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Multiplication Operators On Grand Lorentz Spaces

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Abstract: Let (X,Σ,μ) be a σ-finite measure space, f be a complex-valued measurable function defined on X and $u:X\to\mathbb{C}$ be a measurable function such that $u\cdot f\in M(X,\Sigma)$ whenever $f\in M(X,\Sigma)$ where $M(X,\Sigma)$ is the set of all measurable functions defined on X. This gives rise to a linear transformation $M_u:M(X,\Sigma)\to M(X,\Sigma)$ defined by $M_u(f)=u\cdot f$, where the product of functions is pointwise. In case if $M(X,\Sigma)$ is a topological vector space and M_u is a continuous (bounded) operator, then it is called a multiplication operator induced by u. In this paper, multiplication operators on grand Lorentz spaces are defined and the fundamental properties such as boundedness, closed range, invertibility, compactness and closedness of these are characterized.

Büyük Lorentz Uzaylarında Çarpım Operatörleri

Anahtar Kelimeler
Büyük Lorentz uzayları,
Çarpım Operatörü,
Kompakt(tıkız) operator

Öz: (X,Σ,μ) σ-sonlu bir ölçüm uzayı, $M(X,\Sigma)$, X üzerinde tanımlı tüm ölçülebilir fonksiyonlar ve $u:X\to\mathbb{C}$ ölçülebilir bir fonksiyon olsun. X üzerinde tanımlı kompleks değerli ölçülebilir herhangi bir f fonksiyonu için $u\cdot f\in M(X,\Sigma)$ olduğundan u fonksiyonu $M(X,\Sigma)$ üzerinde $M_u(f)=u\cdot f$, $M_u:M(X,\Sigma)\to M(X,\Sigma)$ şeklinde bir lineer operator tanımlar. Eğer $M(X,\Sigma)$ bir topolojik vektör uzayı ve M_u operatörüde sürekli(sınırlı) bir operatör ise M_u'ya u tarafından indirgenen bir çarpım operatörü denir. Bu çalışmada büyük Lorentz uzaylarında çarpım operatörleri tanımlandı ve sınırlılık, kapalı görüntü, terslenebilirlik, kompaktlık ve kapalılık gibi temel özellikleri karakterize edildi.

1. Introduction

Let (X,Σ,μ) be a σ-finite measure space and f be a complex-valued measurable function defined on X. The distribution function of f is defined by

$$D_f(\lambda)=\mu\left(\left\{x\in X:|f(x)|>\lambda\right\}\right)\text{for all } \lambda\geq 0.$$

By f^*, we mean the non-increasing rearrangement of given function f as

$$f^*(t)=\inf\left\{\lambda>0:D_f(\lambda)\leq t\right\}=\sup\left\{\lambda>0:D_f(\lambda)>t\right\},\ t>0.$$

Also, the average function of f is defined by

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)ds\ ,\ t>0.$$

Note that $D_f(\cdot), f^*(\cdot)$ and $f^{**}(\cdot)$ are non-increasing and right continuous functions on $(0,\infty)$ [2]. For $p,q\in(0,\infty)$, we define

$$\|f\|_{p,q}^* = \begin{cases} \left(\frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & ,q<\infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & ,q=\infty \end{cases} \quad \text{and} \quad \|f\|_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & ,q<\infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & ,q=\infty \end{cases}.$$

For $0<p,q<\infty$, the Lorentz spaces are denoted by $L(p,q)(X,\mu)$ (or in short, $L(p,q)(X)$) is defined to be the vector space of all (equivalence classes of) measurable functions f on X such that $\|f\|_{p,q}^*<\infty$ [2] . We know that $\|f\|_{p,q}^*=\|f\|_p$ and so $L^p(X)=L(p,p)(X,\mu)$ where $L^p(X)$ is the usual Lebesgue space. Also, $L(p,q_1)(X,\mu)\subseteq L(p,q_2)(X,\mu)$ for $q_1\le q_2$. In particular,

$$L(p,q_1)(X)\subset L(p,p)(X)=L^p(X)\subset L(p,q_2)(X)$$

for $0<q_1<p<q_2\le\infty$ ([2, 6]). It is also known that if $1<p<\infty$ and $1\le q\le\infty$, then

$$\|f\|_{p,q}^*\le\|f\|_{p,q}\le\frac{p}{p-1}\|f\|_{p,q}^*$$

for each $f\in L(p,q)(X)$ and $\left(L(p,q)(X),\|\cdot\|_{p,q}\right)$ is a Banach space [2].

The construction of the Lorentz space $L(p,q)(X)$ seems to be inspired by the Lebesgue space $L^p(X)$, where f is replaced by its non-increasing rearrangement and a suitable weight is multiplied. In [3], Iwaniec and Sbordone generalized the notion of Lebesgue space and introduced the so-called grand Lebesgue space denoted by $L^{p)}$, which for $1<p<\infty$ consists of all measurable functions f defined on $(0,1)$ for which

$$\|f\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1|f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}<\infty.$$

The space $L^{p)}$ is a rearrangement invariant Banach function space for $0<\varepsilon<p-1$ and $L^p\subset L^{p)}\subset L^{p-\varepsilon}$ holds [4] . For a measurable function f on $X=(0,1)$, $\|f\|_{p,q)}$ is defined as

$$\|f\|_{p,q)} = \begin{cases} \sup\limits_{0<\varepsilon<q-1}\left(\dfrac{q}{p}\varepsilon\displaystyle\int_0^1 t^{\frac{q}{p}-1}\left(f^*(t)\right)^{q-\varepsilon}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}} & ;1<q<\infty \\ \sup\limits_{0<t<1} t^{\frac{1}{\mathrm{p}}} f^*(t) & ;q=\infty. \end{cases}$$

The grand Lorentz spaces $L_{p,q)}$ consists of those complex-valued measurable functions defined on $X=(0,1)$ such that $\|f\|_{\mathrm{p,q})}<\infty$. Cleary, if $p=q$, then $L_{p,q)}$ is equal to grand Lebesgue space $L_{p)}$ .To see this if one takes $\mathrm{p}\in(1,\infty)$, then

$$\|f\|_{p,p)} = \sup_{0<\varepsilon<p-1}\left(\frac{p}{p}\varepsilon\int_0^1 t^{\frac{p}{p}-1}\left(f^*(t)\right)^{p-\varepsilon}\mathrm{dt}\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{\mu(X)}\int_X\left(f^*(t)\right)^{p-\varepsilon}\mathrm{dt}\right)^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1|f(x)|^{p-\varepsilon}\mathrm{d}x\right)^{\frac{1}{p-\varepsilon}} = \|f\|_{p)}.$$

Now, let's compare the norms of the classical Lorentz space with grand Lorentz spaces.

For $1<p,q<\infty$, let's take a function $f\in L_{p,q)}$. Then we get

$$\|f\|_{p,q)} = \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(f^*(t)\right)^{q-\varepsilon}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}} \le \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1\left(t^{\frac{1}{p}}f^{**}(t)\right)^{q-\varepsilon}\mathrm{t}^{\frac{\varepsilon}{p}-1}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}}$$

and using $t^{\frac{1}{p}}f^{**}(t)\le\left(\frac{q}{p}\right)^{\frac{1}{q}}\|f\|_{pq}$, it can be obtained that

$$\|f\|_{p,q)} \le \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1\left(t^{\frac{1}{p}}f^{**}(t)\right)^{q-\varepsilon}\mathrm{t}^{\frac{\varepsilon}{p}-1}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}}$$

$$\le \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1\left(\left(\frac{q}{p}\right)^{\frac{1}{q}}\|f\|_{pq}\right)^{q-\varepsilon}\mathrm{t}^{\frac{\varepsilon}{\mathrm{p}}-1}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}} \le \left(\frac{q}{p}\right)^{\frac{1}{q}}\|f\|_{pq}.$$

In case of $q=\infty$, we get $\|f\|_{p,\infty)} = \sup\limits_{0<t<1} t^{\frac{1}{p}}f^*(t) \le \sup\limits_{0<t<1} t^{\frac{1}{p}}f^{**}(t) = \|f\|_{p\infty}$.

Example1.1. If E is a finite measurable set in Σ with characteristic function χ_E, then $\chi_E^*(t)=\chi_{[0,\mu(\mathrm{E}))}(t)$ and

$$\|\chi_E\|_{p,q)} = \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\chi_E^*(t)\right)^{q-\varepsilon}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}} = (q-1)\left(\mu(\mathrm{E})\right)^{\frac{q}{p}}$$

for $1<p,q<\infty$. On the other hand, if $q=\infty$, then $\|\chi_E\|_{p,\infty)} = \sup\limits_{0<t<1} t^{\frac{1}{p}}\chi_E^*(t) = \left(\mu(E)\right)^{\frac{1}{p}}$. Therefore

$$\|\chi_E\|_{p,q)} = \begin{cases} (q-1)\left(\mu(E)\right)^{\frac{q}{p}} & ,1<q<\infty \\ \left(\mu(E)\right)^{\frac{1}{p}} & ,q=\infty. \end{cases}$$

Let $u: X \to \mathbb{C}$ be a measurable function such that $u \cdot f \in M(X,\Sigma)$ whenever $f \in M(X,\Sigma)$. This gives rise to a linear transformation $M_u : M(X,\Sigma) \to M(X,\Sigma)$ defined by $M_u(f) = u \cdot f$, where the product of functions is pointwise. In case if $M(X,\Sigma)$ is a topological vector space and M_u is a continuous (bounded) operator, then it is called a multiplication operator induced by u.

Multiplication operators have been studied on various function spaces by various authors such as [1, 5-8]. Along the line of their arguments we will study the multiplication operators on the grand Lorentz spaces $L_{p,q)}$. For this purpose, we will characterize the invertibility of M_u on $L_{p,q)}$ and find necessary and sufficient conditions for Compact multiplication operators.

2. Bounded and Invertible Multiplication Operators

In this section boundedness and invertibility of multiplication operator M_u will be characterized in the terms of the boundedness and invertibility of the measurable function u respectively.

Example 2.1. Consider the complex-valued square integrable functions on the interval $[-3,1]$. For each $k \geq 3$, define a sequence $(f_k)_{k\in\mathbb{N}}$ by $f_k : [-3,1] \to \mathbb{C}$, $f_k(x) = x^{-\frac{1}{k}}$. Since $\| f_k \|_2 = \frac{k}{k-2}\left[1-(-3)^{\frac{k-2}{k}}\right] < \infty$ for all $k \geq 3$, we can say that $f_k \in L_2([-3,1])$ for all $k \geq 3$. Let $u:[-3,1] \to \mathbb{C}$ be a measurable function and define $M_u : L_2([-3,1]) \to L_2([-3,1])$ as $M_u(f) = u \cdot f$ for all $f \in L_2([-3,1])$. If we examine the example, then we have that u must be invertible for M_u is invertible. It can also be observed that M_u is one to one (injective) on the set $supp(u) = \{x \in X : u(x) \neq 0\}$.

Remark 2.2. In general, the multiplication operators on measurable spaces is not injective. Indeed, for a measurable space (X,Σ,μ), let $G = X - supp(u)$ with $\mu(G) > 0$. Then we have $(\chi_G \cdot u)(x) = \chi_G(x) \cdot u(x) = 0$ for all $x \in X$. This implies that $M_u(\chi_G) = 0$ and $Ker M_u \neq \{0\}$. Hence M_u is not injective.

On the contrary, if M_u is injective, then $\mu(X - supp(u))$ must be zero. On the other hand, if $\mu(X - supp(u)) = 0$ and μ is a complete measure, then $M_u(f) = 0$ implies that $f(x) \cdot u(x) = 0$ for all $x \in X$ and so $\{x \in X : f(x) \neq 0\} \subseteq X - supp(u)$ and $f = 0$ (a.e.) on X.

Proposition 2.3. M_u is injective on $K = L_{p,q)}(supp(u)) = \left\{ f\chi_{supp(u)} : f \in L_{p,q)}(X) \right\}$.

Proof. To show that the operator M_u is injective, it is enough to show that $Ker M_u = \{0\}$. Indeed, if $M_u(\overline{f}) = 0$ with $\overline{f} \in K$, then $\overline{f}(x) \cdot u(x) = f(x) \cdot \chi_{supp(u)}(x) \cdot u(x) = 0$ for all $x \in X$. From this, we get $f(x) \cdot u(x) = 0$ for all $x \in supp(u)$ and so $f(x) = 0$. Therefore $\overline{f} = 0$ and $Ker M_u = \{0\}$.

Theorem 2.4. The linear transformation $M_u : f \to u \cdot f$ on grand Lorentz spaces $L_{p,q)}$ is bounded for $1 < p,q \leq \infty$ if and only if u is essentially bounded. Moreover $\|M_u\| = \|u\|_\infty$.

Proof. Suppose that u is essentially bounded i.e. $u \in L_\infty(\mu)$ and $f \in L_{p,q)}$. Since $|u(x)| \leq \|u\|_\infty$ for all $x \in X$, it can be written that $|(u \cdot f)(x)| \leq \|u\|_\infty |f(x)|$ and $\{x \in X : |(u \cdot f)(x)| > \lambda\} \subseteq \{x \in X : \|u\|_\infty |f(x)| > \lambda\}$. Therefore

$$D_{u\cdot f}(\lambda)=D_{M_u(f)}(\lambda)\le D_f\left(\frac{\lambda}{\|u\|_\infty}\right)$$

and

$$\left\{\lambda>0: D_f\left(\frac{\lambda}{\|u\|_\infty}\right)\le t\right\}\subseteq\left\{\lambda>0: D_{M_u(f)}(\lambda)\le t\right\}, t>0.$$

By using the definition of rearrangement, we have $\left(M_u(f)\right)^*\le\|u\|_\infty f^*(t)$ and

$$\begin{aligned}\|M_u(f)\|_{p,q)} &= \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\left(M_u(f)\right)^*(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}\\ &\le \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\|u\|_\infty f^*(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}\\ &=\|u\|_\infty \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(f^*(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}=\|f\|_{p,q)}\|u\|_\infty\end{aligned}$$

Consequently $\|M_u(f)\|_{p,q)}\le\|u\|_\infty\|f\|_{p,q)}$. Also for $q=\infty$, we have

$$\|M_u(f)\|_{p,q)}=\sup_{0<t<1}t^{\frac{1}{p}}\left(M_u(f)\right)^*(t)\le\sup_{0<t<1}t^{\frac{1}{p}}\|u\|_\infty f^*(t)=\|u\|_\infty\|f\|_{p,q)}.$$

Thus, for any $f\in L_{p,q)}$, for all $1<p,q\le\infty$ we obtain

$$\|M_u(f)\|_{p,q)}\le\|u\|_\infty\cdot\|f\|_{p,q)} \tag{2.1}$$

Conversely, suppose that M_{u} is a bounded operator on grand Lorentz spaces for $1<q<\infty$. If u is not an essentially bounded function, then we can write a set $G_k=\{x\in X: |u(x)|>k\}$ which has a positive measure for all $k\in\square$. Since the non-increasing rearrangement of the characteristic function χ_{G_k} is $\left(\chi_{G_k}\right)^*(t)=\chi_{[0,\mu(G_k))}(t)$, we can get

$$\left\{x\in X: k\cdot\chi_{G_k}(x)>\lambda\right\}\subseteq\left\{x\in X:\left(\mathrm{u}\cdot\chi_{G_k}\right)(x)>\lambda\right\}$$

and $D_{k\cdot\chi_{G_k}}(\lambda)\le D_{M_u\left(\chi_{G_k}\right)}(\lambda)$. Therefore

$$\left\{\lambda>0: D_{M_{\mathrm{u}}\left(\chi_{G_k}\right)}(\lambda)\le t\right\}\subseteq\left\{\lambda>0: D_{k\cdot\chi_{G_k}}(\lambda)\le t\right\}$$

for all $t>0$ and $\mathit{inf}\left\{\lambda>0: D_{k\cdot\chi_{G_k}}(\lambda)\le t\right\}\le \mathit{inf}\left\{\lambda>0: D_{M_{\mathrm{u}}\left(\chi_{G_k}\right)}(\lambda)\le t\right\}$.

As a result, $\left(M_u\left(\chi_{G_k}\right)\right)^*(t) \geq k \cdot \left(\chi_{G_k}\right)^*(t)$ and so

$$\left\|M_u\left(\chi_{G_k}\right)\right\|_{p,q)} = \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\left(M_u\left(\chi_{G_k}\right)\right)^*(t)\right)^{q-\varepsilon} dt\right)^{\frac{1}{q-\varepsilon}}$$

$$\geq \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\left(k\cdot\chi_{G_k}\right)^*(t)\right)^{q-\varepsilon} dt\right)^{\frac{1}{q-\varepsilon}} = k\left\|\chi_{G_k}\right\|_{p,q)}.$$

Besides these, for $q=\infty$ we have

$$\left\|M_u\left(\chi_{G_k}\right)\right\|_{p,q)} = \sup_{0<t<1} t^{\frac{1}{p}}\left(M_u\left(\chi_{G_k}\right)\right)^*(t)$$

$$\geq \sup_{0<t<1} t^{\frac{1}{p}} k\left(\chi_{G_k}\right)^*(t) = k\cdot\left\|\chi_{G_k}\right\|_{p,q)}.$$

This contradicts the boundedness of M_u. Hence u must be essentially bounded. Now for any $\delta>0$, let $S=\left\{x\in X: |u(x)|\geq\|u\|_\infty-\delta\right\}$. Then

$$\left\{x\in X: \left(\|u\|_\infty-\delta\right)\chi_S(x)>\lambda\right\}\subseteq\left\{x\in X: \left(u\cdot\chi_S\right)(x)>\lambda\right\}$$

and $D_{(\|u\|_\infty-\delta)\chi_S}(\lambda)\leq D_{u\cdot\chi_S}(\lambda)$. Therefore

$$\left\{\lambda>0: D_{u\cdot\chi_S}(\lambda)\leq t\right\}\subseteq\left\{\lambda>0: D_{(\|u\|_\infty-\delta)\chi_S}(\lambda)\leq t\right\}.$$

By using the infimum property, we get

$$inf\left\{\lambda>0: D_{u\cdot\chi_S}(\lambda)\leq t\right\}\geq inf\left\{\lambda>0: D_{(\|u\|_\infty-\delta)\chi_S}(\lambda)\leq t\right\}$$

and $\left(M_u\left(\chi_S\right)\right)^*(t)\geq\left(\|u\|_\infty-\delta\right)\left(\chi_S\right)^*(t)$ so $\|M_u\|\geq\left(\|u\|_\infty-\delta\right)$. As a result, $\|M_u\|\geq\|u\|_\infty$ and $\|M_u\|=\|u\|_\infty$ with 2.1.

We already know that if X and Y are Banach spaces and $F\in B(X,Y)$, then F is bounded below if and only if F is 1-1 and has closed range. According to this knowledge, we can give the following corollary.

Corollary 2.5. $M_u: L_{p,q)}(supp(u))\to L_{p,q)}(supp(u))$ has closed range if and only if M_u is bounded below on $L_{p,q)}(supp(u))$.

This result is clear. Since M_u is 1-1 on $L_{p,q)}(supp(u))$ by Proposition 2.3. Moreover, if μ is a complete measure and $u\neq 0$ a.e. on X, then we have the following result.

Corollary 2.6. If μ is a complete measure and $u\neq 0$ a.e. on X, then $M_u: L_{p,q)}(X,\Sigma,\mu)\to L_{p,q)}(X,\Sigma,\mu)$ has closed range if and only if M_u is bounded below on $L_{p,q)}(X,\Sigma,\mu)$.

Theorem 2.7. The set of all multiplication operators on the grand Lorentz spaces $L_{p,q)}$ for $1<p,q<\infty$ is a maximal abelian subalgebra of $B\left(L_{p,q)},L_{p,q)}\right)$, Banach algebra of all bounded linear operators on $L_{p,q)}$.

Proof. Let $H=\{M_u : u\in L_\infty\}$. Then H is a vector space under operations $+:\underset{(M_u,M_v)\,\mapsto\, M_{u+v}}{HxH \to H}$, $\cdot:\underset{(k,M_u)\,\mapsto\, M_{ku}}{FxH\to H}$ and a subalgebra of $B(L_{p,q)},L_{p,q)})$. Consider the composition of operators such as $M_u\circ M_v=M_{uv}$, where $M_u,M_v\in H$. Let $u,v\in L_\infty$. Then $|u(x)|\le\|u\|_\infty$ and $|v(x)|\le\|v\|_\infty$ implies that $\|uv\|_\infty\le\|u\|_\infty\|v\|_\infty$ and so the product is an inner operation, moreover the composition is associative, commutative and distributive respect to the sum and the scaler product, thus we can conclude that H is a subalgebra of $B(L_{p,q)},L_{p,q)})$. Let T be any operator on $L_{p,q)}$ such that $T\circ M_u=M_u\circ T$ for every $u\in L_\infty(\mu)$.Consider the unit function $e:X\to\mathbb{R}$ defined by $e(x)=1$ for all $x\in X$ and $v=Te$. Then $T(\chi_E)=T(M_{\chi_E}e)=M_{\chi_E}(T(e))=\chi_E v=M_v\circ\chi_E$ for all measurable set $E\in\Sigma$. Consequently $T=M_v$. Now, let us check that $v\in L_\infty(\mu)$ or not. If possible, the set $G_k=\{x\in X: |v(x)|>k\}$ has a positive measure for each $k\in\mathbb{N}$. Then

$$\left\|T\left(\chi_{G_k}\right)\right\|_{p,q)}=\left\|M_v\left(\chi_{G_k}\right)\right\|_{p,q)}\ge k\cdot\left\|\chi_{G_k}\right\|_{p,q)}.$$

Therefore T is an unbounded operator that is a contradiction to the fact that T is bounded. Therefore $v\in L_\infty(\mu)$ and M_v is bounded by Theorem 2.4.Now, let $f\in L_{p,q)}$ and $(s_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of measurable simple functions such that $\lim_{n\to\infty}s_n=f$. Then $T(f)=T\left(\lim_{n\to\infty}s_n\right)=\lim_{n\to\infty}T(s_n)=\lim_{n\to\infty}M_v(s_n)=M_v\lim_{n\to\infty}(s_n)=M_v(f)$. Therefore, we can conclude that $T\in H=\{M_u : u\in L_\infty\}$.

Corollary 2.8. The multiplication operator M_u on $L_{p,q)}$ for $1<p,q<\infty$ is invertible if and only if u is invertible in L_∞.

Proof. Let M_u be invertible. Then there exists a $T\in B(L_{p,q)},L_{p,q)})$ such that $T\circ M_u=M_u\circ T=I$. Let $M_v\in H$. Then $M_u\circ M_v=M_v\circ M_u$ and

$$T\circ M_v\circ(M_u\circ T)=(T\circ M_u)\circ M_v\circ T\Rightarrow T\circ M_v=M_v\circ T.$$

Therefore, we can conclude that T commute with H and so $T\in H$ by Theorem 2.7. Then there exists a $w\in L_\infty$ such that $T=M_w$ and $M_u\circ M_w=M_w\circ M_u=I$. This implies that $uw=wu=1$ a.e, which means that u is invertible on L_∞. On the other hand, assume that u is invertible on L_∞, that is $\frac{1}{u}\in L_\infty$. Then $M_u\circ M_{1/u}=M_{1/u}\circ M_u=I$ which means that M_u is invertible on $B(L_{p,q)},L_{p,q)})$.

3. Compact Multiplication Operators

In this section we will characterize the compact multiplication operators. A compact operator is a linear operator L from a Banach space X to another Banach space Y, such that the image under L of any bounded subset of X is a relatively compact subset (has compact closure) of Y. Such an operator is necessarily a bounded operator, and so continuous

Definition 3.1. Let T be an operator. A subspace K of a normed space X is said to be invariant under T (or simply T-invariant) whenever $T(K)\subseteq K$.

Lemma 3.2. Let $T:A\to A$ be an operator. If T is compact and N is a closed T-invariant subspace of A, then $T|_N$ is also compact.

Proof. Let $(g_k)_{k\in\mathbb{N}}$ be a bounded sequence in $N\subseteq A$. Then compactness property of T implies that there exists a subsequence $(g_{k_n})_{n\in\mathbb{N}}$ of $(g_k)_{k\in\mathbb{N}}$ such that $(T(g_{k_n}))_{n\in\mathbb{N}}$ converges in. *A.* .Since $(g_{k_n})_{n\in\mathbb{N}}\subset N$ and $(T(g_{k_n}))_{n\in\mathbb{N}}\subseteq T(N)$, then $(T(g_{k_n}))_{n\in\mathbb{N}}$ converges on N. Hence $T|_N$ is compact.

Theorem3.3. Let M_u be a compact operator. Let $G_\delta(u)=\{x\in X : |u(x)|\geq\delta\}$ and $L_{p,q)}(G_\delta(u))=\{f\chi_{G_\delta(u)} : f\in L_{p,q)}\}$ for any $\delta>0$. Then $L_{p,q)}(G_\delta(u))$ is closed invariant subspace of $L_{p,q)}$ under M_u. Moreover M_u is a compact operator on $L_{p,q)}(G_\delta(u))$.

Proof. We first show that $L_{p,q)}(G_\delta(u))$ is a subspace of $L_{p,q)}$. Let $\tilde{f},\tilde{g}\in L_{p,q)}(G_\delta(u))$ and $a,b\in\mathbb{C}$. Since $\tilde{f}=f\chi_{G_\delta(u)}$ and $\tilde{g}=g\chi_{G_\delta(u)}$ for any $f,g\in L_{p,q)}$, we have $a\tilde{f}+b\tilde{g}=af\chi_{G_\delta(u)}+bg\chi_{G_\delta(u)}=(af+bg)\chi_{G_\delta(u)}$. By the definition of $M_u : L_{p,q)}(G_\delta(u))\to L_{p,q)}(X,\Sigma,\mu)$, we have $M_u(\tilde{f})=u\cdot\tilde{f}=u\cdot f\chi_{G_\delta(u)}$. Therefore $L_{p,q)}(G_\delta(u))$ is an invariant subspace of $L_{p,q)}$ under M_u. Now, let us show that $\overline{L_{p,q)}(G_\delta(u))}\subseteq L_{p,q)}(G_\delta(u))$. Let $\tilde{g}$ be in $\overline{L_{p,q)}(G_\delta(u))}$. Then there exists a sequence $\tilde{g}_k$ in $\overline{L_{p,q)}(G_\delta(u))}$ such that $\tilde{g}_k\to\tilde{g}$ where $g_k\in L_{p,q)}$ and $\tilde{g}_k=g_k\chi_{G_\delta(u)}$ for each $k\in\mathbb{N}$.

Since $\tilde{g}_k$ is a Cauchy sequence in $\overline{L_{p,q)}(G_\delta(u))}$, it can be written that for all $\varepsilon>0$, there exists a $k_0\in\mathbb{N}$ such that $\|\tilde{g}_k-\tilde{g}_r\|_{p,q)}<\varepsilon$ for all $k,r>k_0$. Hence for all $k,r>k_0$, we can find a $\delta>0$ such that

$$\delta(g_k-g_r)\leq(g_k-g_r)\chi_{G_\delta(u)} \text{ and } \delta(g_k-g_r)^*\leq(g_k-g_r)^*\chi_{[0,\mu(G_\delta(u)))}$$

Then $\|g_k-g_r\|_{p,q)}\leq\alpha\|\tilde{g}_k-\tilde{g}_r\|_{p,q)}$ for any constant α. Therefore $\{g_k\}_{k\in\mathbb{N}}$ is also a Cauchy sequence in $L_{p,q)}$. Since $L_{p,q)}$ is a Banach space, we can write that $g_k\to g$ for an element $g\in L_{p,q)}$. Thus

$$\left\|g_k\chi_{G_\delta(u)}-g\chi_{G_\delta(u)}\right\|_{p,q)}\leq\|g_k-g\|_{p,q)}$$

and $\tilde{g}_k\to\tilde{g}$. Consequently $\tilde{g}\in L_{p,q)}(G_\delta(u))$ and $M_u|_{L_{p,q)}(G_\delta(u))}$ is a compact operator by Lemma 3.2.

Theorem 3.4. A multiplication operator M_u on $L_{p,q)}$ is compact if and only if $L_{p,q)}(G_\delta(u))$ is finite dimensional for each $\delta>0$, where

$$G_\delta(u)=\{x\in X : |u(x)|\geq\delta\} \text{ and } L_{p,q)}(G_\delta(u))=\{f\chi_{G_\delta(u)} : f\in L_{p,q)}\}.$$

Proof. If M_u is a compact operator, then $L_{p,q)}(G_\delta(u))$ is a closed invariant subspace of $L_{p,q)}$ under M_u and $M_u|_{L_{p,q)}(G_\delta(u))}$ is a compact operator by Theorem 3.3. Let's take any $x\in X$. If $x\notin G_\delta(u)$ then for each $f\in L_{p,q)}$, we can obtain $\left(M_u|_{L_{p,q)}(G_\delta(u))}(f)\right)^*=\left(u\cdot f\chi_{G_\delta(u)}\right)^*=0$. Therefore $M_u|_{L_{p,q)}(G_\delta(u))}=0$. If $x\in G_\delta(u)$, then $|u(x)|\geq\delta$ and note that $\left|(u\cdot f\chi_{G_\delta(u)})(x)\right|\geq\delta\left|(f\chi_{G_\delta(u)})(x)\right|$, $D_{f\chi_{G_\delta(u)}}\left(\frac{\lambda}{\delta}\right)\leq D_{(u\cdot f\chi_{G_\delta(u)})}(\lambda)$. Therefore $\left\{\lambda>0 : D_{(u\cdot f\chi_{G_\delta(u)})}(\lambda)\leq t\right\}\subseteq\left\{\lambda>0 : D_{f\chi_{G_\delta(u)}}\left(\frac{\lambda}{\delta}\right)\leq t\right\}$ for all $t>0$. By using this inclusion, we have

$$\delta\cdot\left(f\chi_{G_\delta(u)}\right)^*(t)\leq\left(u\cdot f\chi_{G_\delta(u)}\right)^*(t)$$

and

$$\left\|M_u\left(f\chi_{G_\delta(u)}\right)\right\|_{p,q)} = \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\left(M_u\left(f\chi_{G_\delta(u)}\right)\right)^*(t)\right)^{q-\varepsilon}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}}$$
$$\geq \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\delta\cdot\left(f\chi_{G_\delta(u)}\right)^*(t)\right)^{q-\varepsilon}\mathrm{dt}\right)^{\frac{1}{q-\varepsilon}}$$
$$\geq \delta\cdot\left\|f\chi_{G_\delta(u)}\right\|_{p,q)}.$$

Thus, in either case $M_{\mathrm{u}}|_{L_{p,q)}(G_\delta(u))}$ has a closed range in $L_{p,q)}\left(G_\delta(u)\right)$ and invertible. Being compact implies that $L_{p,q)}\left(G_\delta(u)\right)$ is finite dimensional.

Conversely, suppose that $L_{p,q)}\left(G_\delta(u)\right)$ is finite dimensional for each $\delta>0$. In particular, $L_{p,q)}\left(G_{1/n}(u)\right)$ is finite dimensional for each $n\in\mathbb{N}$. Define a sequence $u_n : X\to\mathbb{C}$ as

$$u_n(x)=\begin{cases}u(x), & |u(x)|\geq 1/n\\ 0, & |u(x)|<1/n\end{cases}$$

for all $n\in\mathbb{N}$. Since $u\in L_\infty$, it's easy to see that $u_n\in L_\infty$ for each $n\in\mathbb{N}$. Moreover for any $f\in L_{p,q)}$,

$$D_{(u_n-u)\cdot f}(\lambda)=\mu\left(\left\{x\in X : \left|\left((u_n-u)f\right)(x)\right|>\lambda\right\}\right)$$

and

$$\left((u_n-u)f\right)^*(t)=\inf\left\{\lambda>0 : D_{(u_n-u)\cdot f}(\lambda)\leq t\right\}.$$

For any $\lambda>0$, if $x\in G_{1/n}(u)$ then $\left((u_n-u)f\right)^*(t)=0$ and $(u_n-u)f=0$. If $x\notin G_{1/n}(u)$, then we get $\left((u_n-u)f\right)^*(t)\leq\frac{1}{n}f^*(t)$ and $\left\|M_{(u_n-u)}(f)\right\|_{p,q)}\leq\frac{1}{n}\|f\|_{p,q)}$. This implies that M_{u_n} converges to M_u uniformly. Since $L_{p,q)}\left(G_{1/n}(u)\right)$ is finite dimensional so M_{u_n} is finite rank operator. Therefore, M_{u_n} is a compact operator and so M_u is.

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