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Evaluation of some integral representations for extended Srivastava triple hypergeometric function $H_{C,p,v}(\cdot)$

Showkat Ahmad Dar[1]

[1*]*Department of Applied Sciences and Humanities , Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University) , New Delhi, 110025, India.*
[*]*Corresponding author E-mail: showkatjmi34@gmail.com*

Abstract

Many authors study some integral representations of the function $H_C(\cdot)$. Here, we obtain some integral representations for extended Srivastava triple hypergeometric function $H_{C,p,v}(\cdot)$ involving Meijer's G-function of one variable, confluent hypergeometric and Whittaker functions.

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1. Introduction and Preliminaries

For the sake of conciseness of this paper, we use the following notations
$$\mathbb{N} := \{1,2,...\}; \quad \mathbb{N}_0 := \mathbb{N}\cup\{0\}; \quad \mathbb{Z}_0^- := \mathbb{Z}^-\cup\{0\},$$
where the symbols $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural and integers number; as usual, the symbols $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers.

Hypergeometric functions of a single variable have a long history and arise in numerous branches of mathematics and physics. The Gauss hypergeometric function ${}_2F_1(\cdot)$ [19] is defined for $b_1, b_2 \in \mathbb{C},\ c_1 \in \mathbb{C}\setminus\mathbb{Z}_0^-$

$${}_2F_1\left(\begin{array}{c} b_1, b_2; \\ c_1; \end{array} z\right) = \sum_{\ell=0}^{\infty} \frac{(b_1)_\ell (b_2)_\ell}{(c_1)_\ell} \frac{z^\ell}{\ell!} = \sum_{\ell=0}^{\infty} \frac{(b_1)_\ell\, B(b_2+\ell, c_1-b_2)}{B(b_2, c_1-b_2)} \frac{z^\ell}{\ell!}, \tag{1.1}$$

where $|z|<1,\ \Re(c_1)>\Re(b_2)>0$. Here $(\alpha)_\upsilon$ $(\alpha,\upsilon\in\mathbb{C})$ denotes the Pochhammer's symbol (or the shifted factorial, since $(1)_n = n!$) is defined, in general, by

$$(\alpha)_\upsilon := \frac{\Gamma(\alpha+\upsilon)}{\Gamma(\alpha)} = \begin{cases} 1, & (\upsilon=0\,;\ \alpha\in\mathbb{C}\setminus\{0\}) \\ \alpha(\alpha+1)...(\alpha+n-1), & (\upsilon=n\in\mathbb{N}\,;\ \alpha\in\mathbb{C}). \end{cases} \tag{1.2}$$

and $B(\alpha,\beta)$ denotes the classical Beta function defined by [11, (5.12.1)]

$$B(\alpha,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, & (\Re(\alpha)>0, \Re(\beta)>0), \\ \\ \dfrac{\Gamma(\alpha)(\beta)}{\Gamma(\alpha+\beta)}, & (\Re(\alpha)<0, \Re(\beta)<0), \quad (\alpha,\beta)\in\mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} \tag{1.3}$$

Triple hypergeometric functions (that is functions of three variables x,y and z) have been introduced and investigated. The work of Srivastava and Karlsson [18, Chapter 3] provides a table of 205 distinct triple hypergeometric functions. Srivastava introduced the triple hypergeometric functions H_A, H_B and H_C of the second order in [15, 16]. It is known that H_C and H_B are generalizations of Appell's hypergeometric functions F_1 and F_2, while H_A is the generalization of both F_1 and F_2.

Email addresses: **showkatjmi34@gmail.com (Showkat Ahmad Dar)**

In the present study, we confine our attention to Srivastava's triple hypergeometric function H_C given by [18, p. 43, 1.5(11) to 1.5(13)] (see also [15] and [17, p. 68])

$$\begin{aligned} H_C(b_1,b_2,b_3;c_1;x,y,z): &= \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{m+k}(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_{m+n+k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!} \\ &= \sum_{m,n,k=0}^{\infty} \left[\frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\right]. \end{aligned} \tag{1.4}$$

The convergence region for this series $H_C(\cdot)$ is given in [9, p.243] as $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$,where α, β, γ satisfy the relation $\alpha+\beta+\gamma-2\sqrt{(1-\alpha)(1-\beta)(1-\gamma)} < 2$.
The modified Bessel function of the second kind $K_v(z)$ of order v (also known as the Basset function or Macdonald function) is defined by (see [11, p.251],[14])

$$K_v(z) = \sqrt{\pi}(2z)^v\, e^{-z}\, U\left(v+\frac{1}{2},\ 2v+1;\ 2z\right), \tag{1.5}$$

where $v \in \mathbb{C}\setminus\mathbb{Z}$ in [1], and $U(a,b,z)$ is the confluent hypergeometric function [11, p.322].
Bateman's K-function is given by [17, p.40, eq.(30)],

$$K_v(z) = \frac{1}{\Gamma(1+\frac{v}{2})} W_{\frac{v}{2},\ \pm\frac{1}{2}}(2z). \tag{1.6}$$

The Meijer's G-function is defined by means of the Mellin-Barnes contour integral [17, p.45,eq.(1)]

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{array}{l} \alpha_1,\ldots,\alpha_n;\alpha_{n+1},\ldots,\alpha_p \\ \beta_1,\ldots,\beta_m;\beta_{m+1},\ldots,\beta q \end{array}\right) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\prod_{j=1}^{m}\Gamma(\beta_j-\zeta)\prod_{j=1}^{n}\Gamma(1-\alpha_j+\zeta)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+\zeta)\prod_{j=n+1}^{p}\Gamma(\alpha_j-\zeta)} z^{\zeta}\, d\zeta\ , \tag{1.7}$$

where $z \neq 0$, and m,n,p,q are non negative integers such that $1 \leq m \leq q$; $0 \leq n \leq p$, and $p \leq q$. The integral (1.7) converges in the sector $|arg(z)| < \pi\kappa$ where $\kappa = m+n-\frac{1}{2}(p+q)$ and it is supposed that $\kappa > 0$.

The G function is important in applied mathematics; and developed formulas can be considered as a major key from which a very large number of relations can be deduced for Bessel functions, their combinations and many other related functions. Thus the following list of some particular cases of Meijer's G-function associated with the Bessel function $K_v(z)$ has been obtained mainly from several papers by C. S. Meijer (see also [8, pp.219-220, sec.(5.6), eq.(47-50)]; [17, p.48, eq.(12)])

$$K_v(z) = \sqrt{\pi}\, e^z\, G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{array}{l} \frac{1}{2} \\ v\ ,\ -v \end{array}\right), \tag{1.8}$$

$$= \frac{cos(v\pi)}{\sqrt{\pi}}\, e^{-z}\, G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{array}{l} \frac{1}{2} \\ v\ ,\ -v \end{array}\right), \tag{1.9}$$

$$= z^{-\mu}2^{\mu-1}G^{2,0}_{0,2}\left(\frac{z^2}{4} \,\middle|\, \frac{\mu+v}{2}\ ,\ \frac{\mu-v}{2}\right), \tag{1.10}$$

$$= (2z)^{-\mu}e^z\, G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{array}{l} \mu+\frac{1}{2} \\ \mu+v,\ \mu-v \end{array}\right), \tag{1.11}$$

$$= cos(v\pi)\frac{(2z)^{-\mu}e^z}{\sqrt{\pi}}\, G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{array}{l} \mu+\frac{1}{2} \\ \mu+v,\ \mu-v \end{array}\right), \tag{1.12}$$

$$= \frac{z^{-\mu}4^{\mu-1}}{\pi} G^{4,0}_{0,4}\left(\frac{z^4}{256} \,\middle|\, \frac{\mu+v}{4}\ ,\ \frac{2+\mu+v}{4},\ \frac{\mu-v}{4},\ \frac{2+\mu-v}{4}\right), \tag{1.13}$$

where μ is a free parameter and in all these expressions we have $z \neq 0$.

In the available literature, the hypergeometric series with its generalizations and extensions were given in the following references [2, 3, 6, 10, 12]. Srivastava introduced the triple hypergeometric function $H_C(\cdot)$, together with its integral representations, in [15] and [17]. Recently, S. A. Dar and R. B. Paris in [7], obtained an extension of $H_C(\cdot)$ function, which we denote by $H_{C,p,v}(\cdot)$, based on the extended Beta function $B_{p,v}(x,y)$ [13]. This is given by

$$H_{C,p,v}(b_1,b_2,b_3;c_1;x,y,z) = \sum_{m,n,k=0}^{\infty} \left[\frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B_{p,v}(b_1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\right], \tag{1.14}$$

where the parameters $b_1,b_2,b_3 \in \mathbb{C}$ and $c_1 \in \mathbb{C}\setminus\mathbb{Z}_0^-$. The region of convergence for the Srivastava's hypergeometric series $H_{C,p,v}(\cdot)$ is $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$. This definition clearly reduces to the original classical function when $p = 0 = v$. And its integral representation is given by [7]

$$\begin{aligned} H_{C,p,v}(b_1,b_2,b_3;c_1;x,y,z) &= \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)}\sqrt{\frac{2p}{\pi}}\int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1-\frac{3}{2}}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1-b_1-\frac{3}{2}} \\ &\times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\Big] d\xi dt, \end{aligned} \tag{1.15}$$

Many authors have studied some integral representations of the function $H_C(\cdot)$; see [4, 5]. Here, we obtain some integral representations for extended Srivastava triple hypergeometric function $H_{C,p,v}(\cdot)$ involving Meijer's G-function, confluent hypergeometric and Whittaker functions.

2. Some integral representations for $H_{C,p,v}(\cdot)$ function

Theorem 1 The following integral representations of (p,v)-extended Srivastava triple hypergeometric function holds for $\Re(p)>0,\ \Re(b_j)>0 (j=1,2)$ and $\Re(c_1)>0,\ \Re(c_1-b_1-b_2)>0,$

$$H_{C,p,v}(b_1,b_2,b_3;c_1;x,y,z)$$

$$= \frac{\Gamma(c_1)\sqrt{2p}}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1-\frac{3}{2}}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1-b_1-\frac{3}{2}} \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} exp\left\{\frac{p}{t(1-t)}\right\} G^{2,1}_{1,2}\left(\frac{2p}{t(1-t)}\middle| \begin{matrix}\frac{1}{2}\\ v+\frac{1}{2},-v-\frac{1}{2}\end{matrix}\right)\Big] d\xi dt, \tag{2.1}$$

$$= \frac{\Gamma(c_1)cos(v\sqrt{\pi})\sqrt{2p}}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)\pi} \int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1-\frac{3}{2}}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1-b_1-\frac{3}{2}} \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} exp\left\{-\frac{p}{t(1-t)}\right\} G^{2,1}_{1,2}\left(\frac{2p}{t(1-t)}\middle| \begin{matrix}\frac{1}{2}\\ v+\frac{1}{2},-v-\frac{1}{2}\end{matrix}\right)\Big] d\xi dt, \tag{2.2}$$

$$= \frac{\Gamma(c_1)p^{\frac{1}{2}-\mu}2^{\mu-1}}{\sqrt{\pi}\,\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1+\mu-\frac{3}{2}}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1+\mu-b_1-\frac{3}{2}} \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} G^{2,0}_{0,2}\left(\frac{p^2}{4t^2(1-t)^2}\middle| \frac{2\mu+2v+1}{4},\frac{2\mu-2v-1}{4}\right)\Big] d\xi dt, \tag{2.3}$$

$$= \frac{\Gamma(c_1)(2p)^{\frac{1}{2}-\mu}cos(v\pi)}{\pi\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1+\mu-\frac{3}{2}}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1-b_1+\mu-\frac{3}{2}} \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} exp\left\{\frac{p}{t(1-t)}\right\} G^{2,1}_{1,2}\left(\frac{2p}{t(1-t)}\middle| \begin{matrix}\mu+\frac{1}{2}\\ \mu+v+\frac{1}{2},\ \mu-v-\frac{1}{2}\end{matrix}\right)\Big] d\xi dt, \tag{2.4}$$

$$= \frac{\Gamma(c_1)(2p)^{\frac{1}{2}}4^{\mu-1}}{\pi p^{\mu}\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1+\mu-\frac{3}{2}}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1-b_1+\mu-\frac{3}{2}} \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} G^{4,0}_{0,4}\left(\frac{p^4}{(4t)^4(1-t)^4}\middle| \frac{2\mu+2v+1}{8},\frac{2\mu+2v+5}{8},\frac{2\mu-2v-1}{8},\frac{2\mu-2v+3}{8}\right)\Big] d\xi dt, \tag{2.5}$$

$$= \frac{\Gamma(c_1)(2p)^{v+1}}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1-v-2}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1-v-b_1-2} \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} U\left(v+1;\ 2v+2;\ \frac{2p}{t(1-t)}\right)\Big] d\xi dt, \tag{2.6}$$

$$= \frac{\Gamma(c_1)\sqrt{2p}}{\sqrt{\pi}\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)\Gamma\left(\frac{5+2v}{4}\right)} \int_0^1\int_0^1 \Big[\xi^{b_2-1}t^{b_1-\frac{3}{2}}(1-\xi)^{c_1-b_1-b_2-1}(1-t)^{c_1-b_1-\frac{3}{2}} \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt)-y(1-t)\xi\}^{b_3}} W_{\frac{2v+1}{4},\pm\frac{1}{2}}\left(\frac{2p}{t(1-t)}\right)\Big] d\xi dt, \tag{2.7}$$

where $|x|<1,\ |y|<1,\ |z|<1.$

Proof: The above integral representations (2.1)-(2.5) are obtained by using the eqns(1.8)-(1.13) in the expression of the extended Srivastava function in (1.15). Similarly, other integral representations of $H_{C,p,v}(\cdot)$ associated with the confluent hypergeometric and Whittaker function can be obtained by using (1.5) and (1.6) in (1.15).

3. Conclusion

We obtain some integral representations of the extended Srivastava's triple function $H_{C,p,v}(\cdot)$ associated with Meijer's G- function of one variable, confluent hypergeometric and Whittaker function.

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