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The Spectral Determinations of the Multicone Graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ and $K_w \bigtriangledown S \bigtriangledown S$

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Abstract

Characterizing classes of graphs which are determined by their spectra is often a hard and challenging problem. So, finding and introducing any class of these graphs can be an interesting and important problem. This paper aims to characterize new classes of multicone graphs which are determined by both their adjacency spectra and their Laplacian spectra. A multicone graph is obtained from the join of a clique and a regular graph. Let K_w be a complete graph on w vertices. It is proved that multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ and $K_w \bigtriangledown S \bigtriangledown S$ are determined by both their adjacency spectra and their Laplacian spectra, where P_{17} and S denote Paley graph of order 17 and Schläfli graph, respectively.

Keywords: *Adjacency spectrum, Laplacian spectrum, Multicone graph, Paley graph of order 17, Schläfli graph, DS graph.*
2010 Mathematics Subject Classification: *05C50*

1. Introduction

All graphs considered here are simple and undirected. Let G be a graph with vertex set $V = V(G) = \{v_1, ..., v_n\}$ and edge set $E(G)$. All notions on graphs that are not defined here can be found in [12, 13, 18, 22, 38]. A graph consisting of k disjoint copies of an arbitrary graph G will be denoted by kG. The complement of a graph G is denoted by $\overline{G}$. Let matrix $A(G)$ be the $(0,1)$-adjacency matrix of G and d_k the degree of the vertex v_k. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G, where $D(G)$ is the $n \times n$ diagonal matrix with $V = V(G) = \{d_1, ..., d_n\}$ as diagonal entries (and all other entries 0). Since both matrices $A(G)$ and $L(G)$ are real and symmetric, their eigenvalues are all real numbers. Assume that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_n (= 0)$ are respectively the adjacency eigenvalues and the Laplacian eigenvalues of graph G. Two graphs G and H are said to be cospectral if they have equal spectrum (i.e., equal characteristic polynomial). If G and H are isomorphic, they are necessarily cospectral. A graph G is called DS (**determined by spectrum**) if whenever H is cospectral with G, H must be isomorphic to G. We use the multi-set $\mathrm{Spec}_A(G) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, ..., [\lambda_n]^{m_n}\}$ for stating the adjacency spectrum of G, where m_i denote the multiplicities of λ_i. For the Laplacian spectrum the notation is similar. The Laplacian spectrum of a graph G is denoted by $L(G)$. In particular, we use the following notation on graph operations. We define the *sum* $G + H$ of two vertex-disjoint graphs G and H to be their union; that is, $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H)$. The *join* $G \bigtriangledown H$ (or $G * H$) is obtained from $G + H$ by adding an edge from each vertex of G to each vertex of H, that is, by adding the set of edges $\{vw : v \in V(G), w \in V(H)\}$.

So far numerous examples of cospectral but non-isomorphic graphs are constructed by interesting techniques such as Seidel switching, Godsil-McKay switching, Sunada or Schwenk method. For more information, one may see [16, 32, 33] and the references cited in them. Only a few graphs with very special structures have been reported to be determined by their spectra (see [14, 17, 19, 21, 25, 31, 35, 36] and the references cited in them). Recently Wei Wang and Cheng-Xian Xu have developed a new method in [36] to show that many graphs are determined by their spectrum and the spectrum of their complement.

Van Dam and Haemers [32] conjectured that almost all graphs are determined by their spectra. Nevertheless, the set of graphs that are known to be determined by their spectra is too small. So, discovering classes of graphs that are determined by their spectra can be an interesting problem. The characterization of DS graphs goes back about half of a century and it is originated in Chemistry [20, 29]. About the background of the question "Which graphs are determined by their spectrum?", we refer to [32]. A spectral characterization of multicone graphs is studied in [35, 37]. In [37], Wang, Zhao and Huang investigated on the spectral characterization of multicone

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graphs and also they claimed that friendship graph F_n(that are special classes of multicone graphs) are DS with respect to their adjacency spectra. In addition, Wang, Belardo, Huang and Borovićanin [35] proposed such conjecture on the adjacency spectrum of F_n. This conjecture caused some activity on the spectral characterization of F_n. Finally, Cioabä, Haemers, Vermette and Wong [17] proved that if $n \neq 16$, then friendship graphs F_n are determined by their adjacency spectra. Abdian and Mirafzal [1] characterized new classes of multicone graphs which were DS with respect to their spectra. Abdian [2] characterized two classes of multicone graphs and proved that the join of an arbitrary complete graph and the generalized quadrangle graph $GQ(2,1)$ or $GQ(2,2)$ is determined by its adjacency spectra as well as its Laplacian spectra. This author also proposed four conjectures about adjacency spectrum of complement and signless Laplacian spectrum of these multicone graphs. In [3], the author showed that multicone graphs $K_w \bigtriangledown P_{17}$ and $K_w \bigtriangledown S$ are determined by their adjacency spectra as well as their Laplacian spectra, where P_{17} and S denote Paley graph of order 17 and Schläfli graph, respectively. Also, this author conjectured that these multicone graphs are determined by their signless Laplacian spectra. In [4], the author proved that multicone graphs $K_w \bigtriangledown L(P)$ are determined by both their adjacency spectra and Laplacian spectra, where $L(P)$ denotes the line graph of the Petersen graph. He also proposed three conjectures about the signless Laplacian spectrum and the complement spectrum of these multicone graphs. For further information about some multicone graphs which have been characterized so far see [1–10,18,27,28,31].

In this paper, we present some techniques which enable us to characterize graphs that are DS with respect to their adjacency and Laplacian spectra. that are DS with respect to their adjacency and Laplacian spectra.

The paper is organized as follows. In Subsection 4.1, we show that any graph cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ is DS with respect to its adjacency spectrum. In Subsection 4.2 we prove that multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ are DS with respect to their Laplacian spectra. In Section 5 we show that multicone graphs $K_w \bigtriangledown S \bigtriangledown S$ are DS with respect to their adjacency spectra and their Laplacian spectra. We conclude with final remarks and open problems in Section 6.

2. Preliminaries

In this section we present some results which will play an important role throughout this paper.

Lemma 2.1. *[1–4, 13, 27, 32] Let G be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*

(i) The number of vertices,
(ii) The number of edges.
For the adjacency matrix, the following follows from the spectrum:
(iii) The number of closed walks of any length,
(iv) Being regular or not and the degree of regularity,
(v) Being bipartite or not.
For the Laplacian matrix, the following follows from the spectrum:
(vi) The number of spanning trees,
(vii) The number of components,
$(viii)$ The sum of squares of degrees of vertices.
The adjacency spectrum of Paley graph P_{17} and Schläfli graph S are given below ([32]):
(ix) $\mathrm{Spec}_A(P_{17}) = \left\{ [8]^1, \left[\frac{-1+\sqrt{17}}{2}\right]^8, \left[\frac{-1-\sqrt{17}}{2}\right]^8 \right\}.$
(x) $\mathrm{Spec}_A(S) = \left\{ [10]^1, [1]^{20}, [-5]^6 \right\}.$

Theorem 2.2. *[1–5, 18] If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by:*

$$P_{G_1 \bigtriangledown G_2(y)} = \frac{P_{G_1}(y)P_{G_2}(y)}{(y-r_1)(y-r_2)}((y-r_1)(y-r_2)-n_1n_2).$$

The *spectral radius* of a graph Λ is the largest eigenvalue of adjacency matrix of graph Λ and it is denoted by $\rho(\Lambda)$. A graph is called *bidegreed*, if the set of degrees of its vertices consists of two elements.

Theorem 2.3. *[1–4, 37] Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\rho(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\rho(G) \leq \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Theorem 2.4. *[1–4, 26] Let G and H be two graphs with Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_m$, respectively. Then Laplacian spectra of $\overline{G}$ and $G \bigtriangledown H$ are $n-\lambda_1, n-\lambda_2, ..., n-\lambda_{n-1}, 0$ and $n+m, m+\lambda_1, ..., m+\lambda_{n-1}, n+\mu_1, ..., n+\mu_{m-1}, 0$, respectively.*

Theorem 2.5. *[1–4, 26] Let G be a graph on n vertices. Then n is one of the Laplacian eigenvalue of G if and only if G is the join of two graphs.*

Theorem 2.6. *[1–4, 22] For a graph G, the following statements are equivalent:*

(i) G is d-regular.
(ii) $\rho(G) = d_G$, *the average vertex degree.*
(iii) G has $v = (1,1,...,1)^T$ *as an eigenvector for* $\rho(G)$.

Proposition 2.1. *[1–4, 18, 30] Let* $G - j$ *be the graph obtained from G by deleting the vertex j and all edges containing j. Then* $P_{G-j}(y) = P_G(y)\sum_{i=1}^{m}\frac{\alpha_{ij}^2}{y-\mu_i}$, *where m and* α_{ij} *are the number of distinct eigenvalues and the main angles (see [30]) of graph G, respectively.*

Theorem 2.7. *[1, 12] If G is an r-regular graph with eigenvalues* $\lambda_1(=r), \lambda_2, ..., \lambda_n$, *then* $n-1-\lambda_1, -1-\lambda_2, ..., -1-\lambda_n$ *are the eigenvalues of the complement* $\overline{G}$ *of G.*

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Theorem 3.7. *[1, 12] If G is an r-regular graph with eigenvalues* $\lambda_1(=r), \lambda_2, ..., \lambda_n$, *then* $n-1-\lambda_1, -1-\lambda_2, ..., -1-\lambda_n$ *are the eigenvalues of the complement* $\overline{G}$ *of G.*

Remark 3.8. *For seeing some multicone graphs which have been characterized by similar techniques refer to [?, ?, 1–5, 27, 28].*

4. Main Results

The main goal of this section is to prove that multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ are DS with respect to their adjacency spectrum.

4.1. Graphs cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$

Proposition 4.1. *The adjacency spectrum of multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ is:*

$$\left\{[-1]^{w-1}, \left[\frac{-1+\sqrt{17}}{2}\right]^{16}, \left[\frac{-1-\sqrt{17}}{2}\right]^{16}, [-9]^1, \left[\frac{\theta+\sqrt{\theta^2+4\Gamma}}{2}\right]^1, \left[\frac{\theta-\sqrt{\theta^2+4\Gamma}}{2}\right]^1\right\},$$

where $\theta = w+24$ and $\Gamma = 9w+25$.

Proof By Theorem 3.2 and Lemma 3.1 (ix), $\mathrm{Spec}_A(P_{17} \bigtriangledown P_{17}) =$
$\left\{[25]^1, \left[\frac{-1\pm\sqrt{17}}{2}\right]^{16}, [-9]^1\right\}$. Now, by using again of Theorem 3.2 the result follows. □

In Lemma 3.1, we show that any graph cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ must be bidegreed.

Lemma 4.1. *Let G be cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$. Then $\delta(G) = w+25$.*

Proof Suppose that $\delta(G) = w+25+x$, where x is an integer number. First, it is clear that in this case the equality in Theorem 3.3 happens, if and only if $x = 0$. We show that $x = 0$. By contrary, we suppose that $x \neq 0$. It follows from Theorem 3.3 and Proposition 4.1 that
$\rho(G) = \frac{w+24+\sqrt{8k-4l(w+25)+(w+26)^2}}{2} < \frac{w+24+x+\sqrt{8k-4l(w+25)+(w+26)^2+x^2+(2w+52-4l)x}}{2}$,
where the integer numbers k and l denote the number of edges and the number of vertices of the graph G, respectively. For convenience, we let $B = 8k-4l(w+25)+(w+26)^2 \geq 0$ and $C = w+26-2l$, and also let $g(x) = x^2+(2w+52-4l)x = x^2+2Cx$.

Then clearly

$$\frac{w+24+\sqrt{B}}{2} < \frac{w+24+x+\sqrt{B+g(x)}}{2}$$

Therefore,

$$\sqrt{B} - \sqrt{B+g(x)} < x.$$

We consider two cases:

Case 1. $x < 0$.

Obviously, $|\sqrt{B} - \sqrt{B+g(x)}| > |x|$, since $x < 0$.
By a simple calculation we receive

$$2B + g(x) - 2\sqrt{B(B+g(x))} > x^2.$$

Replacing $g(x)$ by x^2+2Cx, we will have:

$$B + Cx > \sqrt{B(B+x^2+2Cx)}.$$

Obviously $Cx \geq 0$. By a simple computation we get:

$$C^2 > B.$$

Therefore,

$$k < \frac{l(l-1)}{2}.$$

So, if $x < 0$, then G cannot be a complete graph. In other words, if G is a complete graph, then $x > 0$. Or one can say that if G is a complete graph, then:

$$\delta(G) > w+25 \ (\dagger).$$

Case 2. $x > 0$.

In the same way of Case 1 we can conclude that if G is a complete graph, then:

$$\delta(G) < w+25 \ (\ddagger).$$

But, two inequalities (†) and (‡) cannot be true together. Hence we must have $x = 0$.
Therefore, the assertion holds. □

Lemma 4.2. *Let G be cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$. Then G is bidegreed in which any vertex of G is of degree $w+33$ or $w+25$.*

Proof By Theorems 3.3, 3.6 and Lemma 4.1 the proof is straightforwad. □

Proposition 4.2. *The graph $P_{17} \bigtriangledown P_{17}$ is DS with respect to its adjacency spectra.*

Proof Let $Spec_A(G) = Spec_A(P_{17} \bigtriangledown P_{17})$. So, it follows from Theorem 3.7 that $Spec_A(\overline{G}) = Spec_A(\overline{P_{17} \bigtriangledown P_{17}}) = Spec_A(2P_{17})$, since P_{17} is a self-complementary graph; that is, $\overline{P_{17}} = P_{17}$. This follows that $\overline{G} \cong 2P_{17}$ (see Proposition 10 of [32]) and so $G \cong P_{17} \bigtriangledown P_{17}$, as desired.□

Remark 4.3. *In a similar way of Proposition 4.2 one can deduce that the graph $S \bigtriangledown S$ is DS with respect to its adjacency spectra.*

In Lemma 4.4, we prove that the multicone graph $K_1 \bigtriangledown P_{17} \bigtriangledown P_{17}$, the cone of the graph $P_{17} \bigtriangledown P_{17}$, is DS with respect to its adjacency spectrum.

Lemma 4.4. *Any graph cospectral with the multicone graph $K_1 \bigtriangledown P_{17} \bigtriangledown P_{17}$ is DS with respect to its adjacency spectra.*

Proof Let $Spec_A(G) = Spec_A(K_1 \bigtriangledown P_{17} \bigtriangledown P_{17})$. It follows from Lemma 4.2 that G is bidegreed in which any its vertex is of degree 34 or 26. We suppose that G has k vertex (vertices) of degree 34. Therefore, by Lemma 3.1 (ii) and the spectrum of the graph G, $34k + (35-k)26 = 918$. This implies that $k = 1$. So, G has one vertex of degree 34, say j. By Proposition 3.1 $P_{G-j}(\lambda) = (\lambda-\mu_3)^{15}(\lambda-\mu_4)^{15}[\alpha_{1j}^2 A_1 + \alpha_{2j}^2 A_2 + \alpha_{3j}^2 A_3 + \alpha_{4j}^2 A_4]$, where

$\mu_1 = \dfrac{25+\sqrt{761}}{2}$, $\mu_2 = \dfrac{25-\sqrt{761}}{2}$, $\mu_3 = \dfrac{-1+\sqrt{17}}{2}$ and $\mu_4 = \dfrac{-1-\sqrt{17}}{2}$, $\mu_5 = -9$.

$$
\begin{aligned}
A_1 &= (\lambda-\mu_2)(\lambda-\mu_3)(\lambda-\mu_4)(\lambda-\mu_5),\\
A_2 &= (\lambda-\mu_1)(\lambda-\mu_3)(\lambda-\mu_4)(\lambda-\mu_5),\\
A_3 &= (\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_4)(\lambda-\mu_5),\\
A_4 &= (\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_3)(\lambda-\mu_5),\\
A_5 &= (\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_3)(\lambda-\mu_4),
\end{aligned}
$$

We know that $G-j$ has 34 eigenvalues (see Lemma 4.2). In other words, $P_{G-j}(\lambda)$ has 17 roots. Also, by removing the vertex j from graph G the number of edges and triangles which are removed from graph G are $|V(G-j) = 34|$ and $|E(G-j)| = 425$, respectively, since by what is proved in the first of this lemma G has one vertex of degree 34 and 34 vertices of degree 26. This means that $G-j$ has 34 vertices of degree 25. Now, by computing the closed walks of lengths 1, 2 and 3 belonging to $G-j$ we have:

$\alpha+\beta+\gamma+25 = -(15\mu_3+15\mu_4)$,
$\alpha^2+\beta^2+\gamma^2+625 = 850-(15\mu_3^2+15\mu_4^2)$,
$\alpha^3+\beta^3+\gamma^3+15625 = 14688-(15\mu_3^3+15\mu_4^3)$,

where α, β and γ are the eigenvalues of graph $G-j$. By solving the above equations $\alpha = \dfrac{-1+\sqrt{17}}{2}$, $\beta = \dfrac{-1-\sqrt{17}}{2}$ and $\gamma = -9$. Hence $Spec_A(G-j) = Spec_A(P_{17} \bigtriangledown P_{17})$ and so $G-j \cong P_{17} \bigtriangledown P_{17}$.
This follows the result. □

Up to now, we show that the multicone graph $K_1 \bigtriangledown P_{17} \bigtriangledown P_{17}$ is DS with respect to its adjacency spectrum. The natural question is; what happen for multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$? the next theorem answers to this question.

Theorem 4.5. *Multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ are DS with respect to their adjacency spectrum.*

Proof We perform the mathematical induction on w. If $w = 1$, by Lemma 4.4 the proof is completed. Let the claim be true for w; that is, if $Spec_A(G_1) = Spec_A(K_w \bigtriangledown P_{17} \bigtriangledown P_{17})$, then $G_1 \cong K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$, where G_1 is an arbitrary graph cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$. We show that the claim is true for $w+1$; that is, if $Spec_A(G) = Spec_A(K_{w+1} \bigtriangledown P_{17} \bigtriangledown P_{17})$, then $G \cong K_{w+1} \bigtriangledown P_{17} \bigtriangledown P_{17}$, where G is a graph. By Lemma 4.2, we can conclude that $G-j \cong K_w \bigtriangledown H$, where H is a 25-regular graph with 34 vertices and j is the vertex of degree $w+34$ belonging to G. On the other hand, by Theorem 3.3 $G-j$ has the eigenvalues $\beta_{1,2} = \dfrac{\Lambda \pm \sqrt{\Lambda^2+4\Gamma}}{2}$, where $\Lambda = w+24$ and $\Gamma = 9w+25$. Now, in a similar manner of Lemma 4.4 for $G-j$ and the closed walks of lengths 1, 2 and 3, we obtain $Spec_A(G-j) = Spec_A(K_w \bigtriangledown P_{17} \bigtriangledown P_{17})$. Hence the induction hypothesis follows that $G-j \cong K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$. This completes the proof. □

In the following we present two alternate proofs of Theorem 4.5.

Proof (the second proof of Theorem 4.5) We perform mathematical induction on w. For $w = 1$, the result follows from Lemma 4.4. Let theorem be true for w; that is, if $Spec_A(G_1) = Spec_A(K_w \bigtriangledown P_{17} \bigtriangledown P_{17})$, then $G_1 \cong K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$, where G_1 is an arbitrary graph cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$. We show that theorem is true for $w+1$; that is, if $Spec_A(G) = Spec_A(K_{w+1} \bigtriangledown P_{17} \bigtriangledown P_{17})$, then $G_1 \cong K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$, where G is a graph. It is clear that G has one vertex and $w+34$ edges more than G_1. On the other hand, it follows from Lemma 4.2 G_1 has w vertices of degree $w+33$ and 34 vertices of degree $w+25$. Also, by this lemma we can conclude that G has w vertices of degree $w+34$ and 34 vertices of degree $w+26$. This means that we must have $G \cong K_1 \bigtriangledown G_1$. Now, the inductive hypothesis follows the result.□

Proof (the third proof of Theorem 4.5) Let $Spec_A(G_1) = Spec_A(K_w \bigtriangledown P_{17} \bigtriangledown P_{17})$. It follows from Lemma 4.2 that G has graph Γ as its subgraph in which degree of any its vertex is $w+33$. In other words, $G \cong K_w \bigtriangledown H$, where H is a subgraph of G. Now, we remove vertices of K_w and we consider 34 another vertices. Degree of graph consists of these vertices is 25. Consider H consisting of these 34 vertices. H is 25-regular and the multiplicity of 25 is 1. By Theorem 3.3 $Spec_A(H) = \left\{ [25]^1, \left[\dfrac{-1\pm\sqrt{17}}{2}\right]^{16}, [-9]^1 \right\} = Spec_A(P_{17} \bigtriangledown P_{17})$. This completes the proof. □

4.2. Graphs cospectral with a multicone graph $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ with respect to Laplacian spectrum

In this subsection, we show that multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ are DS with respect to their Laplacian spectrum.

Theorem 4.6. *Multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ are DS with respect to their Laplacian spectrum.*

Proof We solve the problem by induction on w. If $w = 1$, by Theorems 3.5 and 3.6 the proof is straightforwad. Let the claim be true for w; that is, $\mathrm{Spec}_L(G_1) = \mathrm{Spec}_L(K_w \bigtriangledown P_{17} \bigtriangledown P_{17}) =$
$\left\{[0]^1, [34+w]^{w+1}, \left[\frac{\sqrt{17}+51}{2}+w\right]^{16}, \left[\frac{-\sqrt{17}+51}{2}+w\right]^{16}\right\}$ follows that $G_1 \cong K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$, where G_1 is a graph. We show that the claim is true for $w+1$; that is, we show that $\mathrm{Spec}_L(G) = \mathrm{Spec}_L(K_{w+1} \bigtriangledown P_{17} \bigtriangledown P_{17}) =$
$\left\{[0]^1, [35+w]^{w+1}, \left[\frac{51 \pm \sqrt{17}}{2}+w+1\right]^{16}\right\}$ follows that $G \cong K_{w+1} \bigtriangledown P_{17} \bigtriangledown P_{17}$, where G is a graph. It follows from Lemma 3.5 that G_1 and G are the join of two graphs. On the other hand, G has one vertex and $w+34$ edges more than G_1. Therefore, we must have $G \cong K_1 \bigtriangledown G_1$. Now, the induction hypothsis follows the assertion. □

5. Graphs cospectral with a multicone graph $K_w \bigtriangledown S \bigtriangledown S$

From now on, with the similar arguments of the above results, we characterize another new classes of multicone graphs that are DS with respect to their spectra.

Proposition 5.1. *The adjacency spectrum of multicone graphs $K_w \bigtriangledown S \bigtriangledown S$ is:*

$$\left\{[-1]^{w-1}, [1]^{40}, [-17]^1, [-5]^{12}, \left[\frac{\Lambda+\sqrt{\Lambda^2+4\Gamma}}{2}\right]^1, \left[\frac{\Lambda+\sqrt{\Lambda^2+4\Gamma}}{2}\right]^1\right\},$$

where $\Lambda = 36+w$ and $\Gamma = 17w+37$.

Proof By Lemma 3.1 (x) and Theorem 3.2 the proof is completed. □
Similar to Lemma 3.3 we have the following lemma.

Lemma 5.1. *Let G be cospectral with a multicone graph $K_w \bigtriangledown S \bigtriangledown S$. Then G is bidegreed in which any vertex of G is of degree $w+37$ or $w+53$.*

In the following, we show that any graph cospectral with multicone graph $K_1 \bigtriangledown S \bigtriangledown S$ is DS with respect to its adjacency spectra.

Lemma 5.2. *Multicone graphs $K_1 \bigtriangledown S \bigtriangledown S$ are DS with respect to their adjacency spectra.*

Proof Let G be cospectral with multicone graph $K_1 \bigtriangledown S \bigtriangledown S$. By Lemma 5.1, it is easy to see that G has one vertex of degree 38, say l. Now, it follows from Proposition 3.1 that $P_{G-l}(\lambda) = (\lambda-\mu_3)^{39}(\lambda-\mu_4)^{11}[\alpha_{1j}^2 D_1 + \alpha_{2j}^2 D_2 + \alpha_{3j}^2 D_3 + \alpha_{4j}^2 D_4 + \alpha_{5j}^2 D_5]$, where
$\mu_1 = \frac{37+\sqrt{1585}}{2}$, $\mu_2 = \frac{37-\sqrt{1585}}{2}$, $\mu_3 = 1$, $\mu_4 = -5$ and $\mu_5 = -17$.

$$\begin{aligned}
D_1 &= (\lambda-\mu_2)(\lambda-\mu_3)(\lambda-\mu_4)(\lambda-\mu_5),\\
D_2 &= (\lambda-\mu_1)(\lambda-\mu_3)(\lambda-\mu_4)(\lambda-\mu_5),\\
D_3 &= (\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_4)(\lambda-\mu_5),\\
D_4 &= (\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_3)(\lambda-\mu_5),\\
D_5 &= (\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_3)(\lambda-\mu_4),
\end{aligned}$$

Now, we have:

$$\begin{aligned}
&\varepsilon+\zeta+\vartheta+37 = -(39\mu_3+11\mu_4),\\
&\varepsilon^2+\zeta^2+\vartheta^2+1369 = 1998-(39\mu_3^2+11\mu_4^2),\\
&\varepsilon^3+\zeta^3+\vartheta^3+50653 = 44280-(39\mu_3^3+11\mu_4^3),
\end{aligned}$$

where ε, ζ and ϑ are the eigenvalues of graph $G-j$. By solving the above equations $\varepsilon = 1$, $\zeta = -5$ and $\vartheta = 17$. Hence $\mathrm{Spec}_A(G-j) = \mathrm{Spec}_A(S \bigtriangledown S)$ and so $G-j \cong S \bigtriangledown S$.

This completes the proof. □

Theorem 5.3. *Multicone graphs $K_w \bigtriangledown S \bigtriangledown S$ are DS with respect to their adjacency spectrum.*

Proof We perform the mathematical induction on w. For, $w = 1$ the result follows from Lemma 5.2. Let the claim be true for w; that is, if $\mathrm{Spec}_A(G_1) = \mathrm{Spec}_A(K_w \bigtriangledown S \bigtriangledown S)$, then $G_1 \cong K_w \bigtriangledown S \bigtriangledown S$, where G_1 is an arbitrary graph cospectral with a multicone graph $K_w \bigtriangledown S \bigtriangledown S$. We show that the claim is true for $w+1$; that is, if $\mathrm{Spec}_A(G) = \mathrm{Spec}_A(K_{w+1} \bigtriangledown S \bigtriangledown S)$, then $G \cong K_{w+1} \bigtriangledown S \bigtriangledown S$, where G is a graph. By Lemma 5.1 G_1 has 54 vertices of degree $w+37$ and w vertices of degree $w+53$. Also, by this lemma G has 54 vertices of degree $w+38$ and $w+1$ vertices of degree $w+54$. On the other hand, it is easy and straightforward to see that G has one vertex and $w+54$ edges more than G_1. So, we must have $G \cong K_1 \bigtriangledown G_1$. Now, the inductive hypothesis follows the proof.□

In this subsection, we show that multicone graphs $K_w \bigtriangledown S \bigtriangledown S$ are DS with respect to their Laplacian spectrum.

Theorem 5.4. *Multicone graphs $K_w \bigtriangledown S \bigtriangledown S$ are DS with respect to their Laplacian spectrum.*

Proof We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, $\text{Spec}_L(G_1) = \text{Spec}_L(K_w \bigtriangledown S \bigtriangledown S) = \left\{[w+54]^{w+1}, [w+42]^{12}, [w+36]^{40}, [0]^1\right\}$ follows that $G_1 \cong K_w \bigtriangledown S \bigtriangledown S$, where G_1 is a graph. We show that the claim is true for $w+1$; that is, we show that $\text{Spec}_L(G) = \text{Spec}_L(K_{w+1} \bigtriangledown S \bigtriangledown S) = \left\{[w+55]^{w+2}, [w+43]^{12}, [w+37]^{40}, [0]^1\right\}$ follows that $G \cong K_{w+1} \bigtriangledown S \bigtriangledown S$, where G is a graph. By Theorem 3.5 G_1 and G are the join of two graphs. On the other hand, G has one vertex and $w+54$ edges more than G_1. Therefore, $G \cong K_1 \bigtriangledown G_1$. Now, the inductive hypothesis completes the proof. □

6. Concluding remarks and some open problems

In this study, we prove that multicone graphs $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ and $K_w \bigtriangledown S \bigtriangledown S$ are determined by their adjacency spectra as well as their Laplacian spectra. Now, we pose the following conjectures.

Conjecture 6.1. *Graphs $\overline{K_w \bigtriangledown P_{17} \bigtriangledown P_{17}}$ and $\overline{K_w \bigtriangledown S \bigtriangledown S}$ are DS with respect to their adjacency spectrum.*

Conjecture 6.2. *Multicone graphs $K_w \bigtriangledown S \bigtriangledown S$ and $K_w \bigtriangledown P_{17} \bigtriangledown P_{17}$ are DS with respect to their signless Laplacian spectrum.*

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