Area of a Triangle in Terms of the \( m \)-Generalized Taxicab Distance

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Abstract

In this paper, we give three area formulas for a triangle in the \( m \)-generalized taxicab plane in terms of the \( m \)-generalized taxicab distance. The two of them are \( m \)-generalized taxicab versions of the standard area formula for a triangle, and the other one is an \( m \)-generalized taxicab version of the well-known Heron’s formula.

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1. Introduction

Taxicab geometry was introduced by Menger [11], and developed by Krause [10], using the taxicab metric which is the special case of the well-known \( l_p \)-metric (also known as Minkowski distance) for \( p = 1 \). In this geometry, circles are squares with each diagonal is parallel to a coordinate axis. Afterward, in [15] Lawrence J. Wallen defined the (slightly) generalized taxicab metric, in which circles are rhombuses with each diagonal is also parallel to a coordinate axis. Finally, \( m \)-generalized taxicab metric is defined in [3], for any rhombus (so, any square) to be a circle instead of rhombuses having each diagonal parallel to a coordinate axis. In the last case, for any real number \( m \) and positive real numbers \( u \) and \( v \), the \( m \)-generalized taxicab distance between points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) in \( \mathbb{R}^2 \) is defined by

\[
d_{T(m)}(P_1, P_2) = (u|x_1 - x_2| + m|y_1 - y_2|) + v|m(x_1 - x_2) - (y_1 - y_2)|/(1 + m^2)^{1/2}.
\]

In addition, as a special case of \( d_{T(m)} \) for \( u = v = 1 \),

\[
d_{T(1)}(P_1, P_2) = ((x_1 - x_2) + m|y_1 - y_2|) + |m(x_1 - x_2) - (y_1 - y_2)|/(1 + m^2)^{1/2}
\]

is called the \( m \)-taxicab distance between points \( P_1 \) and \( P_2 \), while the well-known Euclidean distance between \( P_1 \) and \( P_2 \) is

\[
d_{E}(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}.
\]

The \( m \)-generalized taxicab unit circle is a rhombus with diagonals having slopes of \( m \) and \(-1/m \), and with vertices \( A_1 = \left( \frac{1}{m}, \frac{1}{m} \right) \), \( A_2 = \left( -\frac{1}{m}, \frac{1}{m} \right) \), \( A_3 = \left( \frac{1}{m}, -\frac{1}{m} \right) \) and \( A_4 = \left( -\frac{1}{m}, -\frac{1}{m} \right) \), where \( k = (1 + m^2)^{1/2} \); if \( u = v \), then \( m \)-generalized taxicab unit circle is a square with vertices \( A_1, A_2, A_3 \) and \( A_4 \). The \( m \)-generalized taxicab distance between two points is invariant under all translations. In addition, if \( u \neq v \), then the \( m \)-generalized taxicab distance between two points is invariant under rotations of \( \pi \) radian around a point and reflections in lines parallel to the lines with slope \( m \) and \(-1/m \); if \( u = v \), then rotations of \( \pi/2 \), \( \pi \) and \( 3\pi/2 \) radians around a point, and reflections in lines parallel to the lines with slope \( m \), \(-1 \) or \(-m \) (see [3], [4] and [6]).

Since the distance function is different from that of Euclidean geometry, it is interesting to study the \( m \)-generalized taxicab analogues of topics that include the distance concept in Euclidean geometry. In this paper, we give area formulas for a triangle in the \( m \)-generalized taxicab plane in terms of the \( m \)-generalized taxicab distance. One can see from Figure 1 that there are triangles whose \( m \)-generalized taxicab lengths of corresponding sides are the same, while areas of these triangles are different, in the \( m \)-generalized taxicab plane. So, how can one compute the area of a triangle in the \( m \)-generalized taxicab plane? In this study, we present three formulas to compute the area of a triangle in the \( m \)-generalized taxicab plane. Henceforth, we use \( u' = u/(1 + m^2)^{1/2} \) and \( v' = v/(1 + m^2)^{1/2} \) to shorten phrases.
Theorem 2.1. Let $ABC$ be a triangle with area $\mathcal{A}$. The following theorem gives the first $m$-generalized taxicab version of the area formula.

2. The $m$-generalized taxicab version of standard area formula

It is well-known that the standard area formula for triangle $ABC$ is $\mathcal{A} = \frac{ah}{2}$, where $a = d_E(B,C)$ and $h = d_E(A,B)$ where $H$ is the orthogonal projection of the point $A$ on the line $BC$. Here, we give two $m$-generalized taxicab versions of this formula in terms of the $m$-generalized taxicab distance, depending on choice of $h = d_{T^m}(A,H)$ or $h' = d_{T^m}(A,BC)$. The following equation given in [3], which relates the Euclidean distance to the $m$-generalized taxicab distance between two points in the Cartesian coordinate plane, plays an important role in the first $m$-generalized taxicab version of the area formula.

Proposition 2.1. For any two points $A$ and $B$ in $\mathbb{R}^2$ that do not lie on a vertical line, if $n$ is the slope of the line through $A$ and $B$, then

$$d_E(A,B) = \mu(n)d_{T}(A,B)$$

(2.1)

where $\mu(n) = (1 + n^2)^{1/2}/(u'M + v'M)$. If $A$ and $B$ lie on a vertical line, then

$$d_E(A,B) = 1/(u'M + v'M)d_{T}(A,B).$$

(2.2)

Notice that $\mu(m) = \frac{1}{m}$ and if $m \neq 0$, then $\mu(1/m) = \frac{1}{m}$. Therefore, if $l_A$ is the line through $A$ with slope $m$, and $l_B$ is the line through $B$ and perpendicular to the line $l_A$, then

$$d_{T}(A,B) = ud_E(A,l_B) + vd_E(B,l_A).$$

In addition, for any non-zero real number $n$, if $a = v$ then $\mu(n) = \mu(1/n)$.

The following theorem gives the first $m$-generalized taxicab version of the standard area formula of a triangle.

Theorem 2.1. Let $ABC$ be a triangle with area $\mathcal{A}$ in the $m$-generalized taxicab plane, let $H$ be orthogonal projection of the point $A$ on the line $BC$, let $n$ be the slope of the line $BC$, and let $a = d_{T}(B,C)$ and $h = d_{T}(A,H)$.

(i) If $BC$ is parallel to a coordinate axis, then

$$\mathcal{A} = ah/2(u'M + v'M).$$

(2.3)

(ii) If $BC$ is not parallel to any coordinate axis, then

$$\mathcal{A} = [\mu(n)\mu(1/n)]ah/2.$$  

(2.4)

Proof. Let $a = d_E(B,C)$ and $h = d_E(A,B)$. Then, $\mathcal{A} = ah/2$.

(i) If $BC$ is parallel to $x$-axis, then $AH$ is parallel to $y$-axis and

$$a = [1/(u'M + v'M)]a$$

and $h = [1/(u'M + v'M)]h$.

If $BC$ is parallel to $y$-axis, then $AH$ is parallel to $x$-axis and

$$a = [1/(u'M + v'M)]a$$

and $h = [1/(u'M + v'M)]h$.

Hence, we get

$$\mathcal{A} = ah/2(u'M + v'M).$$

(ii) Let $BC$ not be parallel to any coordinate axis, and let $n$ be the slope of the line $BC$. Then, the slope of the line $AH$ is $(-1/n)$. Therefore $a = \mu(n)a$ and $h = \mu(-1/n)h$, hence

$$\mathcal{A} = [\mu(n)\mu(-1/n)]ah/2.$$ 

In the $m$-generalized taxicab plane, $m$-generalized taxicab distance from a point $P$ to a line $l$ is naturally defined by

$$d_{T}(P,l) = \min_{Q \in l} \{d_{T}(P,Q)\}.$$ 

(2.5)

In the following proposition, we give a formula for $d_{T}(P,l)$, similar to the Euclidean geometry.
Proposition 2.2. Given a point \( P = (x_0, y_0) \) and a line \( l : ax + by + c = 0 \) in the \( m \)-generalized taxicab plane. The \( m \)-generalized taxicab distance from the point \( P \) to the line \( l \) can be calculated by the following formula:

\[
d_{T_m}(P, l) = (1 + m^2)^{1/2} |ax_0 + by_0 + c| / \max \left\{ \frac{|a + bm|}{u}, \frac{|am - b|}{v} \right\}.
\]

(2.6)

Proof. It is clear that if \( P \) is on line \( l \), then equation holds. Let \( P \) not be on line \( l \). To find the minimum \( m \)-generalized taxicab distance from the point \( P \) which is off the line \( l \), we let define tangent line to an \( m \)-generalized taxicab circle with center \( P \) and radius \( r \), as a line whose \( m \)-generalized taxicab distance from \( P \) is equal to \( r \), being natural analogue to the Euclidean geometry. Then, we expand an \( m \)-generalized taxicab circle with center \( P \) until the line \( l \) becomes a tangent to the \( m \)-generalized taxicab circle (see Figure 2). It is clear to see that a line can only be a tangent to an \( m \)-generalized taxicab circle at one vertex or two vertices (that is, at one edge). Since corresponding vertices of expanding \( m \)-generalized taxicab circle are on line through \( P \) and parallel to line \( mx - y = 0 \) or \( x + my = 0 \), if \( l \) is a tangent to the \( m \)-generalized taxicab circle with center \( P \), then \( P_1 = \left( \frac{bmy_0 - by_0 - c}{a + bm}, \frac{-amy_0 + ax_0 - cm}{a + bm} \right) \) or \( P_2 = \left( \frac{bmy_0 + bmy_0 + cm}{b - am}, \frac{-amy_0 + ax_0 - c}{b - am} \right) \) is a tangent point, which are intersection points of the line \( l \) and \( mx - y = 0 \) or \( x + my = 0 \), respectively (see Figure 2). Therefore, \( d_{T_m}(P, l) = \min \{d_{T_m}(P, P_1), d_{T_m}(P, P_2)\} \).

\[\text{Figure 2}\]

\[ax + by + c = 0\]

Since \( d_{T_m}(P, P_1) = \frac{(1 + m^2)^{1/2} |ax_0 + by_0 + c|}{|a + bm|/u} \) and \( d_{T_m}(P, P_2) = \frac{(1 + m^2)^{1/2} |ax_0 + by_0 + c|}{|am - b|/v} \), one gets

\[
d_{T_m}(P, l) = (1 + m^2)^{1/2} |ax_0 + by_0 + c| / \max \left\{ \frac{|a + bm|}{u}, \frac{|am - b|}{v} \right\}.
\]

(2.6)

The following equation, which relates the Euclidean distance to the \( m \)-generalized taxicab distance from a point to a line in the Cartesian coordinate plane, plays an important role in the second \( m \)-generalized taxicab version of the area formula.

Proposition 2.3. Given a point \( P \) and a line \( l \) which is not vertical in the Cartesian plane, if \( n \) is the slope of the line \( l \), then

\[
d_E(P, l) = \tau(n) d_{T_m}(P, l)
\]

(2.7)

where \( \tau(n) = \max \left\{ \frac{|m - n|}{u}, \frac{|mn + 1|}{v} \right\} / \left[ (1 + n^2)(1 + m^2) \right]^{1/2} \). If \( l \) is vertical, then \( d_E(P, l) = \left[ \max \left\{ \frac{1}{u}, \frac{|m|}{v} \right\} / (1 + m^2)^{1/2} \right] d_{T_m}(P, l) \).

Proof. Let \( P = (x_0, y_0) \) be a point, and \( l : ax + by + c = 0 \) be a line with slope of \( n \), in the Cartesian plane. If \( l \) is not a vertical line, then \( b \neq 0 \) and \( n = -\frac{a}{b} \). Then, one gets

\[
d_E(P, l) = |ax_0 + by_0 + c| / |b| \left( 1 + n^2 \right)^{1/2} \text{ and } d_{T_m}(P, l) = (1 + m^2)^{1/2} |ax_0 + by_0 + c| / |b| \max \left\{ \frac{|m - n|}{u}, \frac{|mn + 1|}{v} \right\}.
\]

Therefore, \( d_E(P, l) = \tau(n) d_{T_m}(P, l) \) where \( \tau(n) = \max \left\{ \frac{|m - n|}{u}, \frac{|mn + 1|}{v} \right\} / \left[ (1 + n^2)(1 + m^2) \right]^{1/2} \). If \( l \) is a vertical line, then \( b = 0 \) and \( a \neq 0 \). Therefore, one gets that

\[
d_E(P, l) = |ax_0 + c| / |a| \text{ and } d_{T_m}(P, l) = (1 + m^2)^{1/2} |ax_0 + c| / |a| \max \left\{ \frac{1}{u}, \frac{|m|}{v} \right\}.
\]

Hence one has

\[
d_E(P, l) = \left[ \max \left\{ \frac{1}{u}, \frac{|m|}{v} \right\} / (1 + m^2)^{1/2} \right] d_{T_m}(P, l).
\]

(2.7)

Notice that \( \tau(m) = \frac{1}{u} \), and if \( m \neq 0 \), then \( \tau(-\frac{1}{m}) = \frac{1}{v} \). The following theorem gives another \( m \)-generalized taxicab version of the standard area formula of a triangle:
Theorem 2.2. Let ABC be a triangle with area \( \mathcal{A} \) in the m-generalized taxicab plane, \( n \) be the slope of the line BC, and let \( a = d_{T_m}(B,C) \) and \( h' = d_{T_m}(A,BC) \). Then

\[
\mathcal{A} = \max \left\{ \frac{|m-n|}{2}, \frac{|m+n+1|}{2} \right\} \frac{ah'}{2u[|mn+1|+v|m-n|]},
\]
(2.8)

If BC is vertical, then

\[
\mathcal{A} = \max \left\{ \frac{1}{2}, \frac{|m|}{2} \right\} \frac{ah'}{2u|m|+v}.
\]
(2.9)

Proof. Let \( a = d_E(B,C) \) and \( h = d_E(A,BC) \). Then, \( \mathcal{A} = ah'/2 \). Let BC not be vertical, and \( n \) be the slope of the line BC. By Proposition 2.1 and Proposition 2.3, \( a = u(n)a \) and \( h = \tau(n)h' \), hence one has

\[
\mathcal{A} = |\mu(n)\tau(n)|ah'/2 = \max \left\{ \frac{|m-n|}{2}, \frac{|m+n+1|}{2} \right\} \frac{ah'/2}{2u[|mn+1|+v|m-n|]}.
\]

If BC is vertical, then \( a = \left| \frac{1}{(u'|m|+v')} \right| a \) and \( h = \max \left\{ \frac{1}{2}, \frac{|m|}{2} \right\} / (1+m^2)^{1/2}h' \). Hence, one has

\[
\mathcal{A} = \max \left\{ \frac{1}{2}, \frac{|m|}{2} \right\} \frac{ah'/2}{2u|m|+v}.
\]

The following corollary follows from Theorem 2.1 and Theorem 2.2.

Corollary 2.1. Let ABC be a triangle with area \( \mathcal{A} \) in the m-generalized taxicab plane, and let \( a = d_{T_m}(B,C) \), \( h = d_{T_m}(A,H) \), and \( h' = d_{T_m}(A,BC) \). If BC is parallel to mx − y = 0 or x + my = 0, then \( h = h' \) and \( \mathcal{A} = ah'/2uv \).

Proof. If BC is parallel to mx − y = 0 or x + my = 0, then \( n = m \) and \( n = -1/m \), respectively, and Equation (2.4) and Equation (2.8) gives \( \mathcal{A} = ah'/2uv = ah'/2uv \), so \( h = h' \).

3. The m-generalized taxicab version of Heron’s formula

It is well-known that if ABC is a triangle with the area \( \mathcal{A} \) in the Euclidean plane, and \( a = d_E(B,C) \), \( b = d_E(A,C) \), \( c = d_E(A,B) \), and \( p = (a+b+c)/2 \), then

\[
\mathcal{A} = \left| \frac{p(p-a)(p-b)(p-c)}{4} \right|^{1/2},
\]
which is known as Heron’s formula. In this section, we give an m-generalized taxicab version of this formula in terms of m-generalized taxicab distance, similar to the one given in [14]. We need following modified definitions given in [14] to give an m-generalized taxicab version of Heron’s formula:

Definition 3.1. Let ABC be any triangle in the m-generalized taxicab plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to lines mx − y = 0 or x + my = 0. A line \( l \) is called a m-base line of ABC if and only if

(1) \( l \) passes through a vertex,
(2) \( l \) is parallel to lines mx − y = 0 or x + my = 0,
(3) \( l \) intersects the opposite side (as a line segment) of the vertex in (1).

Clearly, at least one of vertices of the triangle always has one or two m-base lines. Such a vertex of the triangle is called an m-basic vertex. An m-base segment is a line segment on an m-base line, which is bounded by an m-basic vertex and its opposite side.

Now, we give the m-generalized taxicab version of Heron’s formula:

Theorem 3.2. Let ABC be a triangle, and \( a = d_{T_m}(B,C) \), \( b = d_{T_m}(A,C) \), \( c = d_{T_m}(A,B) \), \( p = (a+b+c)/2 \), and let \( \alpha \) denote the m-generalized taxicab length of a m-base segment of the triangle. Then the area \( \mathcal{A} \) of the triangle is

\[
\mathcal{A} = \left\{ \begin{array}{ll}
\frac{1}{4} \alpha \left( p - (\alpha + \alpha') \right), & \text{if there exists only one m-base line} \\
\frac{1}{4} \alpha \left( p - (\alpha + \alpha' + \alpha'') \right), & \text{if there exist two m-base lines}
\end{array} \right.,
\]
(3.1)

where \( \alpha' = d_{T_m}(D,H) \), \( \alpha'' = d_{T_m}(\text{basic vertex}, H') \), \( D \) is intersection point of the m-base line and the opposite side, \( H \) is point of orthogonal projection of one of the remaining two vertices on the m-base line which is an endpoint of the m-base segment or not on the m-base segment, \( H' \) is point of orthogonal projection of the third vertex on the same m-base line which is an endpoint of the m-base segment or not on the m-base segment.
Proof. Let $ABC$ be a triangle with $m$-base vertex $C$, without loss of generality. Let $H''$ be the point of orthogonal projection of one of the two remaining vertices which is on the $m$-base segment. Two cases are:

(i) Let $ABC$ has only one $m$-base line passing through $C$. Figure 3 and Figure 4 represent all such triangles. Let $h = d_{T(m)}(A, H)$, $h' = d_{T(m)}(B, H'')$, $c_a = d_{T(m)}(A, D)$, and $c_b = d_{T(m)}(B, D)$. Since $c_a + \alpha = b$ and $c_b + a = a + 2h'$, one gets $h' = p - b$. We also have $h = b - (\alpha + \alpha')$. Therefore, $h + h' = p - (\alpha + \alpha')$. Besides, $\mathcal{A} = \frac{1}{2uv} \alpha(h + h')$ by Corollary 2.1. Hence, $\mathcal{A} = \frac{1}{2uv} \alpha(p - (\alpha + \alpha'))$.

(ii) Let $ABC$ has two $m$-base lines passing through $C$. Figure 5 represents all such triangles. Choose an $m$-base line to determine the point $D$. Let $h = d_{T(m)}(B, H)$ and $h' = d_{T(m)}(A, H'')$. Since $a = h + \alpha + \alpha'$, $b = h' + \alpha''$, and $a + b = c$ one gets $h + h' = a + b - (\alpha + \alpha' + \alpha'') = p - (\alpha + \alpha' + \alpha'')$. Besides, $\mathcal{A} = \frac{1}{2uv} \alpha(h + h')$ by Corollary 2.1. Hence, $\mathcal{A} = \frac{1}{2uv} \alpha(p - (\alpha + \alpha' + \alpha''))$.

The following two corollaries give the $m$-generalized taxicab versions of Heron’s formula for some special cases:

**Corollary 3.1.** If one side of a triangle $ABC$, say $BC$, is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and none of the angles $B$ and $C$ is an obtuse angle, then for the area $\mathcal{A}$ of $ABC$,

$$\mathcal{A} = \frac{1}{2uv} a(p - a).$$ (3.2)

Proof. Let $ABC$ be a triangle with $BC$ is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and none of the angles $B$ and $C$ is an obtuse angle. Then, there is only one $m$-base line passing through $B$ or $C$, so $B$ and $C$ are $m$-basic vertices and $BC$ is the $m$-base segment. Then, $\alpha = a, \alpha' = 0$, hence we have $\mathcal{A} = \frac{1}{2uv} a(p - a)$.

**Corollary 3.2.** If one side of a triangle $ABC$, say $BC$, is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and one of the angles $B$ and $C$ is not an acute angle, then for the area $\mathcal{A}$ of $ABC$,

$$\mathcal{A} = \frac{1}{2uv} a(p - (a + \alpha''))$$ (3.3)

where $\alpha'' = d_{T(m)}(\text{basic vertex}, H')$ and $H'$ is the point of orthogonal projection of $A$ on the same $m$-base line which is an endpoint of the $m$-base segment or not on the $m$-base segment.

Proof. Let $ABC$ be a triangle with $BC$ is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and one of the angles $B$ and $C$, let us say $C$, is not an acute angle. Then, there are two $m$-base lines passing through $C$, so $C$ is $m$-basic vertex and $BC$ is an $m$-base segment. Then, $\alpha = a, \alpha' = 0$, hence we have $\mathcal{A} = \frac{1}{2uv} a(p - (a + \alpha''))$.

Note that since the generalized taxicab and so the taxicab distances are special cases of the $m$-generalized taxicab distance, conclusions given here are also true for the generalized taxicab and so the taxicab geometry.
References