An Approximate Error Expression for RQAM Scheme under α-η-μ Fading Conditions

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Abstract— **In this paper, average symbol error rate (ASER) performance of rectangular quadrature amplitude modulation (RQAM) scheme is analyzed over α-η-μ fading channels. First, an ASER expression is derived based on Chernoff approximation of Gaussian *Q*-function. Then, an asymptotic ASER formula is obtained for analyzing system behavior at high signal-to-noise ratio (SNR) regime. The ASER performance is presented for different modulation levels and fading parameter values. In addition, relative truncation error (RTE) is illustrated in order to determine how many terms are needed for the computation of proposed expression. It is shown that analytical results are in close agreement with exact results.**

Index Terms— **Error analysis, α-η-μ distribution, RQAM, Gaussian *Q*-function.**

I. Introduction

QUADRATURE amplitude modulation (QAM) which is known as efficient modulation method for bandwidth has an important role in digital multimedia transmission since it achieves high data rates. Rectangular QAM (RQAM), cross QAM (XQAM) and square QAM (SQAM) are popular QAM methods that are used in high speed communications. RQAM is considered as a generic modulation type and it has practical applications in the field of high speed mobile communications and microwave communications [1].

A number of studies which focus on the performance for RQAM schemes under different fading conditions, have been presented in the literature [2-7]. In [2], a lower bound ASER expression was derived for cooperative diversity systems with RQAM technique over Rayleigh fading channels. In addition, symbol error probability (SEP) and average symbol error rate (ASER) expressions were proposed for RQAM modulated systems under Nakagami-*m* fading conditions [3-6]. In [3], the SEP of RQAM modulation was presented over Nakagami-*m* fading channels in terms of the product of two Gaussian *Q*-functions. The authors in [4] studied the performance of *L*-branch communication system with RQAM scheme in the presence of Nakagami-*m* fading and they proposed an expression for the SEP of the considered system. In [5] and [6], RQAM technique was applied to multiple relay networks and two-way relaying systems operating under Nakagami-*m* fading conditions. Asghari *et al.* analyzed the SEP of RQAM scheme with maximum ratio combining over η-μ fading channels [7]. In [8], the ASER of RQAM and XQAM modulations were investigated based on moment generating function over two-wave with diffuse power fading channels. Lower bound ASER expressions of RQAM and XQAM for AF relaying systems were presented in Rayleigh fading with maximum ratio combining in [9]. In another work [10], the authors derived ASER formulas for hexagonal and rectangular QAM based on cumulative distribution function over Nakagami-*m* fading channels.

In wireless communications, it is important to take the composite fading channels into consideration such as in [11-15] for performance analysis since these fading models are generalized distributions which provide flexibility for reducing other well-known fading channels. However, as far as we know, error performance of RQAM modulated wireless communication systems over α-η-μ fading channels does not exist in literature. α-η-μ fading can be employed in order to reflect small variations in the signal strength and it has special cases including popular fading distributions such as Rayleigh, Nakagami-*m*, Weibull, η-μ, α-μ. Motivated by this, for the first time in the literature, we analyze ASER performance of RQAM scheme over α-η-μ fading channels. Here, we derive a novel ASER expression based on Chernoff approximation of Gaussian *Q*-function. Then, we also obtain an asymptotic ASER expression in order to evaluate the system behavior at high signal-to-noise ratio (SNR) region.

II. System And Channel Models

We consider a single-input single-output wireless communication system that sends a signal x which is modulated according to RQAM scheme. The received signal, y is defined as

$$y = xG + N_0 \tag{1}$$

where G is the fading coefficient of the channel and N_0 is spectral density of noise power. The probability density function (PDF) of intantaneous SNR, γ, for $\alpha-\eta-\mu$ distribution is expressed by

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$$f_\gamma(\gamma) = A\exp\left(-\frac{2\mu h\gamma^{\left(\frac{\alpha}{2}\right)}}{\bar{\gamma}^{\left(\frac{\alpha}{2}\right)}}\right)\gamma^{\left(\frac{\alpha(\mu+0.5)}{2}-1\right)} I_{\mu-\frac{1}{2}}\left(\frac{2\mu H\gamma^{\left(\frac{\alpha}{2}\right)}}{\bar{\gamma}^{\left(\frac{\alpha}{2}\right)}}\right) \quad (2)$$

$$A = \left(\sqrt{\pi}\alpha\mu^{\mu+0.5}h^{\mu}\right)/\left(\Gamma(\mu)H^{\mu-0.5}\bar{\gamma}^{\lambda}\right) \quad (3)$$

where α, η, μ are fading parameters, $\lambda = \alpha(\mu+0.5)/2$, $\Gamma(\cdot)$ is Gamma function, $I_v(\cdot)$ is the modified Bessel function of the first kind, $\bar{\gamma}$ is the average SNR defined by $\bar{\gamma} = E(\gamma)$ and $E(\cdot)$ denotes expectation. h and H parameters are defined in two different formats, respectively, as

$$h = \frac{(1+\eta)^2}{4\eta},\ H = \frac{1-\eta^2}{4\eta},\quad 0<\eta<\infty,\ \mathit{Format}\,1$$
$$h = \frac{1}{1-\eta^2},\ H = \frac{\eta}{1-\eta^2},\quad -1<\eta<1,\ \mathit{Format}\,2 \quad (4)$$

III. Average Symbol Error Rate Analysis

Mathematicaly, the ASER for any kind of modulation method is evaluated by integrating the conditional symbol error rate (SER) of additive White Gaussian noise (AWGN) channels over the PDF of instantaneous SNR as follows

$$P_s(e) = \int_0^\infty P_s(e|\gamma) f_\gamma(\gamma)d\gamma \quad (5)$$

where $P_s(e|\gamma)$ is the conditional SER expression of AWGN channels and $f_\gamma(\gamma)$ is the PDF of the instantaneous SNR. General order RQAM constellations can be obtained by combining two pulse amplitude modulation (PAM) signals as $M_I - PAM$ (in-phase) and $M_Q - PAM$ (quadrature). For M-ary RQAM, the conditional SER in AWGN channels is expressed as

$$P_s(e|\gamma) = 2\left(pQ(a\sqrt{\gamma}) + qQ(b\sqrt{\gamma}) - 2pqQ(a\sqrt{\gamma})Q(b\sqrt{\gamma})\right) \quad (6)$$

where $M = M_I \times M_Q$, $p = 1-(1/M_I)$, $a = \sqrt{6/\left((M_I^2-1)+(M_Q^2-1)\beta^2\right)}$, $q = 1-(1/M_Q)$, $b = \beta a$, $\beta = d_Q/d_I$ is the decision distance ratio of quadrature-to-in-phase components (d_I is the in-phase decision distance and d_Q is the quadrature decision distance) and $Q(\cdot)$ is Gaussian Q-function. Inserting (6) into (5), the ASER can be rewritten as

$$P_s(e) = \underbrace{\int_0^\infty 2pQ(a\sqrt{\gamma}) f_\gamma(\gamma)d\gamma}_{I_1} + \underbrace{\int_0^\infty 2qQ(b\sqrt{\gamma}) f_\gamma(\gamma)d\gamma}_{I_2} - \underbrace{\int_0^\infty 4pqQ(a\sqrt{\gamma})Q(b\sqrt{\gamma}) f_\gamma(\gamma)d\gamma}_{I_3} \quad (7)$$

The integral in (7) is in an intractable format because of including Gaussian Q-function. Therefore, we utilize an upper bound approximation of the Gaussian Q-function which is defined by

$$Q(x) \approx \frac{1}{2}\exp\left(-\frac{x^2}{2}\right) \quad (8)$$

The approximate form given in (8) is known as Chernoff approximation [16]. This approximation facilitates the integration and ASER analysis over fading channels. First, we start by solving I_1 for ASER analysis. By inserting the Chernoff approximation and (2) into the first integral in (7), I_1 can be reexpressed as follows

$$I_1 = pA\int_0^\infty \exp\left(-\frac{a^2\gamma}{2}\right)\exp\left(-\frac{2\mu h\gamma^{(\alpha/2)}}{\bar{\gamma}^{(\alpha/2)}}\right) \times \gamma^{\lambda-1} I_{\mu-0.5}\left(\frac{2\mu H\gamma^{(\alpha/2)}}{\bar{\gamma}^{(\alpha/2)}}\right)d\gamma \quad (9)$$

Now using the infinite series representations of exponential function [17, (1.211.1)] and $I_v(\cdot)$ [17, (8.445)] in (9), we have

$$I_1 = Ap\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{2\mu h}{\bar{\gamma}^{(\alpha/2)}}\right)^k \sum_{m=0}^{\infty}\frac{(\mu H)^u}{m!\Gamma(\mu+m+0.5)} \times \frac{1}{\left(\bar{\gamma}^{(\alpha/2)}\right)^u}\int_0^\infty \exp\left(-\frac{a^2\gamma}{2}\right)\gamma^{v-1}d\gamma \quad (10)$$

where $v = \lambda + (\alpha k/2) + (\alpha u/2)$ and $u = \mu + 2m - 0.5$. After some algebra and by using [17, (3.381.4)], I_1 is derived as

$$I_1 = Ap\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{2\mu h}{\bar{\gamma}^{(\alpha/2)}}\right)^k \sum_{m=0}^{\infty}\frac{(\mu H)^u \Gamma(v)\left(a^2/2\right)^{-v}}{m!\Gamma(\mu+m+0.5)\left(\bar{\gamma}^{(\alpha/2)}\right)^u} \quad (11)$$

Then, I_2 and I_3 are obtained by using the same analytical steps as used for I_1, respectively, as

$$I_2 = Aq\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{2\mu h}{\bar{\gamma}^{(\alpha/2)}}\right)^k \sum_{m=0}^{\infty}\frac{(\mu H)^u}{\Gamma(\mu+m+0.5)} \times\frac{\Gamma(v)}{m!\left(\bar{\gamma}^{(\alpha/2)}\right)^u\left(b^2/2\right)^v} \quad (12)$$

$$I_3 = Apq\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{2\mu h}{\bar{\gamma}^{(\alpha/2)}}\right)^k \sum_{m=0}^{\infty}\frac{(\mu H)^u}{\Gamma(\mu+m+0.5)} \times\frac{\Gamma(v)}{m!\left(\bar{\gamma}^{(\alpha/2)}\right)^u\left(\left(a^2+b^2\right)/2\right)^v} \quad (13)$$

Finally, the ASER expression is found as

$$P_s(e) = A\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{2\mu h}{\bar{\gamma}^{(\alpha/2)}}\right)^k \sum_{m=0}^{\infty}\frac{(\mu H)^u\Gamma(v)}{m!\Gamma(\mu+m+0.5)\left(\bar{\gamma}^{(\alpha/2)}\right)^u} \times\left\{\frac{p}{\left(a^2/2\right)^v}+\frac{q}{\left(b^2/2\right)^v}-\frac{pq}{\left(\left(a^2+b^2\right)/2\right)^v}\right\} \quad (14)$$

Concerning about the truncation error which arises from the infinite series involved in (14), we evaluated the ASER expression in (14) for several values of the upper limits of infinite summations. Table I tabulates the ASER values and relative truncation error (RTE) for 4X2 QAM at $\bar{\gamma}=20$ dB when K and M terms are used.

TABLE I
ASER AND RTE VALUES FOR 4X2 QAM WITH $\alpha=1.5$, $\eta=0.5$, $\mu=1.5$ AND $\beta=1$

M	*K*	ASER	RTE
24	2	**0.016957**241605913	2.8012×10^{-4}
24	3	**0.0169572**95835405	3.1980×10^{-6}
24	4	**0.016957296**375839	3.1870×10^{-8}
24	5	**0.0169572963807**01	2.8671×10^{-10}
24	6	**0.01695729638074**1	2.3755×10^{-12}
24	7	**0.016957296380742**	1.8369×10^{-14}
24	8	**0.016957296380742**	1.3382×10^{-16}

RTE can be evaluated by following the same procedure in [18]. In Table I, one can see that the decimal places remained same even if the upper limits were increased. Hereby, Table I shows that the number of enough terms in order to compute the derived ASER expression in (14) with a negligible truncation error are $M=24$ and $K=8$. In Table II, we illustrate the ASER and RTE values for 8X4 QAM scheme at $\bar{\gamma}=40$ dB. Again, it can be easily seen that fewer terms are enough to evaluate the derived ASER expression at high SNR.

From Table II, only *K*=4 terms are needed for the evaluation of infinite series when *M*=8 where the RTE value decreases to 7.5473×10^{-16} . For all practical cases, (14) can be computed with a negligible truncation error without compromising numerical precision. Table I and Table II state that the derived expression is in rapidly convergent form.

TABLE II
ASER AND RTE VALUES FOR 8X4 QAM WITH $\alpha=1.5$, $\eta=0.5$, $\mu=1.5$ AND $\beta=1$

M	*K*	ASER	RTE
8	2	**3.68877767066**7582 $\times10^{-5}$	4.5747×10^{-8}
8	3	**3.68877767069095**5 $\times10^{-5}$	6.3366×10^{-12}
8	4	**3.688777670690958** $\times10^{-5}$	7.5473×10^{-16}
8	5	**3.688777670690958** $\times10^{-5}$	8.0208×10^{-20}

IV. Asymptotic Analysis

To analyze system behavior at the high SNR regime $(\bar{\gamma}\to\infty)$, we derive an asymptotic ASER expression for the considered system. Firstly, we expand the infinite series of exponential function and $I_v(\cdot)$ at zero point as

$$\exp\left(-2\mu h\gamma^{(\alpha/2)}/\bar{\gamma}^{(\alpha/2)}\right)\approx 1 \quad (15)$$

$$I_{\mu-\frac{1}{2}}\left(\frac{2H\mu\gamma^{(\alpha/2)}}{\bar{\gamma}^{(\alpha/2)}}\right)\approx\frac{\left(H\mu\gamma^{(\alpha/2)}/\bar{\gamma}^{(\alpha/2)}\right)^{\mu-0.5}}{\Gamma(\mu+0.5)} \quad (16)$$

Thus, the asymptotic PDF expression becomes

$$f_\gamma^a(\gamma)=\frac{A\gamma^{\left(\frac{\alpha(\mu+0.5)}{2}-1\right)\left(\frac{\alpha}{2}(\mu-0.5)\right)}}{\Gamma(\mu+0.5)}\left(\mu H\bar{\gamma}^{(-\alpha/2)}\right)^{\mu-0.5} \quad (17)$$

Then, substituting (17) and Chernoff approximation of Gaussian *Q*-function into (7), we have

$$P_s^a(e)=\frac{A}{\Gamma(\mu+0.5)}\left(\frac{\mu H}{\bar{\gamma}^{(\alpha/2)}}\right)^{\mu-0.5}\left\{p\int_0^\infty\exp\left(-\frac{\gamma a^2}{2}\right)\gamma^{\alpha\mu-1}d\gamma +q\int_0^\infty\exp\left(-\frac{\gamma b^2}{2}\right)\gamma^{\alpha\mu-1}d\gamma-pq\int_0^\infty\exp\left(-\frac{\gamma\left(a^2+b^2\right)}{2}\right)\gamma^{\alpha\mu-1}d\gamma\right\} \quad (18)$$

By using [17, (3.381.4)], the asymptotic ASER expression results in

$$P_s^a(e) = \frac{A\Gamma(\alpha\mu)}{\Gamma(\mu+0.5)}\left(\frac{\mu H}{\bar{\gamma}^{(\alpha/2)}}\right)^{\mu-0.5}\left\{\frac{p}{\left(a^2/2\right)^{\alpha\mu}}+\frac{q}{\left(b^2/2\right)^{\alpha\mu}} - \frac{pq}{\left(\left(a^2+b^2\right)/2\right)^{\alpha\mu}}\right\} \quad (19)$$

V. RESULTS

Here, ASER performance results are illustrated based on the proposed analytical expressions in comparison with exact results to validate the ASER expressions derived in this paper. The derived ASER expression includes convergent infinite series, which is truncated by K and M number of finite terms. Analytical results of (14) were obtained by setting the upper limits of the infinite summations to $K=10$ and $M=30$. Fig. 1 shows the ASER performance of 4X2 QAM scheme for several values of fading parameters. As can be seen from Fig. 1, the analytical results of (14) match closely with the exact results while the asymptotic results are very tight at high SNR regime. A transition from $\alpha=1$, $\eta=0.5$ and $\mu=1.5$ to $\alpha=1.5$, $\eta=0.3$ and $\mu=1.5$ provides a considerable performance improvement which is more than 5 dB even if the value of η decreases from 0.5 to 0.3 with fixed value of μ.

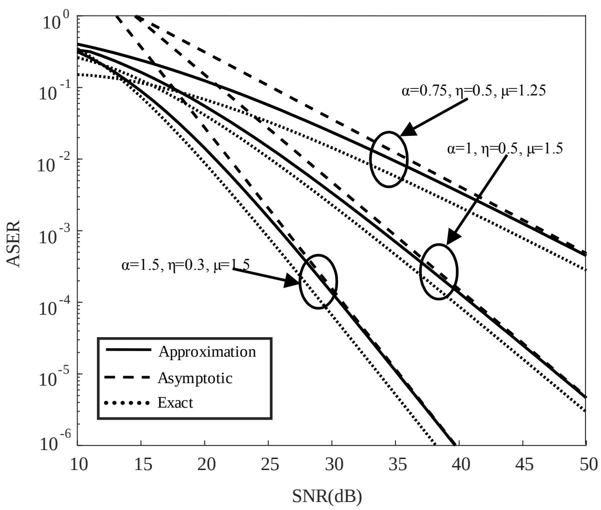

Fig.1. ASER performance of 4X2 QAM scheme with β=1

In Fig. 2, the ASER performance of 8X4 QAM modulated wireless systems is presented. Again, it can be seen that the exact results and approximated results are in close agreement. Moreover, the asymptotic results become tight with the approximate results at high SNR regime. For the case of $\alpha=1.5$, $\eta=0.3$ and $\mu=1.5$, 4X2 QAM scheme provides $P_s(e)=10^{-6}$ at 40 dB while the same ASER value is obtained at 46 dB with 8X4 QAM scheme. As expected, when the constellation size inceases, the performance decreases.

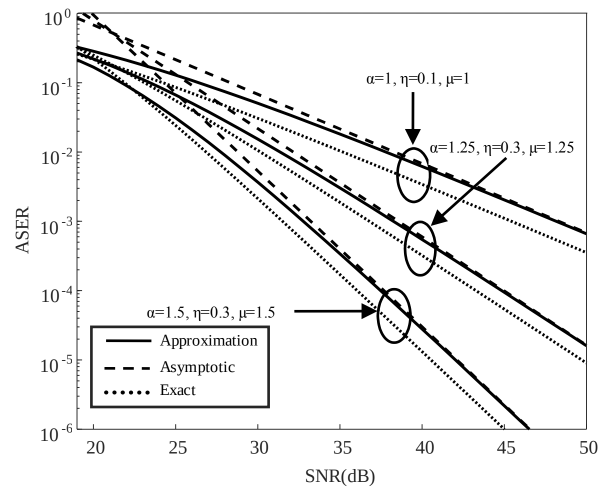

Fig.2. ASER performance of 8X4 QAM scheme with β=1.

VI. CONCLUSION

We have derived approximated and asymptotic ASER expressions for wireless communication systems using RQAM scheme over $\alpha-\eta-\mu$ fading channels. The proposed approximate expression is in rapidly convergent form and its analytical results show close agreement to the exact ones. In addition, the asymptotic results are also tight with the approximate results at high SNR. In addition, it should be highlighted that using more terms for the infinite series does not have any influence in the 15th decimal place of the results and RTE values are decreasing rapidly to the negligible levels as given in Table I and Table II. As a result, one can easily obtain the ASER performance of the considered system over well-known fading channels such as Rayleigh, Nakagami-*m*, Weibull and so on by using the flexibility of $\alpha-\eta-\mu$ fading.

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