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A Survey on the Base, Rolling and Roulette Curves on Generalized Complex Number Plane $\mathbb{C}_J$

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Abstract

In this study, we define a base curve, a rolling curve and a roulette on generalized complex number plane ($\mathfrak{p}$-complex plane) $\mathbb{C}_J$. We examine the third one of these curves under the condition that two others are given. We also re-obtain the Euler Savary's formula in $\mathbb{C}_J$ as a generalization of the Euler Savary's formula for complex plane $\mathbb{C}$, hyperbolic plane $\mathbb{H}$ and dual plane $\mathbb{D}$.

Keywords: *Base curve, rolling curve, roulette, associated curve, Generalized complex number plane, Euler Savary's formula.*
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1. Introduction

Different definitions of the imaginary unit i from $i^2 = -1$ give rise to interesting and useful complex number systems. The scientists have modified the original definition of product of complex numbers. W. Clifford developed the double complex numbers (perplex numbers [1], split-complex numbers [2] or hyperbolic numbers [2]-[3], [4]-[8]) by requiring that $i^2 = 1$. E. Study, [9] added another variant to the collection of complex products. The dual numbers provide the condition that $i^2 = 0$ [3]. The ordinary, dual and double numbers are significant members of a two-parameter family of complex number systems often called binary numbers or generalized complex numbers [10]. Ordinary, dual, and double numbers are usually denoted by different imaginary units, i, ε and j, respectively and $i^2 = -1$, $\varepsilon^2 = 0\,(\varepsilon \neq 0)$ and $j^2 = 1\,(j \neq \pm 1)$. For generalizing this unit, we will take J for three number systems. So, the generalized complex numbers have the form [11]

$$z = x + Jy, \ \ (x, y \in \mathbb{R}) \ \text{ where } J^2 = i\mathfrak{q} + \mathfrak{p}, \ \ (\mathfrak{p}, \mathfrak{q} \in \mathbb{R}).$$

The generalized complex number systems are isomorphic (as rings) to the ordinary, dual and double numbers when $\mathfrak{p} + \mathfrak{q}^2/4$ is negative, zero, and positive, respectively [3].
By taking $J^2 = \mathfrak{p}; \mathfrak{q} = 0$ and $-\infty < \mathfrak{p} < \infty$, generalized complex number system can be represented as follows:

$$\mathbb{C}_{\mathfrak{p}} = \{x + Jy : \ x, y \in \mathbb{R}, \ J^2 = \mathfrak{p}\}.$$

$\mathbb{C}_{\mathfrak{p}}$ is called $\mathfrak{p}$-complex plane [10]. Moreover, the set $\mathbb{C}_J$ is defined

$$\mathbb{C}_J = \left\{x + Jy : \ x, y \in \mathbb{R}, \ J^2 = \mathfrak{p}, \ \mathfrak{p} \in \{-1, 0, 1\}\right\} \tag{1.1}$$

such that $\mathbb{C}_J \subset \mathbb{C}_{\mathfrak{p}}$.
The set $\mathbb{C}_J$ is just the real numbers extended to include the unipotent J such that $\mathbb{C}_J := \mathbb{P}_\varepsilon[J]$, where $\mathbb{P}_\varepsilon$ represents affine Cayley-Klein planes [12]. This yields by the same extension of the set of ordinary (complex) numbers $\mathbb{C}$, dual numbers $\mathbb{D}$ and double (hyperbolic) numbers $\mathbb{H}$ such that $\mathbb{C} := \mathbb{R}[i]$, $\mathbb{D} := \mathbb{R}[\varepsilon]$ and $\mathbb{H} := \mathbb{R}[j]$, respectively [13]-[14]. The $\mathfrak{p}$-complex numbers system play the same role for Cayley-Klein geometry like that played by ordinary numbers in the Euclidean geometry [3], [12]. The Cayley-Klein plane geometries first introduced by F. Klein in 1871 and A. Cayley, and they are number of geometries including Euclidean, Galilean, Minkowskian and Bolyai-Lobachevsikan [15]-[16]. Moreover, I. M. Yaglom distinguished these geometries by choosing one of three ways of measuring length (parabolic, elliptic or hyperbolic) between two points on a line and one of the three ways of measuring angles (parabolic, elliptic or hyperbolic) between two lines [12]. This gives nine ways of measuring lengths and angles. Many recent research are conducted in Cayley-Klein planes in terms of their group structure, group contraction, relationship between kinematic groups in [17]-[22].

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On other hand, the curvature theory states the properties of a point path in planar motion has been extensively worked by many approaches over the past decades. In this theory, two curves which are called a *base curve* and *rolling curve* is denoted by α_B and γ_R, are considered, respectively. Let assume that a point Q which is linked to rolling curve γ_R and rolling curve γ_R rolls without splitting along the base curve α_B. Then, the locus of the point Q makes a curve which is called *roulette* is denoted by (Q). For instance, if α_B is a parabola, γ_R is a an equal parabola and Q is the vertex of the rolling parabola γ_R, then the roulette β_Q is the cissoid of Diocles.

Euler Savary's formula is a very famous theorem which gives relation between curvature of roulette and curvatures of these base curve and rolling curve. This formula has been worked extensively under the two and three- dimensional motions by many researchers: Alexander and Maddocks, [23], Buckley and Whitfield, [24], Dooner and Griffis, [25], Ito and Takahaski, [26], Pennock and Raje, [27], Ersoy and Akyiğit, [28], and Wang at all, [29].

Nowadays, Euler Savary's formula has been obtained in quite studies and worked in many different plane geometries. Firstly, Müller, [30] obtained Euler Savary's formula for one parameter motion in Euclidean plane $\mathbb{E}^2$ in 1959. Secondly, in 1983, Röschel, [31]; developed a formula by using different method from Müllers' analog to the formula of Euler-Savary in Isotropic plane (or called Galilean plane). Then, in 2002, Aytun, [32]; studied the Euler Savary's formula for the one parameter Lorentzian motions as using Müller's Method [30]. In 2003, Ikawa, [33]; gave the Euler-Savary formula on Minkowski without using Müller's Method [30], and also examined a new way for a generalization of the Euler Savary's formula in the Euclidean plane in this article. Similarly, Yüce, [34] ,[35]; obtained the Euler Savary's formula for the one parameter on Galilean Plane $\mathbb{G}^2$ analog [30] (or [32]) and [33].

The investigation of the theory of Euler Savary's formula in number systems is attracted by many researchers. For this respect, in 2010, Masal at all., [36], expressed Euler Savary's formula for one parameter motion in the complex plane $\mathbb{C}$. Moreover, Yüce, [37], studied this formula on complex plane by using Ikawa's method. As a similar way, Ersoy, [28] obtained Euler Savary's formula for one parameter homothetic motion on the hyperbolic plane $\mathbb{H}$. Then, Yüce, [38], investigate Euler Savary's formula on the dual plane $\mathbb{D}$, analog to [30] and [33]. Moreover, Yüce, [39]; expressed the Euler Savary's formula for the one-parameter homothetic motions in the generalized complex number plane $\mathbb{C}_J$ bu using the Müller's method, [30].

In the light of these existing studies, in this paper, we will re-obtain Euler Savary's formula in $\mathbb{C}_J$ by considering a base curve, a rolling curve and roulette by taking into Ikawa's method account. We will also examine the third one of these curves under the condition that two others are given. This generalization of Euler Savary's formula in $\mathbb{C}_J$ gives the opportunity to obtain it in complex plane, dual plane, and hyperbolic plane by taking $\mathfrak{p} = \{-1, 0, 1\}$, respectively.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{C}_J$ from a open interval I into $\mathfrak{p}$-complex plane be a planar curve with arc length parameter s which is defined by $\alpha(s) = \alpha_1(s) + J\alpha_2(s)$. Then, the unit tangent vector of the curve α at the point $\alpha(s)$ is defined by

$\mathbf{t}(s) = \alpha_1'(s) + J\alpha_2'(s)$.

Frenet formulas of the curve α have the following equations

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= \mathfrak{p}\kappa\mathbf{t}, \end{aligned}$$

where $\mathbf{n}$ is the unit normal vector of the curve α and $\kappa(s) = \frac{d\theta}{ds}$ is curvature of α. Also, θ is a slope angle of α. By using these equations, we can define a new curve

$$\beta_A(s) = \alpha(s) + x(s)\mathbf{t}(s) + y(s)\mathbf{n}(s) \tag{2.1}$$

which is called *associated curve* where $x(s), y(s) \in \mathbb{R}$. By differentiating equation (2.1) with respect to s, we have

$$\frac{d(\beta_A(s))}{ds} = \left(1 + \mathfrak{p}\kappa(s)y(s) + \frac{dx(s)}{ds}\right)\mathbf{t}(s) + \left(\kappa(s)x(s) + \frac{dy(s)}{ds}\right)\mathbf{n}(s).$$

However, associated curve is written as

$\beta_A(s) = x(s) + Jy(s)$,

with respect to frame $\{\alpha(s); \mathbf{t}(s), \mathbf{n}(s)\}$ on the $\mathfrak{p}-$complex plane $\mathbb{C}_J$. The expression $x(s) + Jy(s)$ is called *relative coordinate of* $\beta_A(s)$ with respect to frame $\{\alpha(s); \mathbf{t}(s), \mathbf{n}(s)\}$. Moreover, the velocity vector of associated curve $\beta_A(s)$ is given by

$$\frac{d(\beta_A(s))}{ds} = v_1(s) + Jv_2(s)$$

with respect to frame $\{\alpha(s); \mathbf{t}(s), \mathbf{n}(s)\}$, where

$$v_1(s) + Jv_2(s) := 1 + \mathfrak{p}\kappa(s)y(s) + \frac{dx(s)}{ds} + J\left(\kappa(s)x(s) + \frac{dy(s)}{ds}\right). \tag{2.2}$$

Furthermore, the Frenet frame $\{\mathbf{t}_A, \mathbf{n}_A\}$ of β_A with the arc length parameter s_A of β_A can be given as below:

$$\begin{aligned} \mathbf{t}_A' &= \kappa_A(s_A)\mathbf{n}_A(s_A) \\ \mathbf{n}_A' &= \mathfrak{p}\kappa_A(s_A)\mathbf{t}_A(s_A), \end{aligned}$$

where κ_A is the curvature of β_A. Let ω be a slope angle of β_A then, we can write

$$\kappa_A(s_A) = \frac{d\omega}{ds_A} = \frac{d\omega}{ds}\frac{ds}{ds_A} \tag{2.3}$$

$$= \left(\kappa + \frac{d\phi}{ds}\right)\frac{1}{\|\beta_A'(s)\|}, \tag{2.4}$$

where $\phi = \omega - \theta$.

3. The Base, Rolling and Roulette Curves on $\mathbb{C}_J$

In this section, we will investigate the Euler Savary's formula which gives the relation between curvatures of a base curve, a rolling curve, and roulette. Also, we will query the curves for three situation:

i) Suppose that *the base curve and the rolling curve are given.*

Let α_B be the base curve with curvature κ_B and γ_R be a rolling curve with curvature κ_R. Assume that Q is a point relative to γ_R and the roulette of the locus of this point Q constitues generally a curve denoted by (Q). And β_Q is an associated curve of α_B. So from the equation (2.2), we can write the relative coordinate $\{x,y\}$ of β_Q with respect to curve α_B such that

$$v_1(s)+Jv_2(s)=1+\mathfrak{p}\kappa_B(s)y(s)+\frac{dx(s)}{ds}+J\left(\kappa_B(s)x(s)+\frac{dy(s)}{ds}\right).$$

Besides, when the rolling curve rolls without splitting along α_B at the each point of contact, we can consider the relative coordinate $\{x,y\}$ is a relative coordinate of β_Q with respect to γ_R for a suitable parameter s_R. In this situation, the associated curve is only a point Q and

$$v_1(s_R)+Jv_2(s_R)=1+\mathfrak{p}\kappa_R(s_R)y(s_R)+\frac{dx(s_R)}{ds_R}+J\left(\kappa_R(s_R)x(s_R)+\frac{dy(s_R)}{ds_R}\right)=0. \tag{3.1}$$

So, we get

$$\frac{dx(s_R)}{ds_R}+J\frac{dy(s_R)}{ds_R}=-1-\mathfrak{p}\kappa_R(s_R)y(s_R)-J\kappa_R(s_R)x(s_R). \tag{3.2}$$

Substituting the last equation into (2.2), we have

$$v_1(s_R)+Jv_2(s_R)=\mathfrak{p}(\kappa_R-\kappa_B)y+J(\kappa_B-\kappa_R)x. \tag{3.3}$$

Assume that the associated curve β_Q is given by

$$\beta_Q=re^{J\phi}$$

on the polar coordinate with respect to $\{\alpha_B(s);x,y\}$ where r is the distance from the origin point $\alpha_B(s)$ to point Q. In this case, we calculate from (3.2)

$$\begin{aligned}\frac{d\beta_Q}{ds_R} &= \frac{dr}{ds_R}e^{J\phi(s)}+Jre^{J\phi(s_R)}\frac{d\phi(s_R)}{ds_R}\\ &= -\mathfrak{p}\kappa_R r\sin_{\mathfrak{p}}\phi-1+J(-\kappa_R r\cos_{\mathfrak{p}}\phi).\end{aligned}$$

If we solve this equation with respect to $r\frac{d\phi}{ds_R}$, then we find

$$r\frac{d\phi}{ds_R}=-\kappa_R r+\mathrm{Im}(\mathrm{e}^{\mathrm{J}\phi}). \tag{3.4}$$

Furthermore, we know that

$$\kappa_Q = \left(\kappa_B+\frac{d\phi}{ds}\right)\frac{1}{|\kappa_B-\kappa_R|\sqrt{x^2-\mathfrak{p}y^2}} \tag{3.5}$$

$$= \left(\kappa_B+\frac{d\phi}{ds}\right)\frac{1}{|\kappa_B-\kappa_R|r}. \tag{3.6}$$

So from the equations (3.4), (3.5), we get

$$r\kappa_Q=\frac{\kappa_B-\kappa_R}{|\kappa_B-\kappa_R|}+\frac{\mathrm{Im}(\mathrm{e}^{\mathrm{J}\phi})}{r|\kappa_B-\kappa_R|}. \tag{3.7}$$

Theorem 3.1. *On the $\mathfrak{p}-$complex plane $\mathbb{C}_J$, assume that a curve γ_R rolls without splitting along a curve α_B. Let β_Q be a locus of a point that is relative to γ_R. Let Q be a point on β_Q and R be a point of contact of α_B and γ_R corresponds to Q relative to the rolling relation. By (r,ϕ) we denote a polar coordinate of Q with respect to the origin R and the base line $\alpha'_B|_R$. Then, curvature κ_B, κ_R and κ_Q of the curves α_B, γ_R, and β_Q, respectively, satisfies the following equation*

$$r\kappa_Q=\pm1+\frac{\mathrm{Im}(\mathrm{e}^{\mathrm{J}\phi})}{r|\kappa_B-\kappa_R|}.$$

Remark 3.2. *In the special case $J^2=p=1$, vectors are classified as timelike, spacelike and null. So, if the associated curve β_Q is spacelike, then on the polar coordinate of the associated curve β_Q can write*

$$\beta_Q=rJe^{J\phi(s)}.$$

Hence, the equation given by above theorem turns out to be the below equation

$$r\kappa_Q=\frac{\kappa_B-\kappa_R}{|\kappa_B-\kappa_R|}-\frac{\mathrm{Im}(\mathrm{Je}^{\mathrm{J}\phi})}{r|\kappa_B-\kappa_R|}. \tag{3.8}$$

Also, if the associated curve β_Q is null, then goes to a contradiction.

In the special case $J^2 = p = -1$, we obtain Euler Savary's formula on $\mathfrak{p}-$complex plane as [37]. In the special case $J^2 = p = 0$, we have Euler Savary's formula on dual plane which was given in [38]. Also, if $J^2 = p = 1$, then we get Euler Savary's formula on hyperbolic plane as similar to isomorphic plane in [33].

ii) Assume that *the base curve and the roulette are given.*

Assume that $\alpha_B(s_B) = u(s_B) + Jv(s_B)$ be a base curve with the arc length parameter s_B. Let us draw the normal to the roulette β_Q for a point R of α_B and the point $Q = x(s_B) + Jy(s_B)$ be the foot of this normal. So length of the normal **RQ** is given by

$$d(Q,R) = \sqrt{\left|(x(s_B) - u(s_B))^2 - \mathfrak{p}(y(s_B) - v(s_B))^2\right|}. \tag{3.9}$$

However, considering the equation (3.9) on the rolling curve γ_R, this equation represents the length of the point Q relative to γ_R and a point of γ_R. So the orthogonal coordinate $f(s_B) + Jg(s_B)$ of γ_R is given by the equations

$$(f(s_B))^2 - \mathfrak{p}(g(s_B))^2 = (x(s_B) - u(s_B))^2 - \mathfrak{p}(y(s_B) - v(s_B))^2$$

and

$$\left(\frac{df}{ds_B}\right)^2 - \mathfrak{p}\left(\frac{dg}{ds_B}\right)^2 = \pm 1.$$

In the special case $J^2 = \mathfrak{p} = -1$, we obtain the rolling curve as in [37]. Also, in the special cases of $J^2 = \mathfrak{p} = 0$ and $J^2 = \mathfrak{p} = 1$, one can be calculate the rolling curve for dual plane and hyperbolic plane, respectively.

iii) Suppose that *the rolling curve γ_R and the roulette β_Q are given.*

Suppose that $\beta_Q(s_A) = x(s_A) + Jy(s_A)$ is roulette with arc length parameter s_A and $\gamma_R(s_R)$ is given by the polar coordinate $r(s_R)$ with the arc length parameter s_R. The normal of β_Q is $\mathbf{n} = \mathfrak{p}y'(s_A) + Jx'(s_A)$, and so, a point $u(s_B) + Jv(s_B)$ of the point curve α_B is given

$$u(s_B) + Jv(s_B) = x(s_A) + Jy(s_A) \pm r(s_R)\mathbf{n}$$

or

$$u(s_B) + Jv(s_B) = x(s_A) \mp \mathfrak{p}r(s_R)y'(s) + J\left(y(s_A) \pm r(s_R)x'(s)\right). \tag{3.10}$$

So, we can write

$$\frac{du}{ds_R} + J\frac{dv}{ds_R} = \frac{dx}{ds_A}\frac{ds_A}{ds_R} + J\frac{dy}{ds_A}\frac{ds_A}{ds_R} \pm \frac{dr}{ds_R}\mathbf{n} \pm r\frac{d\mathbf{n}}{ds_A}\frac{ds_A}{ds_R}$$

or

$$\frac{du}{ds_R} + J\frac{dv}{ds_R} = \frac{dx}{ds_A}\frac{ds_A}{ds_R} + J\frac{dy}{ds_A}\frac{ds_A}{ds_R} \pm \frac{dr}{ds_R}\left(\mathfrak{p}\frac{dy}{ds_A} + J\frac{dx}{ds_A}\right) \pm r\left(\mathfrak{p}\kappa_Q\left(\frac{dx}{ds_A} + J\frac{dy}{ds_A}\right)\right)\frac{ds_A}{ds_R}.$$

Then, we find

$$\frac{du}{ds_R} + J\frac{dv}{ds_R} = \frac{dx}{ds_A}\left(1 \mp r\mathfrak{p}\kappa_Q\right)\frac{ds_A}{ds_R} \pm \mathfrak{p}\frac{dr}{ds_R}\frac{dy}{ds_A} + J\left(\frac{dy}{ds_A}\left(1 \pm r\mathfrak{p}\kappa_Q\right)\frac{ds_A}{ds_R} \pm \frac{dr}{ds_R}\frac{dx}{ds_A}\right),$$

where κ_Q is the curvature of β_Q. Because of the fact that s_R is also the arc length of α_B, we get

$$\left(\frac{du}{ds_R}\right)^2 - \mathfrak{p}\left(\frac{dv}{ds_R}\right)^2 = \pm 1 \tag{3.11}$$

and

$$\left(\frac{ds_A}{ds_R}\right)\left(1 \pm r\mathfrak{p}\kappa_Q\right)^2 + \left(\frac{dr}{ds_R}\right)^2 = \pm 1. \tag{3.12}$$

From this differential equation, we can solve $s_A = s_A(s_R)$. Substituting this equation into (3.10), we can get the orthogonal coordinate of α_B. In the special case $J^2 = \mathfrak{p} = -1$, we obtain the base curve as in [37]. For the special cases of $J^2 = \mathfrak{p} = 0$ and $J^2 = \mathfrak{p} = 1$, one can obtain the base curve in dual and hyperbolic planes from the equations (3.11) and (3.12), respectively.

4. Conclusion

In this work, we consider a base curve, a rolling curve and roulette curve. We examine the third one of these curves under the condition that two others are known on the generalized complex number plane ($\mathfrak{p}$-complex plane) $\mathbb{C}_J$. Furthermore, we have reconsidered Euler Savary's formula on $\mathbb{C}_J$. So, with the aim of the obtained form of the Euler Savary's formula on $\mathbb{C}_J$ gives the opportunity to obtain it in complex plane, dual plane, and hyperbolic plane by taking $\mathfrak{p} = \{-1, 0, 1\}$, respectively.

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