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Contact Cr-Warped Product Submanifolds of Nearly Quasi-Sasakian Manifold

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Abstract

In the present paper, we construct contact CR-warped product submanifolds of nearly quasi Sasakian manifold. We have obtained results on the existence of warped product CR Submanifolds of nearly quasi Sasakian manifold and discuss the characterization result. We also construct the inequality $||h||^2 \geq 2As + 2s||\nabla \ln f||^2$ for contact CR warped products of nearly quasi Sasakian manifolds. The equality cases are also discussed.

Keywords: *Contact CR-submanifold, Warped product, quasi-Sasakian manifold and Nearly quasi-Sasakian manifold.*
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1. Introduction

Bishop and Neill [13] defined and studied warped products with differential point of view. B.Y Chen [5] extended the work of Bishop and Neill and studied the warped product CR-submanifold of Kaehler manifolds and many more [6, 12]. Since then a number of authors extensively studied these results. Quasi Sasakian structure was initiated by Blair [8]. Since then several papers on quasi-Sasakian manifolds have studied by Tanno [24], Kanemaki [15, 16], Oubina [10], Gonzalez and Chinea [11], and the author and et al., [17-23]. CR-submanifold of a Kahlerian manifold has been studied by A. Bejancu [1]. Then A. Bejancu, N. Papaghiue [2] introduced the idea of semi-invariant submanifold of a Sasakian manifold and they obtained several results on this manifolds. Kim [3] extensively studied quasi-Sasakian manifolds and proved that fibred Riemannian spaces with invariant fibres normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosympletic structure.

2. Preliminaries

Let $\bar{M}$ be a real $2n+1$ dimensional differentiable manifold, endowed with an almost contact metric structure (φ, ξ, η, g). Then we have from [7]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta o \varphi = 0, \quad \eta(X) = g(X, \xi) \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(\varphi X, Y) = -g(X, \varphi Y)$$

for any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. An almost contact metric structure (φ, ξ, η, g) on $\bar{M}$ is called quasi-Sasakian manifold if

$$(\bar{\nabla}_X \varphi)Y = \eta(Y)AX - g(AX,Y)\xi, \qquad \varphi AX = A\varphi X$$

where A a symmetric linear transformation field, $\bar{\nabla}$ denotes the Riemannian connection of g on $\bar{M}$. On a quasi-Sasakian manifold $\bar{M}$, we have

$$\bar{\nabla}_X \xi = \varphi AX$$

Further, an almost contact metric manifold $\bar{M}$ on (φ, ξ, η, g) is called nearly quasi-Sasakian manifold if

$$(\bar{\nabla}_X \varphi)Y + (\bar{\nabla}_Y \varphi)X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi \tag{2.2}$$

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The covariant derivative of the tensor filed φ is defined as

$$(\bar{\nabla}_X\varphi)Y = \bar{\nabla}_X\varphi Y - \varphi\bar{\nabla}_X Y \tag{2.3}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas for the nearly quasi-Sasakian manifold are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.5}$$

for any X, $Y\varepsilon TM$ and $N\varepsilon T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X,Y) = g(h(X,Y),N) \tag{2.6}$$

The notion of warped product manifolds was initiated by Bishop and O Neill [13].They defined as follows

Definition 2.1. *Let (N_1,g_1) and (N_2,g_2) be two Riemannian manifolds and f be a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1\times_f N_2 = (N_1\times N_2,g)$, where*

$$g = g_1 + f^2 g_2 \tag{2.7}$$

A warped product manifold $N_1\times_f N_2$ is said to be trivial if the warping function f is constant.

We recall

Lemma 2.2. *Let $M = N_1\times_f N_2$ be a warped product manifold with the warping function f, then*
(i)$\nabla_X Y\varepsilon\Gamma(TN_1)$ is the lift of $\nabla_X Y$ on N_1,
(ii)$\nabla_X Z = \nabla_Z X = (X\ln f)Z$,
(iii)$\nabla_Z\omega = \nabla^{N_2}_Z\omega - g(Z,\omega)\nabla\ln f$
for each $X,Y\varepsilon\Gamma(TN_1)$ and $Z,\omega\varepsilon\Gamma(TN_1)$, where $\nabla\ln f$ is the gradient of $\ln f$ and ∇ and ∇^{N_2} denote the Levi-Civita connections on M and N_2, respectively.

For a Riemannian manifold M of dimension n and a smooth function f on M , we recall ∇f, the gradient of f which is defined by

$$g(\nabla f,X) = X(f) \tag{2.8}$$

for any $X\varepsilon\Gamma(TM)$. As a consequence, we have

$$||\nabla f||^2 = \sum_{i=1}^{n}(e_i(f))^2 \tag{2.9}$$

for an orthonormal frame $\{e_1,...,e_n\}$ on M.

3. Contact CR-Warped product submanifolds

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following:
(i) A submanifold M tangent to ξ is an invariant submanifold if φ preserves any tangent space of M, that is, $\varphi(T_pM)\subset T_pM$, for every $p\varepsilon M$.
(ii) A submanifold M tangent to ξ is an anti-invariant submanifold if φ maps any tangent space of M into the normal space, that is, $\varphi(T_pM)\subset T^{\perp}_pM$, for every $p\varepsilon M$.
Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$, then for every $p\varepsilon M$ there exists a maximal invariant subspace denoted by D_p of the tangent space T_pM of M. If the dimension of D_p is same for all values of $p\varepsilon M$, then D_p gives an invariant distribution D on M.
A submanifold M of an almost contact manifold $\bar{M}$ is said to be a contat CR submanifold if there exists on M a differentiable distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is;
(i) $TM = D\oplus D^{\perp}\oplus\langle\xi\rangle$.
(ii) D is an invariant distribution, i.e., $\varphi D\subseteq TM$
(iii) $D^{\perp}$ is an anti-invariant distribution, i.e., $\varphi D^{\perp}\subseteq T^{\perp}M$.
A contact CR-submanifold is anti-invariant if $D_p = \{0\}$ and invariant if $D^{\perp}_p = \{0\}$ respectively, for every $p\varepsilon M$. It is a proper contact CR-submanifold if neither $D_p = \{0\}$ nor $D^{\perp}_p = \{0\}$, for each $p\varepsilon M$.
If v is the φ-invariant subspace of the normal bundle $T^{\perp}M$, then in case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \varphi D^{\perp}\oplus v \tag{3.1}$$

where v is the φ -invariant normal subbundle of $T^{\perp}M$.

In this section, we investigate the warped products $M = N_\perp \times_f N_T$ and $M = N_T \times_f N_\perp$ where N_T and $N_\perp$ are invariant and anti-invariant submanifolds of a nearly quasi-Sasakian manifold $\bar{M}$, respectively. First we discuss the warped products $M = N_\perp \times_f N_T$, here two possible cases arise:
(i) ξ is tangent to N_T ,
(ii) ξ is tangent to $N_\perp$.
We start with the case (i).

Theorem 3.1. *If $\bar{M}$ be a nearly quasi-Sasakian manifold then there do not exist warped product submanifold $M = N_\perp \times_f N_T$ such that N_T is an invariant submanifold tangent to ξ is anti invariant submanifold, unless $\bar{M}$ is nearly Sasakian.*

Proof. Consider $\xi \varepsilon \Gamma(TN_T)$ and $Z\varepsilon\Gamma(TN_\perp)$, then by the structure equation of nearly quasi-Sasakian manifold, we have $(\bar{\nabla}_Z\varphi)\xi + (\bar{\nabla}_\xi\varphi)Z = AZ$. Using (2.4), we obtain $-\varphi\bar{\nabla}_Z\xi + \bar{\nabla}_\xi\varphi Z - \varphi\bar{\nabla}_\xi Z = AZ$. Then from Lemma 2.1(ii) and (2.5), we derive

$$\bar{\nabla}_\xi\varphi Z - 2\varphi h(Z,\xi) = AZ \tag{3.2}$$

Taking the inner product with φZ in (3.2) and then using (2.2) and the fact that $\xi\varepsilon\Gamma(TN_T)$, we get $||Z||^2 = 0$ hence we conclude that M is invariant, which proves the theorem. Now, we will discuss the other case, when ξ is tangent to $N_\perp$. □

Theorem 3.2. *If $\bar{M}$ be a nearly quasi-Sasakian manifold then there do not exist warped product submanifolds $M = N_\perp \times_f N_T$ such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\bar{M}$, unless $\bar{M}$ is nearly cosympletic.*

Proof. Let $\xi\varepsilon\Gamma(TN_T)$ and $Z\varepsilon\Gamma(TN_\perp)$, then we have $(\bar{\nabla}_X\varphi)\xi + (\bar{\nabla}_\xi\varphi)X = AX$. Using (2.4), we get

$$-\varphi\bar{\nabla}_X\xi + \bar{\nabla}_\xi\varphi X - \varphi\bar{\nabla}_\xi X = AX. \tag{3.3}$$

Taking the inner product with X in (3.3) and using (2.2), (2.5), Lemma 2.1 (ii) and the fact that ξ is tangent to $N_\perp$, we obtain $||X||^2 = 0$, for some smooth function on $\bar{M}$. Thus, we conclude that M is anti-invariant submanifold of a nearly quasi-Sasakian manifold $\bar{M}$ otherwise $\bar{M}$ is nearly cosympletic.
Now, we will discuss the warped product $M = N_\perp \times_f N_T$ such that the structure vector field ξ is tangent to $N_\perp$. □

Theorem 3.3. *If $\bar{M}$ be a nearly quasi-Sasakian manifold then there do not exist warped product submanifolds $M = N_\perp \times_f N_T$ such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\bar{M}$.*

Proof. If we consider $X\varepsilon\Gamma(TN_T)$ and the structure vector field ξ is tangent to $N_\perp$, then by (2.3), we have $(\bar{\nabla}_X\varphi)\xi + (\bar{\nabla}_\xi\varphi)X = AX$. Using (2.4), we obtain $\bar{\nabla}_\xi\varphi X - \varphi\bar{\nabla}_X\xi - \varphi\bar{\nabla}_\xi X = AX$. Then by (2.5) and Lemma 2.1 (ii), we derive

$$(\varphi X\ln f)\xi - 2\varphi h(X,\xi) + h(\varphi X,\xi) = AX \tag{3.4}$$

Hence, the result is obtained by taking the inner product with ξ in (3.4). □

If we consider the structure vector field ξ tangent to N_T for the warped product $M = N_\perp \times_f N_T$, then we prove the following result for later use.

Lemma 3.4. *If $M = N_\perp \times_f N_T$ be a contact CR-warped product submanifold of a nearly quasi-Sasakian manifold $\bar{M}$ such that N_T and $N_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively, then*
(i) $\xi(\ln f) = 0$
(ii) $g(h(X,Z),\varphi\omega) = g(h(X,\omega),\varphi Z)$
(iii)$g(h(X,\omega),\varphi Z) = g(h(X,Z),\varphi\omega) = \eta(X)g(AZ,\omega) - (\varphi X\ln f)g(Z,\omega)\}$
(iv)$g(h(\xi,Z),\varphi\omega) = g(AZ,\omega)$

Proof. If ξ is tangent to N_T, then for any $Z\varepsilon\Gamma(TN_\perp)$, we have $(\bar{\nabla}_\xi\varphi)Z + (\bar{\nabla}_Z\varphi)\xi = AZ$. Then from (2.4), (2.5) and Lemma 2.1 (ii), we obtain

$$2(\xi\ln f)\varphi Z + 2\varphi h(Z,\xi) - \bar{\nabla}_\xi\varphi Z = AZ \tag{3.5}$$

Taking the inner product with ϕZ in (3.5) and using (2.2), we derive

$$2(\xi\ln f)||Z||^2 - g(\bar{\nabla}_\xi\varphi Z,\varphi Z) = 0 \tag{3.6}$$

On the other hand, by the property of Riemannian connection, we have $\xi g(\varphi Z,\varphi Z) = 2g(\bar{\nabla}_\xi\varphi Z,\varphi Z)$. By (2.2) and the property of Riemannian connection, we get

$$g(\bar{\nabla}_\xi Z,Z) = g(\bar{\nabla}_\xi\varphi Z,\varphi Z) \tag{3.7}$$

Using this fact in (3.6) and then from (2.5) and Lemma 2.1 (ii), we deduce that $\xi(\ln f)||Z||^2 = 0$ for any $Z\varepsilon\Gamma(TN_\perp)$, which gives (i). For the other parts of the lemma, we have $(\bar{\nabla}_X\varphi)Z + (\bar{\nabla}_Z\varphi)X = \eta(X)AZ$, for any $X\varepsilon\Gamma(TN_T)$ and $Z\varepsilon\Gamma(TN_\perp)$. Using (2.4), (2.5) and (2.6), we derive

$$\eta(X)AZ = -A_{\varphi Z}X + \nabla^\perp_X\varphi Z - 2(X\ln f)\varphi Z + (\varphi X\ln f)Z + h(\varphi X,Z) - 2\varphi h(X,Z) \tag{3.8}$$

Thus, the second part can be obtained by taking the inner product in (3.8) with Y , for any $Y\varepsilon\Gamma(TN_T)$. Again, taking the inner product in (3.8) with W for any $W\varepsilon\Gamma(TN_\perp)$, we get

$$\eta(X)g(AZ,\omega) = -g(h(X,\omega),\varphi Z) + (\varphi X\ln f)g(Z,\omega) + 2g(h(X,Z),\varphi\omega) \tag{3.9}$$

By polarization identity, we get

$$\eta(X)g(AZ,\omega) = -g(h(X,Z),\varphi\omega) + (\varphi X \ln f)g(Z,\omega) + 2g(h(X,\omega),\varphi Z) \tag{3.10}$$

Then from (3.9) and (3.10), we obtain

$$g(h(X,Z),\varphi\omega) = g(h(X,\omega),\varphi Z) \tag{3.11}$$

which is the first equality of (iii). Using (3.11) either in (3.9) or in (3.10), we get the second equality of (iii). Now, for the last part, replacing X by ξ in the third part of this lemma. This proves the lemma completely. Now, we have the following characterization theorem. □

Theorem 3.5. *If M be a contact CR-submanifold of a nearly quasi-Sasakian manifold $\bar{M}$ with integrable invariant and anti-invariant distribution $D\oplus\langle\xi\rangle$ and $D^{\perp}$ then M is locally a contact CR-warped product if and only if the shape operator of M satisfies*

$$A_{\varphi\omega}X = -(\varphi X\mu)\omega + \eta(X)A\omega \quad \forall X\varepsilon\Gamma(D\oplus\langle\xi\rangle), \quad \omega\varepsilon\Gamma(D^{\perp}) \tag{3.12}$$

for some smooth function μ on M satisfying $V(\mu) = 0$ for every $V\varepsilon\Gamma(D^{\perp})$.

Proof. Direct part follows from the Lemma 3.1 (iii). For the converse, suppose that M is contact CR-submanifold satisfying (3.12), then we have $g(h(X,Y),\varphi\omega) = g(A_{\varphi\omega}X,Y) = 0$ for any $X,Y\varepsilon\Gamma(D\oplus\langle\xi\rangle)$ and $\omega\varepsilon\Gamma(D^{\perp})$. Using (2.2) and (2.5), we get $g(\bar{\nabla}_X Y,\varphi\omega)X = -g(\varphi\bar{\nabla}_X Y,\omega) = 0$. Then from (2.4), we obtain

$$g((\bar{\nabla}_X\varphi)Y,\omega) = g(\bar{\nabla}_X\varphi Y,\omega) \tag{3.13}$$

Similarly, we have

$$g((\bar{\nabla}_Y\varphi)X,\omega) = g(\bar{\nabla}_Y\varphi X,\omega) \tag{3.14}$$

Then from (3.13) and (3.14), we derive

$$g((\bar{\nabla}_X\varphi)Y + (\bar{\nabla}_Y\varphi)X,\omega) = g(\bar{\nabla}_X\varphi Y + \bar{\nabla}_Y\varphi X,\omega) \tag{3.15}$$

Using (2.3) and the fact that ξ is tangent to N_T , then by orthogonality of two distributions, we obtain

$$g(\bar{\nabla}_X\varphi Y + \bar{\nabla}_Y\varphi X,\omega) = 0 \tag{3.16}$$

This means that $\nabla_X\varphi Y + \nabla_Y\varphi X\varepsilon\Gamma(D\oplus\langle\xi\rangle)$, for any $X,Y\varepsilon\Gamma(D\oplus\langle\xi\rangle)$, that is $D\oplus\langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. So far as the anti-invariant distribution $D^{\perp}$ is concerned, it is integrable on M (cf. [16], Theorem 8.1). Let $N_{\perp}$ be the leaf of $D^{\perp}$ and h^* be the second fundamental form of $N_{\perp}$ in M. Then for any $X\varepsilon\Gamma(D\oplus\langle\xi\rangle)$, and $Z,W\varepsilon\Gamma(D^{\perp})$, we have $g(h^*(Z,\omega),\varphi X) = g(\nabla_Z\omega,\varphi X)$. Using (2.2), (2.4) and (2.5), we obtain $g(h^*(Z,\omega),\varphi X) = g((\bar{\nabla}_Z\varphi)\omega,X) - g(\bar{\nabla}_Z\varphi\omega,X)$. Then from (2.6) and (2.7), we get

$$g(h^*(Z,\omega),\varphi X) = g((\bar{\nabla}_Z\varphi)\omega,X) + g(A_{\varphi\omega}X,Z) \tag{3.17}$$

Using (3.12), we derive

$$g(h^*(Z,\omega),\varphi X) = g((\bar{\nabla}_Z\varphi)\omega,X) + \{\eta(X)A - (\varphi X)\mu\}g(Z,\omega) \tag{3.18}$$

Similarly, we obtain

$$g(h^*(Z,\omega),\varphi X) = g((\bar{\nabla}_\omega\varphi)Z,X) + \{\eta(X)A - (\varphi X)\mu\}g(Z,\omega) \tag{3.19}$$

Then from (3.18) and (3.19), we get

$$2g(h^*(Z,\omega),\varphi X) = g((\bar{\nabla}_Z\varphi)\omega + (\bar{\nabla}_\omega\varphi)Z,X) + 2\{\eta(X)A - (\varphi X)\mu\}g(Z,\omega) \tag{3.20}$$

Using the structure equation of nearly quasi-Sasakian manifold and the fact that ξ is tangent to N_T, we obtain

$$2g(h^*(Z,\omega),\varphi X) = -g(AZ,\omega)g(\xi,X) + 2\{\eta(X)A - (\varphi X)\mu\}g(Z,\omega) \tag{3.21}$$

That is

$$g(h^*(Z,\omega),\varphi X) = (\varphi X)\mu g(Z,\omega) \tag{3.22}$$

Using (2.9), we derive

$$g(h^*(Z,\omega),\varphi X) = g(\nabla\mu,\varphi X)g(Z,\omega) \tag{3.23}$$

From the last relation, we obtain that

$$h^*(Z,\omega) = (\nabla\mu)g(Z,\omega) \tag{3.24}$$

The above relation shows that the leaves of $D^{\perp}$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover, the condition $V\mu = 0$, for any $V\varepsilon\Gamma(D^{\perp})$ implies that the leaves of $D^{\perp}$ are extrinsic spheres in M, that is the integral manifold $N_{\perp}$ of $D^{\perp}$ is umbilical and its mean curvature vector field is non zero and parallel along $N_{\perp}$. Hence, by a result of [11] M is locally a warped product $M = N_T \times_f N_{\perp}$, where N_T and $N_{\perp}$ denote the integral manifolds of the distributions $D\oplus\langle\xi\rangle$ and $D^{\perp}$, respectively and f is the warping function. □

4. Inequality for Contact CR-Warped products

For contact CR-Warped products in nearly quasi-Sasakian manifold, we have the following,

Theorem 4.1. *If $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly quasi-Sasakian manifold $\bar{M}$ such that N_T is an invariant submanifold tangent to ξ and $N_\perp$ an anti-invariant submanifold of $\bar{M}$, then*
(i) The second fundamental form of M satisfies the inequality

$$||h||^2 \geq 2As + 2s||\nabla \ln f||^2 \tag{4.1}$$

where s is the dimension of $N_\perp$ and $\nabla \ln f$ is the gradient of $\ln f$.
(ii) If the equality sign of (4.1) holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is a minimal submanifold in $\bar{M}$.

Proof. Let $\bar{M}$ be a $(2n+1)$-dimensional nearly quasi-Sasakian manifold and $M = N_T \times_f N_\perp$ be an m-dimensional contact CR-warped product submanifolds of $\bar{M}$ and $dimN_T = 2p+1$ and $dimN_\perp = s$, then $m = 2p+1+s$. Let $\{e_1 ...., e_p; \phi e_1 = e_{p+1}, ..., \phi e_p = e_{2p}, e_{2p+1} = \xi\}$ and $\{e_{(2p+1)+1}, ..., e_m\}$ be the local orthonormal frames on N_T and $N_\perp$, respectively. Then the orthonormal frames in the normal bundle $T^\perp M$ of $\phi D^\perp$ and v are $\{\phi e_{(2p+1)+1}, ..., \phi e_m\}$ and $\{e_{m+s+1}, ..., e_{2n+1}\}$, respectively. Then the length of second fundamental form h is defined as

$$||h||^2 = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 \tag{4.2}$$

For the assumed frames, the above equation can be written as

$$||h||^2 = \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 + \sum_{r=m+s+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 \tag{4.3}$$

The first term in the right hand side of the above equality is the $\phi D^\perp$-component and the second term is v-component. If we equate only the $\phi D^\perp$-component, then we have

$$||h||^2 \geq \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 \tag{4.4}$$

For the given frame of $\phi D^\perp$, the above equation will be

$$||h||^2 \geq \sum_{k=(2p+1)+1}^{m} \sum_{i,j=1}^{m} g(h(e_i, e_j), \phi e_k)^2$$

Let us decompose the above equation in terms of the components of $h(D,D), h(D,D^\perp)$ and $h(D^\perp, D^\perp)$, then we have

$$||h||^2 \geq \sum_{k=2p+2}^{m} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \phi e_k)^2 + 2\sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2 \tag{4.5}$$

$$+ \sum_{k=2p+2}^{m} \sum_{i,j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2$$

By Lemma 3.1 (ii), the first term of the right hand side of (4.5) is identically zero and we shall compute the next term and will left the last term

$$||h||^2 \geq 2 \sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2$$

As $j, k = 2p+2, ..., m$ then the above equation can be written for one summation as

$$||h||^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

Making use of Lemma 3.1 (iii), the above inequality will be

$$||h||^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} [\eta(e_i) g(Ae_j, e_k) - (\varphi e_i \ln f) g(e_j, e_k)]^2 \tag{4.6}$$

The above expression can be written as

$$||h||^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} \eta(e_i)^2 g(Ae_j, e_k)^2 + 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\varphi e_i \ln f)^2 g(e_j, e_k)^2 \tag{4.7}$$

$$-4 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} \eta(e_i)(\varphi e_i \ln f) g(e_j, e_k) g(Ae_j, e_k)$$

The last term of (4.7) is identically zero for the given frames. Thus, the above relation gives

$$||h||^2 \geq 2\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i \ln f)^2 g(e_j,e_k)^2 + 2s \tag{4.8}$$

On the other hand, from (2.10), we have

$$||\nabla \ln f||^2 = \sum_{i=1}^{p}(e_i \ln f)^2 + \sum_{i=1}^{p}(\phi e_i \ln f)^2 + (\xi \ln f)^2 \tag{4.9}$$

Now, the equation (4.8) can be modified as

$$||h||^2 \geq 2As + 2\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\varphi e_i \ln f)^2 g(e_j,e_k)^2$$

$$+2\sum_{j,k=2p+1}^{m}(\xi \ln f)^2 g(e_j,e_k)^2 - 2\sum_{j,k=2p+1}^{m}(\xi \ln f)^2 g(e_j,e_k)^2$$

or

$$||h||^2 \geq 2As - 2\sum_{j,k=2p+2}^{m}(\xi \ln f)^2 g(e_j,e_k)^2 + 2\sum_{j,k=2p+2}^{m}(\xi \ln f)^2 g(e_j,e_k)^2$$

$$+4\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}(\phi e_i \ln f)^2 g(e_j,e_k)^2$$

Therefore, using Lemma 3.1 (i) and (4.9), we arrive at

$$||h||^2 \geq 2As + 2s||\nabla \ln f||^2$$

which is the inequality (4.1). Let h^* be the second fundamental form of $N_\perp$ in M, then from (3.24), we have

$$h^*(Z,\omega) = g(Z,\omega)\nabla \ln f \tag{4.10}$$

for any $Z,W\varepsilon\Gamma(D^\perp)$. Now, assume that the equality case of (4.1) holds identically. Then from (4.3), (4.5) and (4.7), we obtain

$$h(D,D) = 0, \quad h(D^\perp,D^\perp) = 0, \quad h(D,D^\perp) \subset \phi D^\perp \tag{4.11}$$

Since N_T is a totally geodesic submanifold in M (by Lemma 2.1 (i)), using this fact with the first condition in (4.11) implies that N_T is totally geodesic in $\bar{M}$. On the other hand, by direct calculations same as in the proof of Theorem 3.4, we deduce that $N_\perp$ is totally umbilical in M. Therefore, the second condition of (4.11) with (4.10) implies that $N_\perp$ is totally umbilical in $\bar{M}$. Moreover, all three conditions of (4.11) imply that M is minimal submanifold of $\bar{M}$. This completes the proof of the theorem.

□

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