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DINI-TYPE HELICOIDAL HYPERSURFACE IN 4-SPACE

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Abstract

We define Dini-type helicoidal hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$. We calculate the Gauss map, Gaussian curvature and the mean curvature of the helicoidal hypersurface. Additionally, we find some special relations and symmetries for the curvatures.

Keywords: Dini-type helicoidal hypersurface; Four dimensional Euclidean space; Gauss map.

1. Introduction

After Moore [27,28], Takahashi [32], and also Chen and Piccinni [8], the theory of submanifolds has been studied by many mathematicians. For some papers about the topic, see $[1-7,\ 9-12,\ 14-26,\ 29-31,\ 33-35]$.

In this work, considering Ulisse Dini's paper $[13]$ in Euclidean 3-space $\mathbb{E}^3$, we study Dini-type helicoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give some basic notions of the geometry of the $\mathbb{E}^4$ in this section. In section 2, we define helicoidal hypersurface. Moreover, we give Dini-type helicoidal hypersurface, and calculate its curvatures obtaining some special symmetries in the last section.

Next, we will introduce the first and second fundamental forms, matrix of the shape operator **S**, Gaussian curvature *K*, and the mean curvature *H* of hypersurface $\mathbf{M}=\mathbf{M}(u,v,w)$ in Euclidean 4-space $\mathbb{E}^4$. We shall identify a vector (a,b,c,d) with its transpose $(a,b,c,d)^t$.

Let $\mathbf{M}=\mathbf{M}(u,v,w)$ be an isometric immersion of a hypersurface M^3 in the $\mathbb{E}^4$. The triple vector product of $\vec{x}=(x_1,x_2,x_3,x_4)$, $\vec{y}=(y_1,y_2,y_3,y_4)$, $\vec{z}=(z_1,z_2,z_3,z_4)$ on $\mathbb{E}^4$ is defined as follows:

$$\vec{x}\times\vec{y}\times\vec{z} = (x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2,$$
$$-x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3z_1y_4 - y_1x_4z_3 + x_4y_3z_1,$$
$$x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2z_1y_4 + y_1x_4z_2 - x_4y_2z_1,$$
$$-x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1).$$

For a hypersurface $\mathbf{M} = \mathbf{M}(u, v, w)$ in 4-space, we compute

$$\det I = \det\begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = (EG - F^2)C - A^2G + 2ABF - B^2E,$$

$$\det II = \det\begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = (LN - M^2)V - P^2N + 2PTM - T^2L,$$

where

$$E = \mathbf{M}_u \cdot \mathbf{M}_u,\ F = \mathbf{M}_u \cdot \mathbf{M}_v,\ G = \mathbf{M}_v \cdot \mathbf{M}_v,$$
$$L = \mathbf{M}_{uu} \cdot e,\quad M = \mathbf{M}_{uv} \cdot e,\quad N = \mathbf{M}_{vv} \cdot e,$$
$$A = \mathbf{M}_u \cdot \mathbf{M}_w,\ B = \mathbf{M}_v \cdot \mathbf{M}_w,\ C = \mathbf{M}_w \cdot \mathbf{M}_w,$$
$$P = \mathbf{M}_{uw} \cdot e,\quad T = \mathbf{M}_{vw} \cdot e,\quad V = \mathbf{M}_{ww} \cdot e,$$

and e is the Gauss map

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|}.$$

Using $(I)^{-1}.(II)$, we get shape operator matrix $\mathbf{S}$, as follows:

$$\mathbf{S} = \frac{1}{\det I}\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix},$$

where

$$
\begin{aligned}
s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2L,\\
s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2M,\\
s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2P,\\
s_{21} &= ABL - CFL + AFP - BPE + CME - A^2M,\\
s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2N,\\
s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2T,\\
s_{31} &= -AGL + BFL + AFM - BME + GPE - F^2P,\\
s_{32} &= -AGM + BFM + AFN - BNE + GTE - F^2T,\\
s_{33} &= -AGP + BFP + AFT - BTE + GVE - F^2V.
\end{aligned}
$$

Finally, we obtain following formulas of the Gaussian curvature K, and the mean curvature H, respectively,

$$K = \frac{(LN - M^2)V + 2MPT - P^2N - T^2L}{(EG - F^2)C + 2ABF - A^2G - B^2E},$$

and

$$H = \frac{(EN + GL - 2FM)C + (EG - F^2)V - A^2N - B^2L - 2(APG + BTE - ABM - ATF - BPF)}{3[(EG - F^2)C + 2ABF - A^2G - B^2E]}.$$

When $K = 0$, hypersurface is flat; and $H = 0$, then hypersurface is minimal.

2. Helicoidal Hypersurface

In this section, we define the rotational hypersurface and helicoidal hypersurface in $\mathbb{E}^4$. Let $\gamma: I \subset \mathbb{R} \to \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π. In $\mathbb{E}^4$, a *rotational hypersurface* is defined by a hypersurface rotating profile curve γ around axis ℓ.

Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Resulting hypersurface is called *helicoidal hypersurface* with axis ℓ, pitches $a, b \in \mathbb{R} - \{0\}$. Supposing ℓ is the line spanned by the vector $(0,0,0,1)^t$, we consider following orthogonal matrix:

$$Z(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $v, w \in \mathbb{R}$. The matrix Z supplies the following equations, simultaneously,

$$Z\ell = \ell, \quad ZZ^t = Z^tZ = I_4, \quad \det Z = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_4-axis of $\mathbb{E}^4$. The profile curve is given by $\gamma(u) = (u, 0,0, \varphi(u))$, where $\varphi(u): I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function for all $u \in I$. Therefore, the helicoidal hypersurface, spanned by the vector $(0,0,0,1)$, is defined by as follows:

$$\mathbf{H}(u, v, w) = Z\gamma^t + (av + bw)\ell^t,$$

where $u \in I$, $v,w \in [0,2\pi]$, $a,b \in \mathbb{R} - \{0\}$. More clear form of the helicoidal hypersurface in 4-space is given by as follows:

$$\mathbf{H}(u, v, w) = \begin{pmatrix} u\cos v\cos w \\ u\sin v\cos w \\ u\sin w \\ \varphi(u) + av + bw \end{pmatrix}.$$

3. Dini-Type Helicoidal Hypersurface

We consider Dini-type helicoidal hypersurface as follows:

$$\mathfrak{D}(u, v, w) = \begin{pmatrix} \sin u\cos v\cos w \\ \sin u\sin v\cos w \\ \sin u\sin w \\ \varphi(u) + av + bw \end{pmatrix},$$

where $u \in \mathbb{R} - \{0\}$, $v,w \in [0,2\pi]$. Using the first differentials of $\mathfrak{D}$ with respect to u,v,w, we get the first quantities as follows

$$I = \begin{pmatrix} \varphi'^2 + \cos^2 u & a\varphi' & b\varphi' \\ a\varphi' & (a^2 - \cos^2 u)\cos^2 w & ab \\ b\varphi' & ab & b^2 + \sin^2 u \end{pmatrix},$$

and have

$$\det I = \sin^2 u[(\cos^2 u\cos^2 w + a^2)\cos^2 u - \varphi'^2\cos^2 w],$$

where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$. Using the second differentials of $\mathfrak{D}$ with respect to u,v,w, we have the second quantities as follows

$$II = \begin{pmatrix} -\frac{\sin^2 u \cos w(\varphi'' \cos u + \varphi' \sin u)}{\sqrt{\det I}} & \frac{a \cos^2 u \sin u \cos w}{\sqrt{\det I}} & \frac{b \sin u \cos^2 u \cos w}{\sqrt{\det I}} \\ \frac{a \cos^2 u \sin u \cos w}{\sqrt{\det I}} & \frac{\sin^2 u \cos^2 w(b \cos u \sin w - \varphi' \sin u \cos w)}{\sqrt{\det I}} & -\frac{a \sin^2 u \cos u \sin w}{\sqrt{\det I}} \\ \frac{b \sin u \cos^2 u \cos w}{\sqrt{\det I}} & -\frac{a \sin^2 u \cos u \sin w}{\sqrt{\det I}} & \frac{\varphi' \sin^3 u \cos w}{\sqrt{\det I}} \end{pmatrix},$$

and we get

$$\det II = \frac{\begin{pmatrix} \varphi'^2 \varphi'' \sin^8 u \cos u \cos^5 w - b\varphi' \varphi'' \sin^7 u \cos^2 u \sin w \cos^4 w \\ +a^2 \varphi'' \sin^6 u \cos^3 u \sin^2 w \cos w \\ +\varphi'^3 \sin^9 u \cos^5 w - b\varphi'^2 \sin^8 u \cos u \sin w \cos^4 w \\ +(a^2 \sin^7 u \cos^2 u \sin^2 w \cos w - a^2 \sin^5 u \cos^4 u \cos^3 w + b^2 \sin^5 u \cos^4 u \cos^5 w)\varphi' \\ -2a^2 b \sin^4 u \cos^5 u \sin w \cos^2 w - b^3 \sin^4 u \cos^5 u \sin w \cos^4 w \end{pmatrix}}{(\det I)^{3/2}}.$$

The Gauss map e of the helicoidal hypersurface $\mathfrak{D}$ is

$$e_{\mathfrak{D}} = \frac{1}{\sqrt{\det I}} \begin{pmatrix} (\varphi' \sin u \cos v - a \cos u \sin v - b \cos u \cos v \sin w \cos w) \sin u \\ (\varphi' \sin u \sin v - a \cos u \cos v - b \cos u \sin v \sin w \cos w) \sin u \\ (\varphi' \sin u \sin w + b \cos u \cos w) \sin u \cos w \\ -\sin^2 u \cos u \cos w \end{pmatrix}.$$

Finally, we calculate the Gaussian curvature of $\mathfrak{D}$, as follows:

$$K = \frac{\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7}{[\sin^2 u((\cos^2 u \cos^2 w + a^2) \cos^2 u - \varphi'^2 \cos^2 w)]^{5/2}},$$

where

$$\begin{aligned}
\alpha_1 &= \sin^8 u \cos u \cos^5 w, \\
\alpha_2 &= -b \sin^7 u \cos^2 u \sin w \cos^4 w, \\
\alpha_3 &= a^2 \sin^6 u \cos^3 u \sin^2 w \cos w, \\
\alpha_4 &= \sin^9 u \cos^5 w, \\
\alpha_5 &= -b \sin^8 u \cos u \sin w \cos^4 w, \\
\alpha_6 &= a^2 \sin^7 u \cos^2 u \sin^2 w \cos w - a^2 \sin^5 u \cos^4 u \cos^3 w + b^2 \sin^5 u \cos^4 u \cos^5 w, \\
\alpha_7 &= -2a^2 b \sin^4 u \cos^5 u \sin w \cos^2 w - b^3 \sin^4 u \cos^5 u \sin w \cos^4 w,
\end{aligned}$$

and we calculate the mean curvature of $\mathfrak{D}$, as follows:

$$H = \frac{\beta_1\varphi'' + \beta_2\varphi'^3 + \beta_3\varphi'^2 + \beta_4\varphi' + \beta_5}{3[\sin^2 u((\cos^2 u\cos^2 w + a^2)\cos^2 u - \varphi'^2\cos^2 w)]^{3/2}},$$

where

$$\begin{aligned}
\beta_1 &= [(b^2 + \sin^2 u)\cos^2 w + a^2]\sin^4 u\cos u\cos w,\\
\beta_2 &= -2\sin^3 u\cos^3 w,\\
\beta_3 &= -b\sin^4 u\cos u\sin w\cos^2 w,\\
\beta_4 &= 2\Big[\Big(-\tfrac{\cos^4 u}{2} + \tfrac{b^2}{2} + b^2\cos^2 u + \tfrac{1}{2}\Big)\cos^2 w + a^2\Big(\cos^2 u + \tfrac{1}{2}\Big)\Big]\sin^2 u\cos w,\\
\beta_5 &= [(b^2 + \sin^2 u)\cos^2 w + a^2]\sin^3 u\cos^3 u.
\end{aligned}$$

Hence, we have following theorems:

Theorem 1. *Let* $\mathfrak{D} : M^3 \to \mathbb{E}^4$ *be an isometric immersion. If* M^3 *is minimal, then we get*

$$\beta_1\varphi'' + \beta_2\varphi'^3 + \beta_3\varphi'^2 + \beta_4\varphi' + \beta_5 = 0.$$

Theorem 2. *Let* $\mathfrak{D} : M^3 \to \mathbb{E}^4$ *be an isometric immersion.* If M^3 *is flat, then we have*

$$\alpha_1\varphi'^2\varphi'' + \alpha_2\varphi'\varphi'' + \alpha_3\varphi'' + \alpha_4\varphi'^3 + \alpha_5\varphi'^2 + \alpha_6\varphi' + \alpha_7 = 0.$$

Solutions of these two eqs. are attracted problem.

Now, taking $\varphi(u) = \cos u + ln\left(\tan\frac{u}{2}\right)$ in Theorem 1, and Theorem 2, we obtain following corollaries:

Corollary 1. *When Dini-type helicoidal hypersurface* $\mathfrak{D}$ *has* $H = 0$ *in 4-space, then we have*

$$\sum_{i=0}^{6}\Phi_i\tan^i\left(\frac{u}{2}\right) = 0,$$

where

$$\begin{aligned}
\Phi_6 &= \beta_2,\\
\Phi_5 &= 2\beta_1 - 6\beta_2\sin u + 2\beta_3,\\
\Phi_4 &= 9\beta_2 - 6\beta_2\cos 2u - 8\beta_3\sin u + 4\beta_4,\\
\Phi_3 &= -8\beta_1\cos u - 18\beta_2\sin u + 2\beta_2\sin 3u + 8\beta_3 - 4\beta_3\cos 2u - 8\beta_4\sin u + 8\beta_5,\\
\Phi_2 &= 9\beta_2 - 6\beta_2\cos 2u - 8\beta_3\sin u + 4\beta_4,\\
\Phi_1 &= 2\beta_1 - 6\beta_2\sin u + 2\beta_3,\\
\Phi_0 &= \beta_2.
\end{aligned}$$

Corollary 2. *When Dini-type helicoidal hypersurface $\mathfrak{D}$ has $K = 0$ in 4-space, then we get*

$$\sum_{j=0}^{8} \Psi_j \tan^j\left(\frac{u}{2}\right) = 0,$$

where

$$\begin{aligned}
\Psi_8 &= \alpha_1, \\
\Psi_7 &= -4\alpha_1 \sin u + 2\alpha_2 + 2\alpha_4, \\
\Psi_6 &= \begin{pmatrix} 2\alpha_1 - 4\alpha_1 \cos u + 4\alpha_1 \sin^2 u - 4\alpha_2 \sin u \\ +4\alpha_3 - 12\alpha_4 \sin u + 4\alpha_5 \end{pmatrix}, \\
\Psi_5 &= \begin{pmatrix} -4\alpha_1 \sin u + 16\alpha_1 \cos u \sin u + 2\alpha_2 - 8\alpha_2 \cos u \\ +6\alpha_4 + 24\alpha_4 \sin^2 u - 16\alpha_5 \sin u + 8\alpha_6 \end{pmatrix}, \\
\Psi_4 &= \begin{pmatrix} -8\alpha_1 \cos u - 16\alpha_1 \cos u \sin^2 u + 16\alpha_2 \cos u \sin u \\ -16\alpha_3 \cos u - 24\alpha_4 \sin u - 16\alpha_4 \sin^3 u \\ +8\alpha_5 + 16\alpha_5 \sin^2 u - 16\alpha_6 \sin u + 16\alpha_7 \end{pmatrix}, \\
\Psi_3 &= \begin{pmatrix} -4\alpha_1 \sin u + 16\alpha_1 \cos u \sin u + 2\alpha_2 - 8\alpha_2 \cos u \\ +6\alpha_4 + 24\alpha_4 \sin^2 u - 16\alpha_5 \sin u + 8\alpha_6 \end{pmatrix}, \\
\Psi_2 &= \begin{pmatrix} 2\alpha_1 - 4\alpha_1 \cos u + 4\alpha_1 \sin^2 u - 4\alpha_2 \sin u \\ +4\alpha_3 - 12\alpha_4 \sin u + 4\alpha_5 \end{pmatrix}, \\
\Psi_1 &= -4\alpha_1 \sin u + 2\alpha_2 + 2\alpha_4, \\
\Psi_0 &= \alpha_1.
\end{aligned}$$

Remark 1. *From Corollary 1, and Corollary 2, we obtain following special symmetries for Dini-type helicoidal hypersurface $\mathfrak{D}$, respectively,*

$$\Phi_6 = \Phi_0,\ \Phi_5 = \Phi_1,\ \Phi_4 = \Phi_2,$$

and

$$\Psi_8 = \Psi_0,\ \Psi_7 = \Psi_1,\ \Psi_6 = \Psi_2,\ \Psi_5 = \Psi_3.$$

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