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Common Fixed Point Theorems Satisfying Implicit Relations on 2-cone Banach Space with an Application

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Abstract
In this paper, we discuss the existence and uniqueness of common fixed-point theorems satisfying implicit relations on 2-cone Banach spaces. Modifying obtained new contractive conditions, we also give an application to the fixed-circle problem.

Keywords: *common fixed point; 2-cone Banach space; 2-cone normed space; fixed circle.*

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1. Introduction and preliminaries

In 2007, Huang and Zhang [3] introduced the concept of a cone metric space and proved fixed point theorems for contraction mappings such as:

Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X$$

has a unique fixed point.

In [4], Karapınar established some fixed-point theorems in cone Banach space in 2009. Ahmet Şahiner and Tuba Yiğit initiated the concept of a 2-cone Banach space and proved some fixed-point theorems [16]. Krishnakumar and Dhamodharan proved some common fixed-point theorems on contractive modulus in 2-cone Banach space [5].

In this paper, following the idea which was given in [14], we establish some common fixed-point theorems for a self-mapping satisfying implicit relations which are contractive conditions in 2-cone Banach spaces. Now we recall some known definitions and basic facts.

Definition 1.1. [3] Let E be the real Banach space. A subset P of E is called a cone if and only if

1. P is closed, nonempty and $P \neq 0$
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

From now on we suppose that E is a Banach space, P is a cone in E with $intP = \emptyset$ and $\leq$ is partial ordering with respect to P.

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Example 1.1. Let $K > 1$ be given. Consider the real vector space with

$$E = \left\{ ax + b : a, b \in R; x \in \left[1 - \frac{1}{k}, 1\right] \right\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.2. [3] Let X be a nonempty set. If the mapping $d : X \times X \to E$ satisfies

1. $d(x, y) > 0$ and $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

then (X, d) is called a cone metric space (CMS).

Example 1.2. [3] Let $E = \mathbb{R}^2$

$$P = \{(x, y) : x, y \geq 0\}.$$

$X = \mathbb{R}$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|),$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.3. [4] Let X be a vector space over $\mathbb{R}$. If the mapping $\|.\|_c : X \to E$ satisfies

1. $\|x\|_c \geq 0$ for all $x \in X$,
2. $\|x\|_c = 0$ if and only if $x = 0$,
3. $\|x + y\|_c \leq \|x\|_c + \|y\|_c$ for all $x, y \in X$,
4. $\|kx\|_c = |k|\|x\|_c$ for all $k \in \mathbb{R}$ and for all $x \in X$,

then $\|.\|_c$ is called a cone norm on X, and the pair $(X, \|.\|_c)$ is called a cone normed space (CNS).

Remark 1.1. [1] Each cone normed space is cone metric space with metric defined by

$$d(x, y) = \|x - y\|_c.$$

Example 1.3. [15] Let $X = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$ and $\|(x, y)\|_c = (a|x|, b|y|), a > 0, b > 0$. Then $(X, \|.\|_c)$ is a cone normed space over $\mathbb{R}^2$.

Example 1.4. [2] Let $E = l_1$, $P = \{\{x_n\} \in E : x_n \geq 0$, for all $n\}$ and $(X, \|.\|)$ be a normed space and $\|.\|_c : X \to E$ defined by $\|x\|_c = \left\{ \frac{\|x\|}{2^n} \right\}$. Then P is a normal cone with constant normal $M = 1$ and $(X, \|.\|_c)$ is a cone normed space.

Definition 1.4. [1] Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X. Then $\{x_n\}_{n \geq 0}$ converges to x whenever for every $c \in E$ with $0 \ll E$, there is a natural number $N \in \mathbb{N}$ such that $\|x_n - x\|_c \ll c$ for all $n \geq N$. It is denoted by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

Definition 1.5. [1] Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X. $\{x_n\}_{n \geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in \mathbb{N}$, such that $\|x_n - x_m\|_c \ll c$ for all $n, m \geq N$.

Definition 1.6. [1] Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X. $(X, \|.\|_c)$ is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called cone Banach spaces.

Lemma 1.1. *[4] Let $(X, \|.\|_c)$ be a CNS, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then* *i.* *the sequence* $\{x_n\}$ *converges to* x *if and only if* $\|x_n - x\|_c \to 0$ *as* $n \to \infty$,

ii. *the sequence* $\{x_n\}$ *is Cauchy if and only if* $\|x_n - x_m\|_c \to 0$ *as* $n, m \to \infty$,

iii. *the sequence* $\{x_n\}$ *converges to* x *and the sequence* $\{y_n\}$ *converges to* y, *then* $\|x_n - y_n\|_c \to \|x - y\|_c$.

Definition 1.7. [16] Let X be a linear space over $\mathbb{R}$ with dimension greater then or equal to 2, E be Banach space with the norm $\|.\|$ and $P \subset E$ be a cone. If the function

$$\|.,.\| : X \times X \to (E, P, \|.\|)$$

satisfies the following axioms then $(X, \|.,.\|_c)$ is called a 2-cone normed space:

1. $\|x, y\|_c \geq 0$ for all $x, y \in X$, $\|x, y\|_c = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\|_c = \|y, x\|_c$ for all $x, y \in X$,
3. $\|\alpha x, y\|_c = |\alpha| \|x, y\|_c$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$,
4. $\|x, y + z\|_c \leq \|x, y\|_c + \|y, z\|_c$ for all $x, y, z \in X$.

If we fix $\{u_1, u_2, ..., u_d\}$ to be a basis for X, we can give the following lemma.

Lemma 1.2. *[16] Let* $(X, \|.,.\|_c)$ *be a 2-cone normed space. Then a sequence* $\{x_n\}$ *converges to* $x \in X$ *if and only if for each* $c \in E$ *with* $c \gg 0$ *(*0 *is zero element of* E*) there exists an* $N = N(c) \in \mathbb{N}$ *such that* $n > N$ *implies* $\|x_n - x, u_i\|_c \ll c$ *for every* $i = 1, 2, ..., d$.

Lemma 1.3. *[16] Let* $(X, \|.,.\|_c)$ *be a 2-cone normed space. Then a sequence* $\{x_n\}$ *converges to* x *in* X *if and only if* $\lim_{n\to\infty} \max \|x_n - x, u_i\|_c = 0$.

Definition 1.8. [16] A 2-cone normed space $(X, \|.,.\|_c)$ is a 2-cone Banach space if any Cauchy sequence in X is convergent to an x in X.

Theorem 1.1. *[17] Any 2-cone normed space* X *is a cone normed spaces and its topology agrees with the norm generated by* $\|.\|_c^\infty$, *where the function* $\|.\|_c^\infty : X \to (E, P, \|.\|)$ *is defined by*

$$\|.\|_c^\infty := \max\{\|x, u_i\|_c : i = 1, 2, \ldots, d\}.$$

2. Main results

In this section, we prove some common fixed-point theorems on 2-cone Banach spaces. To do this, we define some notions and give some necessary examples.

Definition 2.1. Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T be a self-mapping of X. If T satisfies the condition

$$\|Tx - Ty, u\|_c \leq h_1 \|x - y, u\|_c$$

for all $x, y, u \in X$ and some $0 < h_1 < 1$ then it is called 2-Banach contraction.

Definition 2.2. Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T be a self mapping of X. A mapping T is said to be 2-Zamfirescu type contraction if it satisfies at least one of the conditions for all $x, y, u \in X$ and some $h_1 \in (0, 1)$, $h_2, h_3 \in \left(0, \frac{1}{2}\right)$:

1. $\|Tx - Ty, u\|_c \leq h_1 \|x - y, u\|_c$,
2. $\|Tx - Ty, u\|_c \leq h_2(\|x - Ty, u\|_c + \|y - Tx, u\|_c)$,
3. $\|Tx - Ty, u\|_c \leq h_3(\|x - Tx, u\|_c + \|y - Ty, u\|_c)$.

Definition 2.3. Let X be a 2-cone Banach space and T be a self mapping of X. T is said to be continuous at x if for all sequence $\{x_n\}$ in X with $\|x_n, u\|_c \to \|x, u\|_c$ implies that $\|Tx_n, u\|_c \to \|Tx, u\|_c$.

Lemma 2.1. *Let X and Y be two 2-cone Banach spaces and T be a linear map from X into Y. The following properties are equivalent*:

i *(Continuity at a point) Given $0 \ll c$ there is a $0 \ll s$ such that $\|Tx - Tx_0, u\|_c \ll c$ whenever $\|x - x_0, u\|_c \ll s$ for some $x_0 \in X$.*

ii *(Continuity at zero) For $0 \ll c$ there is a $0 \ll s$ such that $\|Tx, u\|_c \ll c$ whenever $\|x, u\|_c \ll s$.*

iii *(Continuity at every point of x) Given $0 \ll c$ there is a $0 \ll s$ such that $\|Tx - Ty, u\|_c \ll c$ whenever $\|x - y, u\|_c \ll s$ for some $x \in X$.*

Proof. Assume that (i) is true. For some $x_0 \in X$ and for every $0 \ll c$ there is a $0 \ll s$ such that $\|Tx - Tx_0, u\|_c \ll c$ whenever $\|x - x_0, u\|_c \ll s$. Then for every $z \in X$ with $\|z, u\|_c \ll s$ we have $\|T(z + x_0) - Tx_0, u\|_c \ll c$ because $\|(z + x_0) - x_0, u\|_c \ll t$, where T is linear map then $\|Tz, u\|_c \ll c$ whenever $\|z, u\|_c \ll s$ and we have shown that (i) implies (ii).
Assume that (ii) is true. For every $x \in X$ and $0 \ll c$, there exits a $0 \ll s$ such that $\|Tz, u\|_c \ll c$ whenever $\|z, u\|_c \ll s$ then we have $\|T(y - x), u\|_c \ll s$. If we take $y - x$ in place of z then we have (ii) implies (iii) since T is linear map. Clearly (iii) implies (i). Thus (i), (ii) and (iii) are equivalent. □

Definition 2.4. Let Φ be the class of continuous functions $\varphi : P^4 \to P$ non-decreasing in the first argument and if φ satisfies one of the following conditions for $x, y \in P$:

a. (a_1) $x \leq \varphi\left(y, x, y, \frac{x+y}{2}\right)$ or (a_2) $x \leq \varphi(x, y, y, x)$.

b. (b_1) $x \leq \varphi\left(y, \frac{x+y}{2}, 0, x + y\right)$ or (b_2) $x \leq \varphi(x, y, x, x)$.

then there exists a real number $0 < h < 1$ such that $x \leq hy$.

Now we define the following conditions:
Condition (I): Let X be a 2-cone Banach space (with $\dim X \geq 2$) and S, T be two self-mappings of X such that for all $x, y, u \in X$ satisfying the condition:

$$\|Sx - Ty, u\|_c \leq \varphi\left(\|x - y, u\|_c, \|x - Sx, u\|_c, \|y - Ty, u\|_c, \frac{\|x - Ty, u\|_c + \|y - Sx, u\|_c}{2}\right).$$

Condition (II): Let X be a 2-cone Banach space (with $\dim X \geq 2$) and S, T be two self-mappings of X such that for all $x, y, u \in X$ satisfying the condition:

$$\|Sx - Ty, u\|_c \leq \varphi\left(\|x - y, u\|_c, \frac{\|x - Sx, u\|_c + \|y - Ty, u\|_c}{2}, 0, \|x - Ty, u\|_c + \|y - Sx, u\|_c\right).$$

Theorem 2.1. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and S, T be two continuous self-mappings of X satisfying the condition (I). Then S and T have a unique common fixed point in X.*

Proof. For a given $x_0 \in X$ and $n \geq 1$, take $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general we define a sequence of elements of X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n} = Tx_{2n-1}$ for $n = 0, 1, 2, 3, \cdots$. Now for all $u \in X$, by condition (I), we obtain

$$\begin{aligned}
\|x_{2n+1} - x_{2n}, u\|_c &= \|Sx_{2n} - Tx_{2n-1}, u\|_c \\
&\leq \varphi\left(\begin{array}{c} \|x_{2n} - x_{2n-1}, u\|_c, \|x_{2n} - Sx_{2n}, u\|_c, \|x_{2n-1} - Tx_{2n-1}, u\|_c, \\ \frac{\|x_{2n} - Tx_{2n-1}, u\|_c + \|x_{2n-1} - Sx_{2n}, u\|_c}{2} \end{array}\right) \\
&= \left(\begin{array}{c} \varphi\|x_{2n} - x_{2n-1}, u\|_c, \|x_{2n} - x_{2n+1}, u\|_c, \|x_{2n-1} - x_{2n}, u\|_c, \\ \frac{\|x_{2n} - x_{2n}, u\|_c + \|x_{2n-1} - x_{2n+1}, u\|_c}{2} \end{array}\right) \\
&= \varphi\left(\begin{array}{c} \|x_{2n} - x_{2n-1}, u\|_c, \|x_{2n} - x_{2n+1}, u\|_c, \|x_{2n} - x_{2n-1}, u\|_c, \\ \frac{\|x_{2n-1} - x_{2n+1}, u\|_c}{2} \end{array}\right) \\
&\leq \varphi\left(\begin{array}{c} \|x_{2n} - x_{2n-1}, u\|_c, \|x_{2n} - x_{2n+1}, u\|_c, \|x_{2n} - x_{2n-1}, u\|_c, \\ \frac{\|x_{2n-1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n+1}, u\|_c}{2} \end{array}\right).
\end{aligned}$$

Hence from Definition 2.4 (a_1), we have

$$\|x_{2n+1} - x_{2n}, u\|_c \leq h\|x_{2n} - x_{2n-1}, u\|_c \text{ where } 0 < h < 1. \tag{2.1}$$

Similarly, we have

$$\|x_{2n} - x_{2n-1}, u\|_c \leq h\|x_{2n-1} - x_{2n-2}, u\|_c. \tag{2.2}$$

Hence, by (2.1) and (2.2), we have

$$\|x_{2n+1} - x_{2n}, u\|_c \leq h^2\|x_{2n-1} - x_{2n-2}, u\|_c.$$

By continuing this process, we get

$$\|x_{2n+1} - x_{2n}, u\|_c \leq h^{2n}\|x_1 - x_0, u\|_c.$$

For every $n > m$, we have

$$\begin{aligned}
\|x_n - x_m, u\|_c &\leq \|x_n - x_{n-1}, u\|_c + \|x_{n-1} - x_{n-2}, u\|_c + \cdots + \|x_{m+1} - x_m, u\|_c \\
&\leq (h^{n-1} + h^{n-2} + \cdots + h^m)\|x_1 - x_0, u\|_c \\
&\leq \left(\frac{h^m}{1-h}\right)\|x_1 - x_0, u\|_c.
\end{aligned}$$

Since $0 < h < 1$, by Definition 2.4, $\left(\frac{h^m}{1-h}\right) << 0$ as $m \to \infty$. Hence $\|x_n - x_m, u\|_c << 0$ as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists a point z in X such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of S and T that $Sz = Tz = z$. Thus z is a common fixed point of S and T.

Uniqueness Let w be another common fixed point of S and T, that is $Sw = Tw = w$. Then, we have

$$\begin{aligned}
\|z - w, u\|_c &= \|Sz - Tw, u\|_c \\
&\leq \varphi\left(\begin{array}{c} \|z - w, u\|_c, \|z - Sz, u\|_c, \|w - Tw, u\|_c, \\ \frac{\|z-Tw,u\|_c + \|w-Sz,u\|_c}{2} \end{array}\right) \\
&\leq \varphi(\|z - w, u\|_c, 0, 0, \|z - w, u\|_c).
\end{aligned} \tag{2.3}$$

By Definition 2.4 (a_2) and the inequality (2.3), we get

$$\|z - w, u\|_c \leq 0.$$

Hence $z = w$ and for all $u \in X$. Thus z is a unique common fixed point of S and T. □

Corollary 2.1. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T be a self-mapping of X satisfying the condition*

$$\|Tx - Ty, u\|_c \leq \varphi\left(\|x - y, u\|_c, \|x - Tx, u\|_c, \|y - Ty, u\|_c, \frac{\|x - Ty, u\|_c + \|y - Tx, u\|_c}{2}\right),$$

for all $x, y, u \in X$. Then T has a unique fixed point in X.

Proof. The proof of corollary has immediately follows from above Theorem 2.1 by taking $S = T$. This completes the proof. □

From the above theorem, we obtain the following results as special cases.

Theorem 2.2. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T, S be two self-mappings of X satisfying the condition*

$$\|Sx - Ty, u\|_c \leq h_1\|x - y, u\|_c,$$

for all $x, y, u \in X$, $0 < h_1 < 1$. Then T and S have a unique common fixed point in X.

Theorem 2.3. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T, S be two self-mappings of X satisfying the condition*

$$\|Sx - Ty, u\|_c \leq h_2(\|x - Ty, u\|_c + \|y - Sx, u\|_c),$$

for all $x, y, u \in X$, $0 < h_2 < \frac{1}{2}$. Then T and S have a unique common fixed point in X.

We prove the following theorem using the condition (II).

Theorem 2.4. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and S, T be two continuous self-mappings of X satisfying the condition (II). Then S and T have a unique common fixed point in X.*

Proof. For a given $x_0 \in X$ and $n \geq 1$, take $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general we define a sequence of elements of X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n} = Tx_{2n-1}$ for $n = 0, 1, 2, 3, \cdots$. Now for all $u \in X$, by condition (II), we obtain

$$\begin{aligned}
\|x_{2n+1} - x_{2n}, u\|_c &= \|Sx_{2n} - Tx_{2n-1}, u\|_c \\
&\leq \varphi\left(\begin{array}{c} \|x_{2n} - x_{2n-1}, u\|_c, \frac{\|x_{2n-1} - Tx_{2n-1}, u\|_c + \|x_{2n} - Sx_{2n}, u\|_c}{2}, \\ \|x_{2n} - Tx_{2n-1}, u\|_c, \|x_{2n-1} - Sx_{2n}, u\|_c \end{array} \right) \\
&= \varphi\left(\begin{array}{c} \|x_{2n} - x_{2n-1}, u\|_c, \frac{\|x_{2n-1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n+1}, u\|_c}{2}, \\ \|x_{2n} - x_{2n}, u\|_c, \|x_{2n} - x_{2n+1}, u\|_c \end{array} \right) \\
&= \varphi\left(\begin{array}{c} \|x_{2n} - x_{2n-1}, u\|_c, \frac{\|x_{2n-1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n+1}, u\|_c}{2}, \\ 0, \|x_{2n} - x_{2n+1}, u\|_c \end{array} \right) \\
&\leq \varphi\left(\begin{array}{c} \|x_{2n} - x_{2n-1}, u\|_c, \frac{\|x_{2n-1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n+1}, u\|_c}{2}, \\ 0, \|x_{2n-1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n+1}, u\|_c \end{array} \right).
\end{aligned}$$

Hence from Definition 2.4 (b_1), we have

$$\|x_{2n+1} - x_{2n}, u\|_c \leq h\|x_{2n} - x_{2n-1}, u\|_c \text{ where } 0 < h < 1. \tag{2.4}$$

Similarly, we have

$$\|x_{2n} - x_{2n-1}, u\|_c \leq h\|x_{2n-1} - x_{2n-2}, u\|_c. \tag{2.5}$$

Hence from (2.4) and (2.5), we have

$$\|x_{2n+1} - x_{2n}, u\|_c \leq h^2\|x_{2n-1} - x_{2n-2}, u\|_c$$

on continuing this process, we get

$$\|x_{2n+1} - x_{2n}, u\|_c \leq h^{2n}\|x_1 - x_0, u\|_c.$$

For every $n > m$, we have

$$\begin{aligned}
\|x_n - x_m, u\|_c &\leq \|x_n - x_{n-1}, u\|_c + \|x_{n-1} - x_{n-2}, u\|_c + \cdots + \|x_{m+1} - x_m, u\|_c \\
&\leq (h^{n-1} + h^{n-2} + \cdots + h^m)\|x_1 - x_0, u\|_c \\
&\leq \left(\frac{h^m}{1-h}\right)\|x_1 - x_0, u\|_c.
\end{aligned}$$

Since $0 < h < 1$, by Definition 2.4, $\left(\frac{h^m}{1-h}\right) << 0$ as $m \to \infty$. Hence $\|x_n - x_m, u\|_c << 0$ as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists a point z in X such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of S and T that $Sz = Tz = z$. Thus z is a common fixed point of S and T.
Uniqueness Let w be another common fixed point of S and T, that is $Sw = Tw = w$. Then, we have

$$\begin{aligned}
\|z - w, u\|_c &= \|Sz - Tw, u\|_c \\
&\leq \varphi\left(\begin{array}{c} \|z - w, u\|_c, \frac{\|w - Tw, u\|_c + \|z - Sz, u\|_c}{2}, \\ \|z - Tw, u\|_c, \|w - Sz, u\|_c \end{array} \right) \\
&\leq \varphi(\|z - w, u\|_c, 0, \|z - w, u\|_c, \|z - w, u\|_c).
\end{aligned} \tag{2.6}$$

By Definition 2.4 (b_2) and the inequality (2.6), we get

$$\|z - w, u\|_c \leq 0.$$

Hence $z = w$ and for all $u \in X$. Thus z is a unique common fixed point of S and T. □

Corollary 2.2. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T be a self-mapping of X satisfying the condition*

$$\|Tx - Ty, u\|_c \leq \varphi\left(\|x-y,u\|_c, \frac{\|x-Tx,u\|_c + \|y-Ty,u\|_c}{2}, \|x-Ty,u\|_c, \|y-Tx,u\|_c\right),$$

for all $x, y, u \in X$. Then T has a unique fixed point in X.

Proof. The proof of corollary has immediately follows from above Theorem 2.4 by taking $S = T$. This completes the proof. □

From the above theorem we obtain the following result as a special case.

Theorem 2.5. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T, S be two self-mappings of X satisfying the condition*

$$\|Sx - Ty, u\|_c \leq h_3(\|x - Sx, u\|_c + \|y - Ty, u\|_c),$$

for all $x, y, u \in X$, $0 < h_3 < \frac{1}{2}$. Then T and S have a unique common fixed point in X.

From Theorem 2.1 and Theorem 2.4, we obtain the following results as special cases.

Theorem 2.6. *Let X be a 2-cone Banach space (with $\dim X \geq 2$) and T, S be two self-mappings of X. A mapping T and S are said to be 2-Zamfirescu type contraction satisfying the at least one of the following conditions is true:*

1. $\|Sx - Ty, u\|_c \leq h_1\|x - y, u\|_c$
2. $\|Sx - Ty, u\|_c \leq h_2(\|x - Ty, u\|_c + \|y - Sx, u\|_c)$
3. $\|Sx - Ty, u\|_c \leq h_3(\|x - Sx, u\|_c + \|y - Ty, u\|_c)$

for all $x, y, u \in X$, $h_1 \in (0, 1)$ and $h_2, h_3 \in \left(0, \frac{1}{2}\right)$. Then T and S have a unique common fixed point in X.

3. An application to the fixed-circle problem

In this section, we give an application to the fixed-circle problem which is a new geometric approach to fixed-point theory raised by Özgür and Taş [8]. More recently, some different solutions of the problem have been investigated with various techniques on metric spaces or some generalized metric spaces (see [6], [7], [9], [10], [11], [12], [13], [18], [19], [20] and [21] for more details). In this context, we obtain new fixed-circle theorems on 2-cone normed spaces. At first, we recall the notion of an open ball and define a circle on a 2-cone normed space.

Definition 3.1. [17] Let $\|.\|_c^\infty : X \to (E, P, \|.\|)$ and $r \in E$ with $r \gg \theta$. Then the set

$$B_{\{u_1,u_2,\ldots,u_d\}}(x_0, r) = \{x : \|x - x_0\|_c^\infty \ll r\}$$

is called an open ball centered at x_0 with radius r.

Definition 3.2. (1) Let $\|.\|_c^\infty : X \to (E, P, \|.\|)$ and $r \in E$ with $r \gg \theta$ or $r = \theta$. Then the set

$$C_{x_0,r}^2 = C_{\{u_1,u_2,\ldots,u_d\}}(x_0, r) = \{x : \|x - x_0\|_c^\infty = r\}$$

is called a circle centered at x_0 with radius r.

(2) Let $\|.\|_c^\infty : X \to (E, P, \|.\|)$ and $r \in E$ with $r \gg \theta$ or $r = \theta$. Then the set

$$B_{\{u_1,u_2,\ldots,u_d\}}[x_0, r] = B_{\{u_1,u_2,\ldots,u_d\}}(x_0, r) \cup C_{x_0,r}^2$$

is called a closed ball centered at x_0 with radius r.

(3) The circle $C_{x_0,r}^2$ (or the closed ball $B_{\{u_1,u_2,\ldots,u_d\}}[x_0, r]$) is called as the fixed circle (or fixed disc) of a self-mapping T if $Tx = x$ for all $x \in C_{x_0,r}^2$ (or $x \in B_{\{u_1,u_2,\ldots,u_d\}}[x_0, r]$), respectively.

We give the following fixed-circle (or fixed-disc) results:

Theorem 3.1. *Let X be a 2-cone normed space (with $\dim X \geq 2$), $T : X \to X$ be a self-mapping, $x_0 \in X$ and*

$$r = \inf_{x \in X} \{\|Tx - x, u\|_c : Tx \neq x\}. \tag{3.1}$$

If T satisfies the following conditions, then $C^2_{x_0,r}$ is a fixed circle of T :
(1) *If $Tx \neq x$ then*

$$\|Tx - x, u\|_c \leq \varphi \left(\begin{array}{c} \|x - x_0, u\|_c , \|Tx - x, u\|_c , \|x - Tx_0, u\|_c , \\ \frac{\|x - Tx_0, u\|_c + \|Tx - x, u\|_c}{2} \end{array} \right),$$

where $\varphi \in \Phi$.
(2) $Tx_0 = x_0$.

Proof. **Case 1:** Let $r = \theta$. Then we have

$$\begin{array}{rcl} \|x - x_0\|_c^\infty & = & \theta \Longrightarrow \max\{\|x - x_0, u_i\|_c : i = 1, 2, \ldots d\} = 0 \\ & \Longrightarrow & \|x - x_0, u_i\|_c = 0 \text{ for all } i = 1, 2, \ldots d \\ & \Longrightarrow & x = x_0 \\ & \Longrightarrow & C^2_{x_0,r} = \{x_0\}. \end{array}$$

Using the condition (2), we know $Tx_0 = x_0$ and so $C^2_{x_0,r}$ is a fixed circle of T.
Case 2: Let $r \gg \theta$ and $x \in C^2_{x_0,r}$ with $Tx \neq x$. By the definition of r, we have $r \leq \|Tx - x, u\|_c$. Using the conditions (1), (2) and the property of φ, we obtain

$$\begin{array}{rcl} \|Tx - x, u\|_c & \leq & \varphi \left(\begin{array}{c} \|x - x_0, u\|_c , \|Tx - x, u\|_c , \|x - Tx_0, u\|_c , \\ \frac{\|x - Tx_0, u\|_c + \|Tx - x, u\|_c}{2} \end{array} \right) \\ & \leq & \varphi\left(r, \|Tx - x, u\|_c , r, \dfrac{r + \|Tx - x, u\|_c}{2}\right). \end{array}$$

From Definition 2.4 (a_1), we have

$$\|Tx - x, u\|_c \leq hr, \, h \in (0, 1),$$

which is a contradiction with the definition of r. Therefore, it should be $Tx = x$. Consequently, $C^2_{x_0,r}$ is a fixed circle of T. □

Corollary 3.1. *Let X be a 2-cone normed space (with $\dim X \geq 2$), $T : X \to X$ be a self-mapping, $x_0 \in X$ and r be defined as in (3.1). If T satisfies the following conditions, then T fixes the closed ball $B_{\{u_1,u_2,\ldots,u_d\}}[x_0, \rho]$ with $\rho \leq r$ (or $B_{\{u_1,u_2,\ldots,u_d\}}[x_0, r]$ is the fixed disc of T) :*
(1) *If $Tx \neq x$ then*

$$\|Tx - x, u\|_c \leq \varphi \left(\begin{array}{c} \|x - x_0, u\|_c , \|Tx - x, u\|_c , \|x - Tx_0, u\|_c , \\ \frac{\|x - Tx_0, u\|_c + \|Tx - x, u\|_c}{2} \end{array} \right),$$

where $\varphi \in \Phi$.
(2) $Tx_0 = x_0$.

Proof. The proof can be easily seen by the similar arguments used in the proof of Theorem 3.1. □

Theorem 3.2. *Let X be a 2-cone normed space (with $\dim X \geq 2$), $T : X \to X$ be a self-mapping, $x_0 \in X$ and r be defined as in (3.1). If T satisfies the following conditions, then $C^2_{x_0,r}$ is a fixed circle of T :*
(1) *If $Tx \neq x$ then*

$$\|Tx - x, u\|_c \leq \varphi \left(\begin{array}{c} \|x - x_0, u\|_c , \frac{\|x - Tx_0, u\|_c + \|Tx - x, u\|_c}{2}, 0, \\ \|Tx - x, u\|_c + \|x - Tx_0, u\|_c \end{array} \right),$$

where $\varphi \in \Phi$.
(2) $Tx_0 = x_0$.

Proof. **Case 1:** Let $r=\theta$. Then we have $C^2_{x_0,r}=\{x_0\}$. Using the condition (2), we know $Tx_0=x_0$ and so $C^2_{x_0,r}$ is a fixed circle of T.

Case 2: Let $r\gg\theta$ and $x\in C^2_{x_0,r}$ with $Tx\neq x$. By the definition of r, we have $r\leq\|Tx-x,u\|_c$. Using the conditions (1), (2) and the property of φ, we obtain

$$\begin{aligned}\|Tx-x,u\|_c &\leq \varphi\begin{pmatrix}\|x-x_0,u\|_c, \frac{\|x-Tx_0,u\|_c+\|Tx-x,u\|_c}{2}, 0, \\ \|Tx-x,u\|_c+\|x-Tx_0,u\|_c\end{pmatrix} \\ &\leq \varphi\left(r, \frac{r+\|Tx-x,u\|_c}{2}, 0, \|Tx-x,u\|_c+r\right).\end{aligned}$$

From Definition 2.4 (b_1), we have

$$\|Tx-x,u\|_c\leq hr, h\in(0,1),$$

which is a contradiction with the definition of r. Therefore, it should be $Tx=x$. Consequently, $C^2_{x_0,r}$ is a fixed circle of T. □

Corollary 3.2. *Let X be a 2-cone normed space (with $\dim X\geq 2$), $T:X\to X$ be a self-mapping, $x_0\in X$ and r be defined as in (3.1). If T satisfies the following conditions, then T fixes the closed ball $B_{\{u_1,u_2,\ldots,u_d\}}[x_0,\rho]$ with $\rho\leq r$ (or $B_{\{u_1,u_2,\ldots,u_d\}}[x_0,r]$ is the fixed disc of T) :*

(1) If $Tx\neq x$ then

$$\|Tx-x,u\|_c\leq\varphi\begin{pmatrix}\|x-x_0,u\|_c, \frac{\|x-Tx_0,u\|_c+\|Tx-x,u\|_c}{2}, 0, \\ \|Tx-x,u\|_c+\|x-Tx_0,u\|_c\end{pmatrix},$$

where $\varphi\in\Phi$.

(2) $Tx_0=x_0$.

Proof. The proof can be easily seen by the similar arguments used in the proof of Theorem 3.2. □

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