ in (3.34), we obtain

$$(k-1)[\{k+a^2-1+\frac{\lambda a^2}{2n}\}\{g(X,U)-\eta(X)\eta(U)\} + (\mu+2a-2)g(hX,U)] = 0, \tag{3.35}$$

i.e., $k = 1$ or by considering $k = -a^2$ (non-Sasakian case) and $\lambda = \frac{2n}{a^2}$, we get $\mu = 2-2a$. Hence by the fact that (from (2.8)) $\mu^* = 0$. Thus we can state the following:

Theorem 3.6. *Let $M(\phi^*,\xi^*,\eta^*,g^*)$ be a (k,μ) contact metric manifold obtained by D_a-homothetic deformation of (k,μ) manifold M. If $M(\phi^*,\xi^*,\eta^*,g^*)$ is h-projectively semisymmetric, then $M(\phi^*,\xi^*,\eta^*,g^*)$ is either Sasakian manifold or $N(k)$-contact manifold provided $k = -a^2$ and $\lambda = \frac{2n}{a^2}$.*

References

[1] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. 509. Springer Verlag, New York, 1973.
[2] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds,* Progress in Mathematics, BirkhauserBoston. Inc., Boston, 2002.
[3] D.E. Blair, *Two remarks on contact metric structures*, Tohoku Math. J., 29 (1977), 319-324.
[4] E. Boeckx, *A full classification of contact metric (k,μ)- spaces,* Illinois J. Math, 44 (2000), 212-219.
[5] J.T. Cho, *A conformally flat (k,μ)-space,* Indian J. Pure Appl. Math. 32 (2001), 501-508.
[6] U.C. De, Y.H. Kim and A.A. Shaikh, *Contact metric manifolds with ξ belonging to (κ,μ)-nullity distribution*, Indian J. Math., 47 (2005), 1-10.
[7] U.C. De, and A. Sarkara, *On the quasi-conformal curvature tensor of a (k,μ)-contact metric manifold,* Math. Reports 14(64), 2 (2012), 115-129.
[8] A. Ghosh, T. Koufogiorgos and R. Sharma, *Conformally flat contact metric manifolds*, J. Geom., 70 (2001), 66-76.
[9] A. Ghosh and R. Sharma, *A classification of Ricci solitons as (k,μ)-contact metrics,* Springer Proceedings in Mathematics and Statistics, Springer Japan, (2014), 349-358.
[10] R.S. Hamilton, *The Ricci flow on surfaces,* Contemporary Mathematics, 71 (1988), 237-262.
[11] T. Ivey, *Ricci solitons on compact 3-manifolds,* Differential Geom. Appl. 3 (1993), 301-307.
[12] J.B. Jun, A. Yildiz and U.C. De, *On ϕ-recurrent (k,μ)-contact metric manifolds.* Bulletin of the Korean Mathematical Society, 45(4) (2008), 689-700.
[13] P. Majhi and G. Ghosh, *Concircular vectors field in (k,μ)-contact metric manifolds.* International Electronic Journal of Geometry, 11(1) (2018), 52-56.
[14] B.J. Papantoniou, *Contact Riemannian manifolds satisfying $R(X,\xi)\cdot R = 0$ and $\xi \in (\kappa,\mu)$-nullity distribution*, Yokohama Math. J., 40 (1993), 149-161.
[15] D.G. Prakasha, C.S. Bagewadi and Venkatesha, *On pseudo projective curvature tensor of a contact metric manifold,* SUT J. Math. 43 (2007), 115-126.
[16] R. Sharma, *Certain results on K-contact and (κ,μ)-contact metric manifolds*, J. Geom., 89 (2008), 138-147.
[17] R. Sharma and T. Koufogiorgos, *Locally symmetric and Ricci symmetric contact metric manifolds,* Ann. Global Anal. Geom., 9 (1991), 177-182.
[18] S. Tanno, *Ricci curvatures of contact Riemannian manifolds,* Tohoku Mathematical Journal, Second Series, (40(3) (1988), 441-448.
[19] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois Journal of Mathematics, 12(4) (1968), 700-717.
[20] M.M. Tripathi, *Ricci solitons in contact metric manifolds,* arXiv:0801.4222v1 [math.DG], 2008.
[21] M.M. Tripathi and. J.S. Kim, *On the concircular curvature tensor of a (k,μ)-manifold,* Balkan J. Geom. Appl. 9(1) (2004), 114-124.
[22] K. Yano and M. Kon, *Structures on manifolds,* Series in Pure Mathematics, World Scientific publishing, Singapore, 3 (1984).