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Riemannian manifolds are KKM spaces

Sehie Park[a,b]

[a] *The National Academy of Sciences, Republic of Korea, Seoul 06579, KOREA;*
[b] *Department of Mathematical Sciences, Seoul National University, Seoul 08826, KOREA*

Abstract

Let (M, g) be a complete, finite-dimensional Riemannian manifold. Based on the fact that any geodesic convex subset of M is a KKM space, we establish the KKM theory on such subsets originated from the Knaster-Kuratowski-Mazurkiewicz theorem in 1929.

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1. Introduction

Since 2006, we have established the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theory on abstract convex spaces; for example, see [18, 19]. In the present article, we are going to establish the basis of the KKM theory on Riemannian manifolds.

In the last few years, a number of researchers are focused on extending several important concepts and techniques of nonlinear analysis in Euclidean spaces to Riemannian manifolds in order to go further in the study of the convex theory, the fixed point theory, the variational inequality and related topics.

In fact, Kristály [11] studied existence and location of Nash equilibrium points for a large class of a finite family of payoff functions whose domains are not necessarily convex in the usual sense. His geometric idea was to embed these non-convex domains into suitable Riemannian manifolds regaining certain geodesic convexity properties of them.

By using recent non-smooth analysis on Riemannian manifolds and a variational inequality for acyclic sets, he gave an efficient location result of Nash equilibrium points. He assumed that the strategy sets K_i (which are not convex in the usual sense) can be embedded into suitable Riemannain manifolds (M_i, g_i) in a geodesic convex way.

Let (M, g) be a complete, finite-dimensional Riemannnian manifold. A *geodesic convex* set $K \subset M$ is a subset such that, for any two points of K there exists a unique geodesic in (M, g) joining them which belongs entirely to K.

The following result is given by Kristály ([11], Proposition 2.2):

Lemma 1.1. *Any geodesic convex set $K \subset M$ is contractible.*

In the present article, we show that we can establish the KKM theory on Riemannian manifolds based on this lemma.

This article is organized as follows: Section 2 is a preliminary on our abstract convex spaces, and to introduce one of the most general KKM type theorems in [20] for abstract convex spaces. In Section 3, we prove several forms of the KKM theorem in the setting of Riemannian manifolds. Finally in Section 4, we introduce the contents of articles related to some known KKM theoretic results in Hadamard manifolds or others which are consequences of the preceding results in the present paper.

2. Preliminary

We follow our previous works and the references therein.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D, such that, for any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map with respect to F*. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E of E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KD}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KD}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$, that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KD}(E, E)$, that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *(partial) KKM space* if it satisfies the (partial) KKM principle, resp.

For typical examples of KKM spaces, see [19] and the references therein. We need the following:

Definition. A triple $(X \supset D; \Gamma)$ is called an H-*space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a *c*-space by Horvath; see [7].

Now we have the following well-known diagram for triples $(E, D; \Gamma)$:

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Convex space} \Longrightarrow \text{H-space}$$
$$\Longrightarrow \text{G-convex space} \iff \phi_A\text{-space} \Longrightarrow \text{KKM space}$$
$$\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}$$

Now we prepare to introduce one of the most general forms of the KKM theorem.
Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Note that Luc et al. showed that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and not conversely in each step.

The following is one of the most general KKM type theorems in [20] for abstract convex spaces:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, Z)$, and $G : D \multimap Z$ a map such that*
    (1) *$\overline{G}$ is a KKM map w.r.t. F; and*
    (2) *there exists a nonempty compact subset K of Z such that either*
      (i) *$K = Z$;*
      (ii) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D \rangle$; or*
      (iii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*
$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have
$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,
(α) if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and
(β) if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

3. Main results

In this section, we prove several forms of the KKM theorem in the setting of Riemannian manifolds.

Definition. The *geodesic convex hull* co A of a subset A of a Riemannian manifold M is the intersection of all geodesic convex subsets of M which contains A. Let $\Gamma_A := \text{co } A$ for any $A \subset M$.

Lemma 3.1. *Any geodesic convex subset X of a Riemannian manifold (M, g) with a nonempty set $D \subset X$ can be made into an H-space $(X, D; \Gamma)$ and hence a KKM space.*

PROOF. For any $A \in \langle D \rangle$, let $\Gamma_A = \Gamma(A) = \text{co } A$. Then each Γ_A is contractible by Lemma 1.1. Moreover, $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. Therefore $(X, D; \Gamma)$ is an H-space, and hence a KKM space by our KKM theory. $\square$

Otherwise, by putting $E = Z$ and $F = 1_E$ in Theorem C, we immediately have the following form of the KKM theorem in the setting of Riemannian manifolds:

Theorem 3.2. *Let $(X, D; \Gamma)$ be as in Lemma 3.1, and $G : D \multimap X$ a map such that*
    (1) *G is a closed-valued KKM map; and*
    (2) *there exists a nonempty compact subset K of X such that either*
      (i) *$K = X$;*
      (ii) *$\bigcap\{G(y) \mid y \in N\} \subset K$ for some $N \in \langle D \rangle$; or*
      (iii) *for each $N \in \langle D \rangle$, there exists a closed compact geodesic convex subset L_N of X containing some*
$D' \subset D$ such that $N \subset D'$, and

$$L_N \cap \bigcap_{y \in D'} G(y) \subset K.$$

Then we have

$$K \cap \bigcap_{y \in D} G(y) \neq \emptyset.$$

The following is a simple observation:

Theorem 3.3. *Let M be a Riemannian manifold and $K \subset M$ a geodesic convex subset. Let $G : K \multimap K$ be a KKM map such that, for each $x \in K$, $G(x)$ is closed. Then $\{G(x) \mid x \in K\}$ has the finite intersection property.*
    Moreover, if there exists $x_0 \in K$ such that $G(x_0)$ is compact, then

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

PROOF. By Lemma 3.1, a geodesic convex subset $K \subset M$ is a KKM space. Hence, by the definition itself, the conclusion follows. □

For open-valued KKM map, we have the following:

Theorem 3.5. *Let M be a Riemannian manifold and $K \subset M$ a geodesic convex subset. Let $G : K \multimap K$ be a KKM map such that, for each $x \in K$, $G(x)$ is open. Then $\{G(x) \mid x \in K\}$ has the finite intersection property.*

PROOF. By Lemma 1.1, K is a KKM space. Hence, by the definition itself, the conclusion follows. □

Recall that, in [18], we derived generalized forms of the Ky Fan minimax inequality, the von Neumann-Sion minimax theorem, the von Neumann–Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for partial KKM spaces. Consequently, our results in [18] unify and generalize most of previously known particular cases of the same nature.

Moreover, in [19], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, the Fan type minimax inequalities, and several variational inequality results for any KKM spaces. Consequently, [19] unifies and enlarges previously known many proper examples of such statements for particular types of partial KKM spaces.

Recall that some corrections on [19] were made in [22].

For any geodesic convex subset X of a Riemannian manifold (M, g) with a nonempty set $D \subset X$, the KKM space $(X, D; \Gamma)$ satisfies all results in [18], [19].

Recall that Hadamard manifolds are all finite-dimensional connected, simply connected, complete Riemannian manifolds of nonpositive curvature, see Reich and Shafrir [25]. Therefore, all results for Riemannian manifolds in this article can be applied to Hadamard manifolds.

4. Applications to known related articles

In this section, we introduce the contents of articles related to some known KKM theoretic results in Hadamard manifolds or others which are consequences of the preceding results in the present paper:

Walter in 1974 [28]

FROM TEXT: For a Riemannian manifold M, a subset $C \subset M$ is called *weakly convex* if, for $p, q \in C$, there is a minimal geodesic $c : [a, b] \to M$ from p to q lying in C. If, in addition, such a minimal geodesic is unique within C then C is called *convex*. The set C is *strongly convex* if, for $p, q \in C$, there is just one minimal geodesic from p to q and if it is in C.

COMMENTS. Terminology for a Riemannian manifold is different according to authors.

Németh in 2003 [17]

ABSTRACT. The notion of variational inequalities is extended to Hadamard manifolds and related to geodesic convex optimization problems. Existence and uniqueness theorems for variational inequalities on Hadamard manifolds are proved. A convexity property of the solution set of a variational inequality on a Hadamard manifold is presented.

COMMENTS. The basic lemma is based on the Brouwer fixed point theorem. At the end of Introduction, Németh stated: "The extension of our results to an arbitrary Riemannian manifold would be desirable. . . . However, we consider our results as an important first step in developing the theory of variational inequalities on Riemannian manifolds."

Now the theory of variational inequalities on Riemannian manifolds can be established as in [19].

Kirk and Panyanak in 2008 [10]

ABSTRACT. A CAT(0) space is a geodesic space for which each geodesic triangle is at least as thin as its comparison triangle in the Euclidean plane. A notion of convergence introduced independently several years ago by Lim and Kuczumow is shown in CAT(0) spaces to be very similar to the usual weak convergence in Banach spaces. In particular many Banach space results involving weak convergence have precise analogues in this setting. At the same time, many questions remain open.

FROM PRELIMINARY REMARKS: For a metric space (X, d), the following concepts are defined: a geodesic (path), a geodesic (or metric) segment, a geodesic space, a uniquely geodesic space, a convex subset, a geodesic triangle, a comparison triangle, and a CAT(0) space.

COMMENTS. Our KKM theoretic results in Section 3 can be applied to CAT(0) spaces. For CAT(0) spaces, see also [8] and the references therein.

Li, Li, Liou, and Yao in 2009 [16]

FROM MR2560235 (2010J:49013). The authors study the existence and uniqueness of solutions to variational inequality problems on Riemannian manifolds, and extend some results obtained by S.Z. Németh [17] on Hadamard manifolds to more general Riemannian manifolds.— Daniel Azagra

COMMENTS. Several variational inequality results in [19] can be applied to Riemannian manifolds.

Zhou and Huang in 2009 [31]

ABSTRACT. In this paper, a new notion of KKM mapping is introduced and a generalized KKM theorem is proved on Hadamard manifolds. As applications, an existence theorem of solution for a generalized mixed variational inequality and a fixed point theorem for a set-valued mapping are obtained on Hadamard manifolds.

COMMENTS. Theorem 3.1 in this paper shows that a geodesic convex subset of an Hadamard manifold is a partial KKM space, and hence, satisfies all results in our [18], [19]. This is applied to a generalized mixed variational inequality and a fixed point theorem of the Fan-Browder type. Note that the results in [31] can be extended to Riemannian manifolds.

Kristály in 2010 [11]

ABSTRACT. Existence and location of Nash equilibrium points are studied for a large class of a finite family of payoff functions whose domains are not necessarily convex in the usual sense. The geometric idea is to embed these non-convex domains into suitable Riemannian manifolds regaining certain geodesic convexity properties of them. By using recent non-smooth analysis on Riemannian manifolds and a variational inequality for acyclic sets, an efficient location result of Nash equilibrium points is given. Some examples show the applicability of our results.

COMMENTS. We borrowed Lemma 1.1 from Kristály ([11], Proposition 2.2), which enables us to show that Riemannian manifolds are KKM spaces.

Colao, Lopez, Marino, and Martin-Marquez in 2012 [5]

ABSTRACT. An equilibrium theory is developed in Hadamard manifolds. The existence of equilibrium points for a bifunction is proved under suitable conditions, and applications to variational inequality, fixed point and Nash equilibrium problems are provided. The convergence of Picard iteration for firmly nonexpansive mappings along with the definition of resolvents for bifunctions in this setting is used to devise an algorithm to approximate equilibrium points.

COMMENTS. Lemma 3.1 is a particular form of our KKM theorem in [19] for a closed convex subset of a Hadamard manifold, and can be extended to a Riemannian manifold. This is applied to equilibrium problems and further applications. Some more comments were given in [21].

Tang and Huang in 2012 [26]

ABSTRACT. The concept of pseudomonotone vector field on Hadamard manifold is introduced. A variant of Korpelevich's method for solving the variational inequality problem is extended from Euclidean spaces to constant curvature Hadamard manifolds. Under a pseudomonotone assumption on the underlying vector field, we prove that the sequence generated by the method converges to a solution of variational inequality, whenever it exists. Moreover, we give an example to show the effectiveness of our method.

COMMENTS. The variational inequality problem on Hadamard manifolds can be extended to the one on Riemannian manifolds.

Yang and Pu in 2012 [29]

ABSTRACT. In this paper, a generalized Browder-type fixed point theorem on Hadamard manifolds is introduced, which can be regarded as a generalization of the Browder-type fixed point theorem for the set-valued mapping on an Euclidean space to a Hadamard manifold. As applications, a maximal element theorem, a section theorem, a Ky Fan-type Minimax Inequality and an existence theorem of Nash equilibrium for non-cooperative games on Hadamard manifolds are established.

COMMENTS. In this paper it is proved that a Fan-Browder type fixed point theorem with strongly geodesic convexity on Hadamard manifolds. It is clear that such results are closely related to the KKM theory on Riemannian manifolds, and consequences of our [19].

Yang and Pu in 2012 [30]

ABSTRACT. In this paper, maximal element theorem on Hadamard manifolds is established. First, we prove the existence of solutions for maximal element theorem on Hadamard manifolds. Further, we prove

that most of problems in maximal element theorem on Hadamard manifolds (in the sense of Baire category) are essential and that, for any problem in maximal element theorem on Hadamard manifolds, there exists at least one essential component of its solution set. As applications, we study existence and stability of solutions for variational relation problems on Hadamard manifolds, and existence and stability of weakly Pareto-Nash equilibrium points for n-person multi-objective games on Hadamard manifolds.

COMMENTS. The maximal element theorem is equivalent to many types of results in KKM spaces; see [19]. Therefore the results in this paper might work for Riemannian manifolds.

Park in 2013 [21]

ABSTRACT. In 2012, Colao, Lopez, Marino, and Martin-Marquez [4] developed an equilibrium theory in Hadamard manifolds. In [21], we show that three of their key results (the KKM lemma, the Ky Fan type minimax inequality, and Nash equilibrium theorem) on Hadamard manifolds can be extended to hyperbolic spaces and are particular ones for abstract convex spaces in the sense of ours. Similarly, most of main theorems in the KKM theory on abstract convex spaces can be applied to hyperbolic spaces and Hadamard manifolds.

COMMENTS. The three main results are consequences of the KKM theory in [19] and can be generalized to Riemannian manifolds.

Tang, Zhou, and Huang in 2013 [27]

ABSTRACT. In this paper, we investigate the proximal point algorithm (in short PPA) for variational inequalities with pseudomonotone vector fields on Hadamard manifolds. Under weaker assumptions than monotonicity, we show that the sequence generated by PPA is well defined and prove that the sequence converges to a solution of variational inequality, whenever it exists. The results presented in this paper generalize and improve some corresponding known results given in literatures.

Zhou and Huang in 2013 [32]

ABSTRACT. In this paper, a relationship between a vector variational inequality and a vector optimization problem is given on a Hadamard manifold. An existence of a weak minimum for a constrained vector optimization problem is established by an analogous to KKM lemma on a Hadamard manifold.

COMMENTS. Based on a KKM lemma on Hadamard manifolds, the authors proved the existence of solutions for the vector variational inequalities on Hadamard manifolds. The results presented in [32] generalize some previous ones from Euclidean spaces to Hadamard manifolds, but are consequences of [19]. This paper can be extended to Riemannian manifolds.

Alghamdi, Kirk, and Shahzad in 2014 [1]

ABSTRACT. The main focus of this paper is on fixed point theory for mapping satisfying local contractive conditions in Banach spaces and in various geodesic spaces. The emphasis is on locally nonexpansive mappings and local contractions.

FROM INTRODUCTION. In its broadest sense, metric fixed point theory has been couched in the setting of a complete metric space. When specific structure is needed the setting has usually been in the frame work of a Banach space. However there are many intermediate classes of spaces for which there are rich sets of examples and applications. These include geodesic spaces, including the hyperbolic spaces, Busemann convex spaces, and CAT(0), as well as the more general length spaces.

COMMENTS. Some of the spaces listed above are (partial) KKM spaces which have rich KKM theoretical results as shown in our [19]. It would be interesting to find more new (partial) KKM spaces.

Ariza-Ruiz, Li, and Lopez-Acedo in 2014 [2]

ABSTRACT. We give a direct proof of Schauder's fixed point theorem in the setting of geodesic metric spaces, generalizing the classical Schauder's theorem and improving a recent version of this theorem in CAT(κ) spaces. As an application we prove an existence result for a variational inequality in the setting of CAT(κ) spaces.

COMMENTS. The classical KKM theorem in 1929 is applied to obtain the main theorem of this paper.

Chen, Huang, and O'Regan in 2014 [4]

ABSTRACT. We introduce a class of functions called geodesic $\mathcal{B}$-preinvex and geodesic $\mathcal{B}$-invex functions on Riemannian manifolds and generalize the notions to the so-called geodesic quasi/pseudo $\mathcal{B}$-preinvex and geodesic quasi/pseudo $\mathcal{B}$-invex functions. We discuss the links among these functions under appropriate conditions and obtain results concerning extremum points of a nonsmooth geodesic $\mathcal{B}$-preinvex function by using the proximal subdifferential. Moreover, we study a differentiable multiobjective optimization problem involving new classes of generalized geodesic $\mathcal{B}$-invex functions and derive Kuhn-Tucker-type sufficient conditions for a feasible point to be an efficient or properly efficient solution. Finally, a Mond-Weir type duality is formulated and some duality results are given for the pair of primal and dual programming.

Kristály in 2014 [12]

ABSTRACT. Motivated by Nash equilibrium problems on 'curved' strategy sets, the concept of Nash-Stampacchia equilibrium points is introduced via variational inequalities on Riemannian manifolds. Characterizations, existence, and stability of Nash-Stampacchia equilibria are studied when the strategy sets are compact/noncompact geodesic convex subsets of Hadamard manifolds, exploiting two well-known geometrical features of these spaces both involving the metric projection map.These properties actually characterize the non-positivity of the sectional curvature of complete and simply connected Riemannian spaces, delimiting the Hadamard manifolds as the optimal geometrical frame work of Nash-Stampacchia equilibrium problems. Our analytical approach exploits various elements from set-valued and variational analysis, dynamical systems, and non-smooth calculus on Riemannian manifolds. Examples are presented on the Poincaré upper-plane model and on the open convex cone of symmetric positive definite matrices endowed with the trace-type Killing form.

Chaipunya and Kumam in 2015 [3]

ABSTRACT. In this paper, we consider the KKM maps defined for a nonself map and the correlated intersection theorems in Hadamard manifolds. We also study some applications of the intersection results. Our outputs improved the results of Raj and Somasundaram [V. Sankar Raj and S. Somasundaram, KKM-type theorems for best proximity points, Appl. Math. Lett., 25(3):496–499, 2012.].

COMMENTS. Particular types of generalized KKM theorems for Hadamard manifolds are applied, and can be extended to Riemannian manifolds.

Lee in 2015 [15]

ABSTRACT. In 2012, Yang and Pu [29] proved an Browder type fixed point theorem with strongly geodesic convexity on Hadamard manifolds. In this paper, we show that their main result can be obtained from the one in the more general space. As applications, we claim that their maximal element theorems, section theorems, Ky Fan type minimax inequality, and equilibrium theorem about non-cooperative games on Hadamard manifold are already obtained in some sense.

COMMENTS. Most of results in [15] works for Riemannian manifolds.

Kristály, Li, Lopez-Acedo, and Nicolae in 2016 [13]

ABSTRACT. Various results based on some convexity assumptions (involving the exponential map along with affine maps, geodesics and convex hulls) have been recently established on Hadamard manifolds. In this

paper, we prove that these conditions are mutually equivalent and they hold, if and only if the Hadamard manifold is isometric to the Euclidean space. In this way, we show that some results in the literature obtained on Hadamard manifolds are actually nothing but their well-known Euclidean counterparts.

Rahimi, Farajzadeh, and Vaezpour in 2017 [24]

ABSTRACT. In this paper, an extension of the Fan-KKM lemma to Hadamard manifolds is established. By using it, some existence results for equilibrium problems on Hadamard manifolds are provided. Finally, as an application of the main results, an existence result of a solution of the mixed variational inequality problem in the setting of Hadamard manifolds is stated.

COMMENTS. This paper is a consequences of our [19]. The results in this paper can be extended to Riemannian manifolds.

Cruz Neto, Jacinto, Soares Jr., and Souza in 2018 [6]

ABSTRACT. We study some conditions for a monotone bifunction to be maximally monotone by using a corresponding vector field associated to the bifunction and vice versa. This approach allows us to establish existence of solutions to equilibrium problems in Hadamard manifolds obtained by perturbing the equilibrium bifunction.

Kim in 2018 [9]

ABSTRACT. In this paper, we provide two basic Fan-Browder type fixed point theorems for multimaps on geodesic convex sets in Hadamard manifolds. Also, an existence theorem of Nash equilibrium for an 1-person game in Hadamard manifolds is established

COMMENTS. Some detailed comments on and improvements of [9] are given in [23]. Now the results can be extended to Riemannian manifolds.

Kumam and Chaipunya in 2018 [14]

ABSTRACT. In this paper, we consider the equilibrium problems and also their regularized problems under the setting of Hadamard spaces. The solution to the regularized problem is represented in terms of resolvent operators. As an essential machinery in the existence of an equilibrium, we first prove that the KKM principle is attained in general Hadamard spaces without assuming the compactness of the closed convex hull of a finite set. We construct the proximal algorithm based on this regularization and give convergence analysis adequately.

COMMENTS. A uniquely geodesic metric space (X, ρ) is a CAT(0) space if each geodesic triangle in X is at least as thin as its comparison triangle in Euclidean plane. A complete CAT(0) space is then called *Hadamard space*. The authors defined the convexity and the *convex hull* co(N) of a subset $N \subset X$.

They showed that, for a closed convex set $K \subset X$, $(K; \text{co})$ is a partial KKM space and hence satisfies results in [19].

Consequently, we can establish the KKM theory of Hadamard spaces.

Park in 2018 [23]

ABSTRACT. In some previous works, it is known that the KKM type results on Hadamard manifolds can be extended to hyperbolic spaces. Such results are the KKM theorem, the Fan-Browder fixed point theorem, Nash equilibrium theorem, variational inequalities, etc. based on our theory of abstract convex spaces. In the present article, we show that our method can be applied some recent works on Hadamard manifolds. Historical remarks are added on the study of the KKM type results on Hadamard manifolds and hyperbolic spaces.

COMMENTS. Instead of Hadamard manifolds, Riemannian manifolds would work in [23].

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