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A Note on the Dunkl-Appell Orthogonal Polynomials

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Abstract

This paper deals with the problem of finding all orthogonal polynomial sets which are also T_μ-Appell where $T_\mu, \mu \in \mathbb{C}$ is the Dunkl operator. The resulting polynomials reduce to Generalized Hermite polynomials $\{\mathscr{H}_n(\mu)\}_{n\geq 0}$.

Keywords: *Orthogonal polynomials, Dual sequence, Dunkl operator, Generalized Hermite Polynomials.*
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1. Introduction and Preliminary Results

Let L be a lowering operator, that is, a linear operator that decreases in one unit the degree of a polynomial and such that $L(1) = 0$. Among such lowering operators, we mention the derivative operator D, the difference operator D_w, the Hahn operator H_q and the Dunkl operator T_μ. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n, n \geq 0$. The sequence $\{P_n\}_{n\geq 0}$ is called L-Appell when $P_n = \frac{L(P_{n+1})}{\alpha_n}, n \geq 0$, with α_n is the normalization coefficient.
A most specific problem is to find the sequences of monic orthogonal polynomials which belong to the L-Appell class. Such characterization takes into account the fact that polynomial set which are obtainable from one another by a linear change of variable are assumed equivalent. For the derivative operator D, it is well known (see [3]) that the Hermite polynomials are the only solution to the last problem. This characterization of the Hermite polynomials was first given by Angelesco [3], and later by other authors (see [2] and [13] for additional references).
For the difference operator D_w, the only solution is the Charlier family (see [6]).
For the Hahn operator H_q, the only solution is the Al-Salam and Carlitz sequence [10].
Lastly, for the Dunkl operator T_μ, the problem was solved by Y. Ben Cheikh and M. Gaied in the positive define case (for $\mu > -\frac{1}{2}$) [5] and by L Kheriji and A . Gherissi in the symmetric case (e.i. $P_n(-x) = (-1)^n P_n(x), n \geq 0$) [9]. The obtained solution is the generalized Hermite polynomials set. In this paper,using duality, we solve the problem in the general case with $\mu \in \mathbb{C}$.
This first section contains preliminary results and notations To be used in the sequel. In the second section, using a technique based on duality, we determine all the sequences of monic orthogonal polynomials which belong to the T_μ-Appell class without the constraint the sequences are symmetric. There's a unique solution, up to affine transformations, it is the set of generalized Hermite orthogonal polynomials. This result generalizes Corollary 2.3. in [9].

We begin by reviewing some preliminary results needed for the sequel. The vector space of polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) is denoted by $\mathbb{P}$ and by $\mathbb{P}'$ its dual space, whose elements are called forms. The set of all nonnegative integers will be denoted by $\mathbb{N}$. The action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ is denoted by $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \in \mathbb{N}$, the moments of u. For any form u, any $a \in \mathbb{C} - \{0\}$ and any polynomial h let $Du = u'$, hu, and $h_a u$ be respectively the forms defined by: $\langle u', f\rangle := -\langle u, f'\rangle$, $\langle hu, f\rangle := \langle u, hf\rangle$, and $\langle h_a u, f\rangle =: \langle u, h_a f\rangle = \langle u, f(ax)\rangle$, $f \in \mathbb{P}$.

Then, it is straightforward to prove that for $f \in \mathbb{P}$ and $u \in \mathbb{P}'$, we have

$$(fu)' = f'u + fu' \ . \tag{1.1}$$

We will only consider sequences of polynomials $\{P_n\}_{n\geq 0}$ such that $\deg P_n \leq n, n \in \mathbb{N}$. If the set $\{P_n\}_{n\geq 0}$ spans $\mathbb{P}$, which occurs when $\deg P_n = n, n \in \mathbb{N}$, then it will be called a polynomial sequence (PS). Along the text, we will only deal with PS whose elements are monic, that is, monic polynomial sequences (MPS). It is always possible to associate to $\{P_n\}_{n\geq 0}$ a unique sequence $\{u_n\}_{n\geq 0}, u_n \in \mathbb{P}'$, called its dual sequence, such that $\langle u_n, P_m\rangle = \delta_{n,m}$ $, n,m \geq 0$, where $\delta_{n,m}$ is the Kronecker's symbol [11].

The MPS $\{P_n\}_{n\geq 0}$ is orthogonal with respect to $u \in \mathbb{P}'$ when the following conditions hold: $\langle u, P_n P_m\rangle = r_n \delta_{n,m}$, $n,m \geq 0$, $r_n \neq 0$, $n \geq$

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0 [7]. In this case, we say that $\{P_n\}_{n\geq 0}$ is a monic orthogonal polynomial sequence (MOPS) and the form u is said to be regular. Necessarily, $u=\lambda u_0, \lambda \neq 0$. Furthermore, we have

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, n \geq 0, \tag{1.2}$$

and the MOPS $\{P_n\}_{n\geq 0}$ fulfils the second order recurrence relation

$$\begin{array}{l} P_0(x)=1 \quad , \quad P_1(x)=x-\beta_0 \\ P_{n+2}=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x) \quad ,\gamma_{n+1}\neq 0, \quad n\geq 0. \end{array} \tag{1.3}$$

A form u is said symmetric if and only if $(u)_{2n+1}=0, n\geq 0$, or, equivalently, in (1.3) $\beta_n = 0, n \geq 0$. Furthermore, the orthogonality is kept by shifting. In fact, let

$$\{\tilde{P}_n := a^{-n}\left(h_a P_n\right)\}_{n\geq 0}, \quad a\neq 0, \tag{1.4}$$

then the recurrence elements $\tilde{\beta}_n$, $\tilde{\gamma}_{n+1}$, $n \geq 0$, of the sequence $\{\tilde{P}_n\}_{n\geq 0}$ are

$$\tilde{\beta}_n = \frac{\beta_n}{a}, \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n\geq 0. \tag{1.5}$$

Let us introduce the Dunkl operator

$$T_\mu(f) = f' + 2\mu H_{-1}f, \quad (H_{-1}f)(x) = \frac{f(x)-f(-x)}{2x}, \quad f\in\mathbb{P}, \mu\in\mathbb{C}.$$

This operator was introduced and studied for the first time by Dunkl [8]. Note that T_0 is reduced to the derivative operator D. The transposed ${}^tT_\mu$ of $T\mu$ is ${}^tT_\mu = -D-H_{-1} = -T_\mu$, leaving out a light abuse of notation without consequence. Thus we have

$$\langle T_\mu u, f\rangle = -\langle u, T_\mu f\rangle, \quad u\in\mathbb{P}', \quad f\in\mathbb{P}, \quad \mu\in\mathbb{C}.$$

In particular, this yields $\langle T_\mu u, x^n\rangle = -\mu_n (u)_{n-1}, n\geq 0$, where $(u)_{-1}=0$ and

$$\mu_n = n+\mu(1-(-1)^n), \quad n\geq 0. \tag{1.6}$$

It is easy to see that

$$T_\mu(fu) = fT_\mu u + f'u + 2\mu\left(H_{-1}f\right)(h_{-1}u), \quad f\in\mathbb{P}, \quad u\in\mathbb{P}', \tag{1.7}$$

$$h_a\circ T_\mu = aT_\mu\circ h_a \quad in\ \mathbb{P}', \quad a\in\mathbb{C}-\{0\}. \tag{1.8}$$

Now, consider a MPS $\{P_n\}_{n\geq 0}$ and let

$$P_n^{[1]}(x,\mu) = \frac{1}{\mu_{n+1}}\left(T_\mu P_{n+1}\right)(x), \quad \mu\neq -n-\frac{1}{2}, \quad n\geq 0. \tag{1.9}$$

Lemma 1.1. *[12] Denoting by $\{u_n^{[1]}(\mu)\}_{n\geq 0}$ the dual sequence of $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$, we have*

$$T_\mu\left(u_n^{[1]}(\mu)\right) = -\mu_{n+1}u_{n+1}, n\geq 0. \tag{1.10}$$

Definition 1.2. *The sequence $\{P_n\}_{n\geq 0}$ is called Dunkl-Appell or T_μ-Appell if $P_n^{[1]}(.,\mu) = P_n, n\geq 0$.*
When $\mu = 0$, we meet the Appell polynomials.

2. The Main Result

Let us recall some results to be used in the sequel. We begin by giving some properties of the Generalized Hermite polynomials $\{\mathscr{H}_n(\alpha)\}_{n\geq 0}$ (see [1, 4] and [7]). They satisfy the recurrence relation (1.3) with

$$\beta_n = 0,\ \gamma_{n+1} = \frac{1}{2}\left(n+1+\alpha\left(1+(-1)^n\right)\right), \quad 2\alpha\neq -2n-1, \quad n\geq 0. \tag{2.1}$$

The sequence $\{\mathscr{H}_n(\alpha)\}_{n\geq 0}$ is orthogonal with respect to $\mathscr{H}(\alpha)$, this last form has the following integral representation [7], p. 157

$$\langle\mathscr{H}(\alpha), f\rangle = \frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_{-\infty}^{+\infty}|x|^{2\alpha}e^{-x^2}f(x)dx\ ,\ \ \Re(\alpha) > -\frac{1}{2}\ ,\ \ f\in\mathbb{P}\ . \tag{2.2}$$

This family reduces to the ordinary Hermite polynomial set when $\alpha = 0$.

Proposition 2.1. *Let $\{P_n\}_{n\geq 0}$ be a MPS and let $\{u_n\}_{n\geq 0}$ be the corresponding dual sequence. The following statements are equivalent*
(a) The sequence $\{P_n\}_{n\geq 0}$ is T_μ-Appell.
(b) The sequence $\{u_n\}_{n\geq 0}$ verifies

$$T_\mu u_n = -\mu_{n+1}u_{n+1}, \quad n\geq 0. \tag{2.3}$$

Proof. $(a) \Longrightarrow (b)$. Let $\{u_n^{[1]}(\mu)\}_{n\geq 0}$ be the dual sequence of $\{P_n^{[1]}(.,\mu)\}_{n\geq 0}$. Then, Definition1.2 results $u_n^{[1]}(\mu) = u_n$ because $\{P_n\}_{n\geq 0}$ has a unique dual sequence. Therefore, using (1.10), we obtain (2.3).
$(b) \Longrightarrow (a)$. From (1.10) and (2.3), we have the following

$$\begin{cases} T_\mu u_n = -\mu_{n+1}u_{n+1}, n \geq 0, \\ T_\mu \left(u_n^{[1]}(\mu)\right) = -\mu_{n+1}u_{n+1}, n \geq 0. \end{cases}$$

Then, we obtain $T_\mu u_n = T_\mu \left(u_n^{[1]}(\mu)\right), n \geq 0$. So, $u_n = u_n^{[1]}(\mu)$ because T_μ is injective in $\mathbb{P}'$. Moreover, we have $\langle u_n, P_m\rangle = \left\langle u_n^{[1]}(\mu), P_m^{[1]}(.,\mu)\right\rangle = \delta_{n,m}, n,m \geq 0$, which gives

$$\left\langle u_n, P_m - P_m^{[1]}(.,\mu)\right\rangle = 0, \quad n,m \geq 0.$$

Since $\{u_n\}_{n\geq 0}$ is a set of linearly independent vectors, we deduce that $P_m = P_m^{[1]}(.,\mu), m \geq 0$. Hence, the sequence $\{P_n\}_{n\geq 0}$ is T_μ-Appell. □

Now, we state our main result:

Theorem 2.2. *The orthogonal polynomial sets which are also T_μ-Appell, up to affine transformations, is the set of the Generalized Hermite polynomials $\{\mathscr{H}_n(\mu)\}_{n\geq 0}$ ($\mu \neq 0, 2\mu \neq -2n-1, n \geq 0$).*

Proof. Suppose that the sequence $\{P_n\}_{n\geq 0}$ is both orthogonal and T_μ-Appell.

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, n \geq 0, \tag{2.4}$$

From (1.2), (1.10) and (2.3) (according to assumptions)

$$\left(\left\langle u_0, P_n^2\right\rangle\right)^{-1} T_\mu (P_n u_0) = -\mu_{n+1}\left(\left\langle u_0, P_{n+1}^2\right\rangle\right)^{-1} P_{n+1}u_0, \quad n \geq 0 \tag{2.5}$$

Then, by (1.3), the last equation becomes

$$T_\mu (P_n u_0) = -\frac{\mu_{n+1}}{\gamma_{n+1}} P_{n+1}u_0, \quad n \geq 0. \tag{2.6}$$

The particular choice of $n = 0$ in (2.6) yields

$$T_\mu u_0 = -(1+2\mu)\gamma_1^{-1} P_1 u_0. \tag{2.7}$$

In accordance with (1.7), we have

$$T_\mu (P_n u_0) = P_n T_\mu u_0 + P_n' u_0 + 2\mu (H_{-1}P_n)(h_{-1}u_0), \quad n \geq 0. \tag{2.8}$$

Then, using (2.7) and (2.8), (2.6) becomes

$$-\frac{\mu_1}{\gamma_1} P_n P_1 u_0 + P_n' u_0 + 2\mu (H_{-1}P_n)(h_{-1}u_0) = -\frac{\mu_{n+1}}{\gamma_{n+1}} P_{n+1}u_0, \quad n \geq 0. \tag{2.9}$$

For $n = 1$, equation (2.9) becomes

$$2\mu (h_{-1}u_0) = \left(\frac{\mu_1}{\gamma_1} P_1^2 - \frac{\mu_2}{\gamma_2} P_2 - 1\right) u_0. \tag{2.10}$$

Thus,

$$-\frac{\mu_1}{\gamma_1} P_n P_1 u_0 + P_n' u_0 + (H_{-1}P_n)\left(\frac{\mu_1}{\gamma_1} P_1^2 - \frac{\mu_2}{\gamma_2} P_2 - 1\right) u_0 = -\frac{\mu_{n+1}}{\gamma_{n+1}} P_{n+1}u_0, \quad n \geq 0. \tag{2.11}$$

By virtue of the regularity of u_0, we get

$$-\frac{\mu_1}{\gamma_1} P_n P_1 + P_n' + (H_{-1}P_n)\left(\frac{\mu_1}{\gamma_1} P_1^2 - \frac{\mu_2}{\gamma_2} P_2 - 1\right) = -\frac{\mu_{n+1}}{\gamma_{n+1}} P_{n+1} \quad n \geq 0. \tag{2.12}$$

The comparison of the coefficients of x^{n+1} in the previous identity leads to

$$-\frac{\mu_1}{\gamma_1} + \frac{1-(-1)^n}{2}\left(\frac{\mu_1}{\gamma_1} - \frac{\mu_2}{\gamma_2}\right) = -\frac{\mu_{n+1}}{\gamma_{n+1}}, \quad n \geq 0.$$

Therefore,

$$\gamma_{2n+1} = \frac{\gamma_1}{\mu_1}\mu_{2n+1}, \quad \gamma_{2n+2} = \frac{\gamma_2}{\mu_2}\mu_{2n+2}, \quad n \geq 0. \tag{2.13}$$

Now we treat the two cases $\beta_0 = 0$ and $\beta_0 \neq 0$ separately.

Case I. $(\beta_0 = 0)$.

In this case, from (1.3), we have $P_1(x) = x$. Then, by (2.7), we obtain

$$\langle T_\mu u_0, x^n \rangle = -(1+2\mu)\gamma_1^{-1} \left\langle u_0, x^{n+1} \right\rangle. \tag{2.14}$$

So, for $n = 0$, we get $(u)_1 = 0$ and for $n \geq 1$, $-\mu_n (u)_{n-1} = -(1+2\mu)\gamma_1^{-1}(u)_{n+1}$. Thus, we deduce $\langle u_0, x^{2n+1} \rangle = 0, \quad n \geq 0$. Then, the form u_0 is symmetric which is equivalent to $\beta_n = 0, n \geq 0$.

Therefore, we deduce $(h_{-1}u_0) = u_0$ and the equation (2.10) becomes

$$2\mu u_0 = \left(\frac{\mu_1}{\gamma_1} P_1^2 - \frac{\mu_2}{\gamma_2} P_2 - 1 \right) u_0. \tag{2.15}$$

Thus, we deduce $\dfrac{\mu_1}{\gamma_1} = \dfrac{\mu_2}{\gamma_2}$. Then, (2.13) becomes

$$\gamma_{2n+1} = \frac{\gamma_1}{\mu_1} \mu_{2n+1}, \quad \gamma_{2n+2} = \frac{\gamma_1}{\mu_1} \mu_{2n+2}, \quad n \geq 0. \tag{2.16}$$

With the choice $a = \sqrt{\dfrac{\gamma_1}{\mu_1}}$ in (1.4)-(1.5) and using the last equation where $\{\mu_n\}_{n\geq 0}$ is given by (1.6) and the fact $\beta_n = 0, n \geq 0$, we get the following canonical case

$$\tilde{\beta}_n = 0,\ \tilde{\gamma}_{n+1} = \frac{1}{2}\left(n+1+\mu\left(1+(-1)^n\right)\right), \quad 2\mu \neq -2n-1, \quad n \geq 0. \tag{2.17}$$

Hence the MOPS $\{\tilde{P}_n\}_{n\geq 0}$ corresponds to the Generalized Hermite polynomials of parameter μ according to (2.1). Indeed $\tilde{P}_n = \mathscr{H}_n(\mu), \quad \mu \neq 0, \quad \mu \neq -n - \frac{1}{2}, \quad n \geq 0$.

Case II. $(\beta_0 \neq 0)$**.**

From the recurrence relation (1.3), we have

$$\begin{cases} P_1(x) = x - \beta_0, \\ P_2(x) = x^2 - (\beta_0 + \beta_1)x + \beta_0\beta_1 - \gamma_1, \\ P_3(x) = x^3 - (\beta_0 + \beta_1 + \beta_2)x + (\beta_2(\beta_0 + \beta_1) + \beta_0\beta_1 - \gamma_1 - \gamma_2)x - \beta_2(\beta_0\beta_1 - \gamma_1) + \gamma_2\beta_0 \end{cases} \tag{2.18}$$

Making $n = 1$ in (2.12) and using (2.18), we get by equating the coefficients of x^2 in the obtained equation

$$(\beta_0 + \beta_1)\frac{\mu_2}{\gamma_2} = (\beta_1 + \beta_2)\frac{\mu_1}{\gamma_1} \tag{2.19}$$

We have $P_1(x) = P_1^{[1]}(x,\mu) = \mu_2^{-1} T_\mu P_2(x)$ because $\{P_n\}_{n\geq 0}$ is T_μ-Appell. Then, using (2.18), we get

$$x - \beta_0 = x - \frac{\mu_1}{2}(\beta_0 + \beta_1).$$

This, leads to

$$\beta_0 + \beta_1 = \frac{2\beta_0}{\mu_1}. \tag{2.20}$$

We have $P_2(x) = P_2^{[1]}(x,\mu) = \mu_3^{-1} T_\mu P_3(x)$. Then, using (2.18) and comparing coefficients of powers of x in both sides of the resulting equation, we find that

$$\beta_0 + \beta_1 = \frac{2(\beta_0 + \beta_1 + \beta_2)}{\mu_3}, \tag{2.21}$$

and

$$\beta_0\beta_1 - \gamma_1 = \frac{\mu_1}{\mu_3}\left(\beta_2(\beta_0 + \beta_1) + \beta_0\beta_1 - \gamma_1 - \gamma_2\right). \tag{2.22}$$

Using (2.20) and taking into account that $\mu_1 = 1 + 2\mu$ and $\mu_3 = 3 + 2\mu$, we get from (2.21)

$$\beta_2 = \beta_0. \tag{2.23}$$

Based on (2.20), (2.23) and the fact $\mu_2 = 2$, (2.22) becomes

$$\frac{\gamma_1}{\mu_1} - \frac{\gamma_2}{\mu_2} = \frac{2\mu\beta_0^2}{\mu_1}. \tag{2.24}$$

But, from (2.20), (2.23), (2.19) and the fact $\beta_0 \neq 0$, we deduce $\dfrac{\gamma_1}{\mu_1} = \dfrac{\gamma_2}{\mu_2}$. Then, (2.24) gives $\dfrac{2\mu\beta_0^2}{\mu_1} = 0$ which is a contradiction if $\mu \neq 0$. This completes the proof of the Theorem2.2. □

Remark 2.3. *The Theorem 2.2 generalizes Corollary 2.3. in [9].*

Remark 2.4. *From* (2.3)*, by induction we can easily prove*

$$u_n = (-1)^n \left(\prod_{k=0}^{n} \mu_k \right)^{-1} T_\mu^n u_0, \quad n \geq 0.$$

Then, using (1.2), (2.2), (2.16) *where* $\gamma_1 = \mu_1$, (2.17) *and the above equation, we can deduce the following Rodrigues formula for Generalized Hermite polynomials*

$$\mathscr{H}_n(\mu) = (-1)^n |x|^{-2\mu} e^{x^2} T_\mu^n \left(|x|^{2\mu} e^{-x^2} \right), \quad n \geq 0.$$

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