

## Coretractable modules relative to a submodule

Research Article

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**Abstract:** Let  $R$  be a ring and  $M$  a right  $R$ -module. Let  $N$  be a proper submodule of  $M$ . We say that  $M$  is  $N$ -coretractable (or  $M$  is coretractable relative to  $N$ ) provided that, for every proper submodule  $K$  of  $M$  containing  $N$ , there is a nonzero homomorphism  $f : M/K \rightarrow M$ . We present some conditions that a module  $M$  is coretractable if and only if  $M$  is coretractable relative to a submodule  $N$ . We also provide some examples to illustrate special cases.

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### 1. Introduction

Throughout this paper  $R$  will denote an arbitrary associative ring with identity and all modules will be unitary right  $R$ -modules unless stated otherwise. Let  $M$  be an  $R$ -module. We use  $\text{End}_R(M)$ ,  $\text{ann}_r(M)$  (in the case  $M$  is a right  $R$ -module),  $\text{ann}_l(M)$  (in the case  $M$  is a left  $R$ -module) to denote the ring of endomorphisms of  $M$ , the right annihilator in  $R$  of  $M$  and the left annihilator in  $R$  of  $M$ , respectively. Let  $M$  be a module and  $K$  a submodule of  $M$ . Then  $K$  is essential in  $M$  denoted by  $K \leq_e M$ , if  $L \cap K \neq 0$  for every nonzero submodule  $L$  of  $M$ . Dually,  $K$  is small in  $M$  ( $K \ll M$ ), in case  $M = K + L$  implies that  $L = M$ . A submodule  $N$  of  $M$  is called *supplement*, if there is a submodule  $K$  of  $M$  such that  $M = N + K$  and  $N \cap K \ll N$ . A module  $M$  is called *supplemented* if every submodule of  $M$  has a supplement in  $M$ . For any unexplained terminology we refer to [3], [9] and [11].

Khuri in [5] introduced the concept of a retractable module. Let  $M$  be a module. Then  $M$  is retractable in case for every nonzero submodule  $N$  of  $M$ , there is a nonzero homomorphism  $f : M \rightarrow N$ , i.e.  $\text{Hom}_R(M, N) \neq 0$ . In the literature, there are some works about retractable modules (see [6, 12, 14]). Amini, Ershad and Sharif in [2] defined a dual notation namely coretractable modules. A module  $M$  is coretractable provided that,  $\text{Hom}_R(M/N, M) \neq 0$  for every proper submodule  $N$  of  $M$ . There are also some papers whose main subject is to study and investigate coretractable modules. We refer readers to [1, 4, 13] for more information about coretractable modules.

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In [10], the author introduced a generalization of coretractable modules via the cosingular submodule. Following [10], a module  $M$  is called  $\overline{Z}(M)$ -coretractable in case, for every proper submodule  $N$  of  $M$  containing  $\overline{Z}(M)$ , there is a nonzero homomorphism  $f : M/N \rightarrow M$ . It is proved in [10, Theorem 2.11] that a ring  $R$  is  $\overline{Z}(R_R)$ -coretractable if and only if every finitely generated free right  $R$ -module  $F$  is  $\overline{Z}(F)$ -coretractable. Also, a characterization of commutative semiperfect Kasch rings is presented via  $\overline{Z}$ -coretractability ([10, Corollary 2.14]). Inspiring by [10], we are interested to study coretractability of modules relative to their submodules. If in the definition of a coretractable module  $M$ , we fix a submodule  $N$  and focus just on nonzero homomorphisms from  $M/K$  to  $M$  where  $K \neq M$  contains  $N$ , we have a special generalization of coretractable modules. We may choose special submodules of a module  $M$  such as  $Soc(M)$ ,  $Rad(M)$  and some others. We present some necessary conditions to prove that when two concepts coretractable and coretractable relative to a submodule coincide. Among them, we show that for a small or a semisimple submodule  $N$  of  $M$ ,  $M$  is coretractable if and only if  $M$  is  $N$ -coretractable. It is also shown that if  $M$  is  $N$ -coretractable and  $N$  is coretractable, then  $M$  is coretractable. For a right ideal  $I$  of  $R$ , we show that  $R_R$  is  $I$ -coretractable if and only if every simple right  $R$ -module that is annihilated by  $I$ , can be embedded in  $R_R$ . As a consequence,  $R_R$  is coretractable if and only if  $R$  is right Kasch.

## 2. Coretractable modules relative to a submodule

In this section we introduce a new generalization of coretractable modules via submodules.

Recall that a module  $M$  is coretractable, in case for every proper submodule  $N$  of  $M$ , there exists a nonzero homomorphism  $f : M/N \rightarrow M$ .

**Definition 2.1.** Let  $M$  be a module and  $N$  a proper submodule of  $M$ . We say  $M$  is  $N$ -coretractable in case for every proper submodule  $K$  of  $M$  containing  $N$ , there is a nonzero homomorphism  $f : M/K \rightarrow M$ . Note that a module  $M$  is coretractable if and only if  $M$  is  $\{0\}$ -coretractable.

Let  $M$  be a module and  $N$  a proper submodule of  $M$ . It is not hard to verify that  $M$  is  $N$ -coretractable if and only if for every proper essential submodule  $K$  of  $M$  containing  $N$ , there is a nonzero homomorphism from  $M/K$  to  $M$ .

Note that if a module  $M$  is  $N$ -coretractable, then for every submodule  $T \subseteq N$ , there is a nonzero homomorphism  $g : M/T \rightarrow M$ . In fact, if  $M$  is  $N$ -coretractable, then for every submodule  $T$  of  $M$ , either contained in  $N$  or containing  $N$ , there will be a nonzero homomorphism from  $M/T$  to  $M$ .

Recall from [7], a ring  $R$  is right (left) Kasch in case every simple right (left)  $R$ -module can be embedded in  $R_R$  ( ${}_R R$ ). In [2, Theorem 2.14], the authors proved that  $R$  is right Kasch if and only if  $R_R$  is coretractable.

Let  $R$  be a right Kasch ring which is not left perfect. Then by [4, Proposition 2.9], there is a right ideal  $I$  of  $R$  such that  $R/I$  is not coretractable while  $R_R$  is coretractable as  $R$  is a Kasch ring (see also [4, Example 2.10]).

**Lemma 2.2.** (1) Let  $N, K, N_i < M$ . Let  $M$  be  $N$ -coretractable. If  $K \supseteq N$ , then  $M$  is  $K$ -coretractable. In particular, if  $M$  is  $N_i$ -coretractable for each  $i \in I$ , then  $M$  is  $(\sum_{i \in I} N_i)$ -coretractable.

(2) Let  $M$  be  $N$ -coretractable. If  $K \leq N$  such that  $K$  contains no nonzero image of any endomorphism of  $M$ , then  $M/K$  is  $N/K$ -coretractable. In a special case, if  $M$  is  $N$ -coretractable such that for every  $f \in End(M)$ ,  $Im f \not\subseteq N$ , then  $M/N$  is coretractable (see [4, Proposition 2.11]).

**Proof.** (1) This is straightforward.

(2) Let  $T/K$  be a proper submodule of  $M/K$  containing  $N/K$ . Then  $N \subseteq T \subset M$ . Since  $M$  is  $N$ -coretractable, there exists a nonzero homomorphism  $g : M/T \rightarrow M$ . Now define  $h : \frac{M/K}{T/K} \rightarrow M/K$  by  $h(x + K + \frac{T}{K}) = g(x + T) + K$  for every  $x \in M$ . If  $Im h = 0$ , then  $Im g \subseteq K$ . Now,  $K$  contains

the image of the endomorphism  $g\circ\pi$  of  $M$  where  $\pi : M \rightarrow M/T$  is the natural epimorphism, this gives a contradiction. Therefore,  $M/K$  is  $N/K$ -coretractable.  $\square$

Let  $R$  be a right Noetherian ring and  $M$  be a  $N$ -coretractable module where  $N$  is a finitely generated proper submodule of  $M$ . Then by Lemma 2.2(2),  $M/N$  is coretractable (see [4, Corollary 2.13]).

**Proposition 2.3.** *Let  $M$  be a module and  $K \leq N < M$ . If  $M/K$  is  $N/K$ -coretractable and  $M/K$  can be embedded in  $M$ , then  $M$  is  $N$ -coretractable. In particular, if  $M = K \oplus K'$  and  $N$  is any submodule of  $M$  such that  $K'$  is  $(N \cap K')$ -coretractable, then  $M$  is  $N$ -coretractable.*

**Proof.** Let  $T$  be a proper submodule of  $M$  containing  $N$ . Then  $T/K$  is a proper submodule of  $M/K$  containing  $N/K$ . By assumption, there is a nonzero homomorphism  $g : \frac{M/K}{T/K} \cong M/T \rightarrow M/K$ . There also exists a monomorphism  $h : M/K \rightarrow M$ . Now, the homomorphism  $hog : M/T \rightarrow M$  is the required one.  $\square$

**Corollary 2.4.** *Let  $M$  be a module and  $N < M$  such that  $M/N$  is coretractable. If  $M/N$  can be embedded in  $M$ , then  $M$  is  $N$ -coretractable. In particular, if  $M$  is supplemented with  $\text{Rad}(M)$  a direct summand of  $M$ , then  $M$  is  $\text{Rad}(M)$ -coretractable.*

**Proof.** This is a special case of Proposition 2.3. The last part follows from the fact that for a supplemented module  $M$ , the module  $M/\text{Rad}(M)$  is coretractable since  $M/\text{Rad}(M)$  is semisimple. In this case  $M$  is  $\text{Rad}(M)$ -coretractable.  $\square$

**Example 2.5.** (1) *Let  $M$  be a coretractable module and  $N < M$ . Then  $M$  is  $N$ -coretractable. In particular, every cogenerator  $M$  in the category of right  $R$ -modules is coretractable relative to every  $N < M$ .*

(2) *Let  $M$  be a module such that for every submodule  $K$  of  $M$  we have  $M/K \cong M$ . Then  $M$  is coretractable relative to each  $N < M$ .*

(3) *Let  $M$  be a module and  $N < M$ . If every proper submodule of  $M$  containing  $N$ , is contained in a proper summand of  $M$ , then  $M$  is  $N$ -coretractable.*

(4) *Let  $M$  be an uniserial module. If  $M$  is coretractable relative to a proper submodule  $N$ , then  $M$  is coretractable.*

The following introduces a  $N$ -coretractable module which is not coretractable. In fact, the class of relative coretractable modules properly contains the class of coretractable modules.

**Example 2.6.** *Let  $P$  be the set of all prime numbers and  $M = \prod_{p \in P} \mathbb{Z}_p$  as an  $\mathbb{Z}$ -module. Take  $N = \{0\} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \dots$  which is a maximal submodule of  $M$ , since  $M/N \cong \mathbb{Z}_2$ . Consider  $g : \mathbb{Z}_2 \rightarrow M$  defined by  $g(x) = (x, 0, 0, \dots)$ . Then  $g$  is a nonzero homomorphism indicating that  $M$  is  $N$ -coretractable. Note that by [2, Example 2.9],  $M$  is not a coretractable  $\mathbb{Z}$ -module.*

**Remark 2.7.** *Let  $M$  be a module and  $N < M$ . If there is not a nonzero homomorphism from  $M/N$  to  $M$ , then  $M$  is not  $N$ -coretractable. For example, let  $M$  be a nonsingular module and  $N$  be a proper submodule of  $M$  such that  $M/N$  is singular. So there does not exist any nonzero homomorphism from  $M/N$  to  $M$ . Now,  $M$  is not  $N$ -coretractable (for example,  $\mathbb{Z}$ -modules  $\mathbb{Q}$  and  $\mathbb{Z}$  can not be  $n\mathbb{Z}$ -coretractable).*

We shall consider some conditions under which the two concepts coretractable and  $N$ -coretractable coincide.

**Lemma 2.8.** *Let  $M$  be a module and  $N < M$ . In each of the following cases  $M$  is  $N$ -coretractable if and only if  $M$  is coretractable.*

- (1)  $N$  is a small submodule of  $M$ .
- (2)  $N$  is a coretractable module.

**Proof.** (1) Let  $M$  be  $N$ -coretractable where  $N \ll M$  and  $K$  be a proper submodule of  $M$ . Since  $N$  is small in  $M$ , we have  $N + K \neq M$ . Now since  $M$  is  $N$ -coretractable, then there is a nonzero homomorphism  $f : M/(N + K) \rightarrow M$ . So that  $Hom_R(M/K, M) \neq 0$ . It follows that  $M$  is coretractable. The converse is clear.

(2) Let  $K$  be a proper submodule of  $M$ . Then either  $K + N \neq M$  or  $K + N = M$ . If  $K + N \neq M$ , then similarly to (1) we have  $Hom_R(M/K, M) \neq 0$ . Now suppose that  $K + N = M$ . Then there is an isomorphism  $h : M/K \rightarrow N/(N \cap K)$  induced from  $M = N + K$ . Since  $N$  is coretractable, there is a nonzero homomorphism  $g : N/(N \cap K) \rightarrow N$ . Therefore,  $jogoh : M/K \rightarrow M$  is a nonzero homomorphism where  $j : N \rightarrow M$  is the inclusion.  $\square$

Recall that a module  $M$  is hollow, provided every proper submodule of  $M$  is small in  $M$ .

**Corollary 2.9.** (1) Let  $M$  be a hollow module and  $N < M$ . Then  $M$  is  $N$ -coretractable if and only if  $M$  is coretractable.

(2) Let  $M$  be a finitely generated module. Then  $M$  is  $Rad(M)$ -coretractable if and only if  $M$  is coretractable.

(3) Let  $N$  be a semisimple submodule of  $M$ . Then  $M$  is  $N$ -coretractable if and only if  $M$  is coretractable.

(4) Let  $M$  be a module. Then  $M$  is  $Soc(M)$ -coretractable if and only if  $M$  is coretractable.

Let  $M$  be a module and  $N$  a submodule of  $M$ . Following [15],  $N$  is  $\delta$ -small in  $M$  (denoted by  $N \ll_\delta M$ ), in case  $M = N + K$  with  $M/K$  singular implies that  $M = K$ . Note that by definitions, every small submodule of  $M$  is  $\delta$ -small in  $M$ . The sum of all  $\delta$ -small submodules of  $M$  is denoted by  $\delta(M)$ . Also  $\delta(M)$  is the reject of the class of all simple singular modules in  $M$ .

**Proposition 2.10.** Let  $M$  be a module and  $N$  be a proper  $\delta$ -small submodule of  $M$ . Then  $M$  is  $N$ -coretractable if and only if  $M$  is coretractable.

**Proof.** Let  $M$  be  $N$ -coretractable and  $K$  be a proper submodule of  $M$ . Suppose that  $M \neq N + K$ . Since  $M$  is  $N$ -coretractable, there is a nonzero homomorphism  $f : M/(N + K) \rightarrow M$ . So that  $f \circ \pi : M/K \rightarrow M$  is the required homomorphism where  $\pi : M/K \rightarrow M/(N + K)$  is natural epimorphism. Otherwise,  $M = N + K$ . Now from [15, Lemma 1.2], there is a decomposition  $M = Y \oplus K$  where  $Y$  is a semisimple projective submodule of  $N$ . Therefore, there is a monomorphism from  $M/K$  to  $M$  since  $K$  is a direct summand of  $M$ . It follows that  $M$  is coretractable.  $\square$

**Proposition 2.11.** Let  $M$  be a module and  $N$  be a proper submodule of  $M$ . If  $M$  is  $N$ -coretractable and  $M/N$  has a maximal submodule, then  $Soc(M) \neq 0$ . In particular, if  $M$  is finitely generated and  $N$ -coretractable, then  $Soc(M) \neq 0$ .

**Proof.** Let  $K/N$  be a maximal submodule of  $M/N$ . Then  $K$  is a maximal submodule of  $M$ . So there is a nonzero homomorphism  $h : M/K \rightarrow M$ . It follows that  $Imh$  is a simple submodule of  $M$ . This completes the proof.  $\square$

The following is an immediate consequence of last proposition.

**Corollary 2.12.** Let  $R$  be a ring such that every cyclic right  $R$ -module is coretractable relative to at least one of its submodules. Then  $R$  is semi-Artinian.

Let  $R$  be a ring. Then  $R$  is called a right  $V$ -ring in case every simple right  $R$ -module is injective. As a generalization of  $V$ -rings,  $R$  is a right generalized  $V$ -ring ( $GV$ -ring for short), if every simple singular right  $R$ -module is injective ([11]).

**Proposition 2.13.** *Let  $R$  be a ring and  $M$  be an indecomposable right  $R$ -module with  $\text{Rad}(M) \neq M$ . If each of the following statements holds, then  $M$  is  $\text{Rad}(M)$ -coretractable if and only if  $M$  is simple.*

- (1)  $R$  is a right GV-ring.
- (2)  $M$  is noncosingular.

**Proof.** (1) Let  $M$  be  $\text{Rad}(M)$ -coretractable. Then for each maximal submodule  $K$  of  $M$  there is a monomorphism  $g : M/K \rightarrow M$ . It follows that  $\text{Img}$  is a simple submodule of  $M$ . Then  $\text{Img}$  is either singular or projective. If  $\text{Img}$  is projective, then  $K$  is a direct summand of  $M$  and hence  $K = 0$  or  $K = M$ . So that  $K = 0$ . If  $\text{Img}$  is singular, it will be injective as  $R$  is right GV. Therefore,  $\text{Img}$  is a summand of  $M$  and since  $g \neq 0$  we conclude that  $\text{Img} = M$ . In both cases,  $M$  is simple. The converse is obvious.

(2) It follows from (1) and the fact that every homomorphic image of  $M$  is noncosingular. □

**Corollary 2.14.** *Let  $R$  be a right V-ring and  $M$  an indecomposable right  $R$ -module. Then  $M$  is coretractable if and only if  $M$  is simple.*

Following [8], a module  $M$  is dual Rickart provided that for every  $f \in \text{End}(M)$ ,  $\text{Im}f$  is a direct summand of  $M$ .

**Remark 2.15.** *Let  $M$  be an indecomposable dual Rickart module with  $\text{Rad}(M) \neq M$ . Then  $M$  is  $(\text{Rad}(M))$ -coretractable if and only if  $M$  is simple. Let  $K$  be a maximal submodule of  $M$ . Then there is a monomorphism  $g : M/K \rightarrow M$ . Consider the endomorphism  $h = g\pi : M \rightarrow M$  where  $\pi : M \rightarrow M/K$  is the natural epimorphism. Then  $\text{Im}h = \text{Img}$  is a summand of  $M$ . So  $\text{Img} = M$  as  $M$  is indecomposable. It follows that  $M$  is simple.*

**Proposition 2.16.** *Let  $M$  be a module and  $L$  a proper submodule of  $M$  such that  $L$  has a supplement  $K$  in  $M$ . If  $M$  is  $L$ -coretractable and  $K$  is fully invariant in  $M$ , then  $K$  is coretractable.*

**Proof.** Let  $K$  be a supplement of  $L$  in  $M$ . Then  $M = K + L$  and  $K \cap L \ll K$ . Let  $N$  be a proper submodule of  $K$ . Then  $N + L$  is a proper submodule of  $M$ . For if,  $N + L = M$ , by modular law  $N + (K \cap L) = K$ , which implies that  $N = K$ , a contradiction. Since  $M$  is  $L$ -coretractable, there is a nonzero homomorphism  $f : M/(N + L) \rightarrow M$ . Since  $K$  is a fully invariant submodule of  $M$ , we have  $f\pi(K) \subseteq K$  where  $\pi : M \rightarrow M/(N + L)$  is the natural epimorphism. Now consider  $h : K/N \rightarrow K$  by  $h(x + N) = f(x + N + L)$  for every  $x \in K$ . It is not hard to verify that  $h$  is well-defined. Now, there is  $y \in M$  such that  $y \notin N + L$  and  $f(y + N + L) \neq 0$ . Now there exists  $k \in K$  and  $l \in L$  such that  $y = k + l$ . It is easy to see that  $h(k + N) = f(k + l + N + L) = f(y + L) \neq 0$ . It follows that  $h$  is nonzero. □

**Corollary 2.17.** ([2, Proposition 2.5]) *Every fully invariant direct summand of a coretractable module is coretractable.*

Let  $M$  be a module. Then  $M$  is called a duo module provided every submodule of  $M$  is fully invariant.

**Corollary 2.18.** *Let  $M$  be a duo module. If  $M$  is coretractable relative to each direct summand of  $M$ , then every direct summand of  $M$  is coretractable.*

**Proposition 2.19.** *Let  $M = M_1 \oplus \dots \oplus M_n$  and  $N < M$ . If each  $M_i$  is  $N \cap M_i$ -coretractable, then  $M$  is  $N$ -coretractable. Especially a finite direct sum of coretractable modules is coretractable.*

**Proof.** The proof is exactly similar to proof of [2, Proposition 2.6]. □

**Proposition 2.20.** *Let  $R$  be a right max ring and  $M = \bigoplus_{i \in I} M_i$  be a direct sum of  $N \cap M_i$ -coretractable right  $R$ -modules where  $N < M$ . Then  $M$  is  $N$ -coretractable. In particular, an arbitrary direct sum of coretractable right  $R$ -modules is coretractable.*

**Proof.** Similar to the proof of [2, Proposition 2.7]. □

Let  $M$  be an  $R$ -module. A submodule  $K$  of  $M$  is said to be dense in  $M$  if, for any  $y \in M$  and  $0 \neq x \in M$ , there exists  $r \in R$  such that  $xr \neq 0$  and  $yr \in K$ . Obviously, any dense submodule of  $M$  is essential in  $M$ . From [7, Proposition 8.6],  $K$  is dense in  $M$  if and only if  $\text{Hom}_R(P/K, M) = 0$  for every submodule  $P \supseteq K$ .

**Remark 2.21.** Let  $M$  be a module and  $N < M$ . If  $N$  is dense in  $M$ , then  $M$  is not  $N$ -coretractable. In fact for a  $N$ -coretractable module  $M$ , we have  $N$  is not dense in  $M$ . This follows from the fact that if  $M$  is  $N$ -coretractable, then there is a nonzero homomorphism from  $M/N$  to  $M$ .

**Proposition 2.22.** Let  $M$  be a module and  $N$  a proper submodule of  $M$ . If  $M$  is quasi-injective or every proper submodule of  $M$  is contained in a maximal submodule, then  $M$  is  $N$ -coretractable if and only if every proper submodule of  $M$  containing  $N$  is not dense in  $M$ .

**Proof.** (1) Let  $M$  be a quasi-injective module such that every proper submodule of  $M$  containing  $N$  is not dense in  $M$ . Suppose that  $K$  is a proper submodule of  $M$  containing  $N$ . Since  $K$  is not dense in  $M$ , there is a  $f : P/K \rightarrow M$  where  $P$  is a submodule of  $M$  containing  $K$ . It follows that  $f \circ \pi : P \rightarrow M$  is a nonzero homomorphism where  $\pi : P \rightarrow P/K$  is the natural epimorphism. Consider the inclusion homomorphism  $j : P \rightarrow M$ . Since  $M$  is quasi-injective, there exists  $h : M \rightarrow M$  such that  $h \circ j = f \circ \pi$ . By defining  $\bar{h} : M/K \rightarrow M$  with  $\bar{h}(m + K) = h(m)$  we conclude that  $M$  is  $N$ -coretractable. Note that  $\bar{h}$  is nonzero. Conversely, if  $M$  is  $N$ -coretractable and  $N \subseteq K < M$ , then there is a homomorphism  $g : M/K \rightarrow M$  which shows that  $K$  is not dense in  $M$ .

(2) Suppose that every submodule of  $M$  is contained in a maximal submodule of  $M$ . Let  $N \subseteq K < M$ . Then there is a maximal submodule  $L$  of  $M$  such that  $K \leq L$ . Since  $L$  is not dense in  $M$ , there is a nonzero homomorphism  $h : M/L \rightarrow M$ . As  $f : M/K \rightarrow M/L$  with  $f(x + K) = x + L$  is a nonzero homomorphism, then  $h \circ f$  is nonzero. It follows that  $M$  is  $N$ -coretractable. The converse is the same as (1).  $\square$

The following presents a characterization of  $I$ -coretractable rings.

**Theorem 2.23.** Let  $R$  be a ring and  $I$  be a proper right ideal of  $R$ . Then the following are equivalent:

- (1)  $R_R$  is  $I$ -coretractable;
- (2) Every  $n$ -generated free right  $R$ -module is  $I^{(n)}$ -coretractable;
- (3) For every right ideal  $T \supseteq I$ ,  $\text{ann}_l(T) \neq 0$ .

**Proof.** (1)  $\Leftrightarrow$  (2) Follows from Proposition 2.19.

(1)  $\Rightarrow$  (3) Let  $T$  be a right ideal of  $R$  containing  $I$ . Since  $R_R$  is  $I$ -coretractable, there is a nonzero homomorphism  $f : R/T \rightarrow R$ . Consider the endomorphism  $g = f \circ \pi : R \rightarrow R$  where  $\pi$  is the natural epimorphism from  $R$  to  $R/T$ . Then there is an element  $0 \neq a \in R$  such that  $g(x) = ax$ . Let  $y \in T$ . Then  $g(y) = ay = 0$  as  $T \subseteq \text{Ker } g$ . This shows that  $0 \neq a \in \text{ann}_l(T)$ .

(3)  $\Rightarrow$  (1) Let  $T$  be a right ideal of  $R$  containing  $I$ . Since  $\text{ann}_l(T) \neq 0$ , there exists an element of  $R$  such as  $a$  that  $aT = 0$  and  $a \neq 0$ . Define  $f : R/T \rightarrow R$  by  $f(x + T) = ax$ . It is easy to check that  $f$  is an  $R$ -homomorphism and in particular  $f \neq 0$ .  $\square$

**Remark 2.24.** Let  $R$  be a ring and  $I \leq R_R$  with  $\text{ann}_l(I) = 0$ . Then  $R_R$  is not  $I$ -coretractable. For example, let  $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$  be the ring of  $2 \times 2$  upper triangular matrices over a field  $K$ . Let  $I = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}$  which is a right ideal of  $R$ . Then  $\text{ann}_l(I) = 0$ . Hence,  $R_R$  is not  $I$ -coretractable. In other words,  $R/J(R)$  is coretractable relative to each of its ideals as  $R/J(R)$  is a semisimple ring. Note that  $J(R) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ .

**Theorem 2.25.** Let  $R$  be a ring and  $I$  be a proper two-sided ideal of  $R$ . Then the following statements are equivalent:

- (1)  $R_R$  is  $I$ -coretractable;
- (2) Every simple right  $R$ -module that is annihilated by  $I$  can be embedded in  $R_R$ .

**Proof.** (1)  $\Rightarrow$  (2) Let  $M \cong R/K$  be a simple right  $R$ -module such that  $MI = 0$ . It follows that  $I \subseteq K$ . Since  $R_R$  is  $I$ -coretractable, there is a nonzero homomorphism  $f : R/K \rightarrow R$ .

(2)  $\Rightarrow$  (1) Let  $T$  be a right ideal of  $R$  containing  $I$ . Now there exists a right maximal ideal  $K$  of  $R$  such that  $I \subseteq T \subseteq K$ . Consider the simple right  $R$ -module  $M = R/K$ . Since  $MI = 0$ , there is a nonzero homomorphism  $g : R/K \rightarrow R$  by assumption. As  $T$  is a submodule of  $K$ , there exists  $f : R/T \rightarrow R/K$  defined by  $f(x + T) = x + K$ . Hence  $gof$  is the desired homomorphism.  $\square$

For a ring  $R$ , Theorem 2.25 implies that  $R_R$  is  $(J(R))$ -coretractable if and only if  $R$  is a right Kasch ring.

In [2, Proposition 4.4], it is shown that if  $R$  is a von Neumann regular ring then  $R$  is right Kasch if and only if  $R$  is semisimple. In the following we shall investigate a more general version.

**Proposition 2.26.** *Let  $R$  be a right GV-ring. Then the following are equivalent:*

- (1)  $R$  is right Kasch;
- (2)  $R$  is semisimple.

**Proof.** (1)  $\Rightarrow$  (2) Let  $R$  be right Kasch. So  $R_R$  is  $J(R)$ -coretractable. Now suppose that  $K$  is an arbitrary maximal right ideal of  $R$ . Then there is a monomorphism  $g : R/K \rightarrow R$ . It follows that  $R/K \cong \text{Im}g$  is a simple right  $R$ -module. So,  $\text{Im}g$  is either singular or projective. In first case  $\text{Im}g$  should be injective as  $R$  is right GV. Therefore,  $\text{Im}g$  is a direct summand of  $R_R$ . Now  $\text{Im}g$  is singular projective which implies that  $\text{Im}g = 0$ , a contradiction. So that  $\text{Im}g$  and hence every simple right  $R$ -module will be projective. This shows that  $R$  is semisimple.

(2)  $\Rightarrow$  (1) It is obvious.  $\square$

**Corollary 2.27.** *Let  $R$  be a right V-ring. Then  $R$  is a Kasch ring if and only if  $R$  is semisimple.*

**Example 2.28.** (1) Let  $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$  where  $K$  is a field. Then  $J(R) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ . It is easy to check that  $R$  is a semilocal ring as  $R/J(R) \cong K \times K$  which is a semisimple ring. Now by [3, Exercise 10, Page 113],  $\text{Soc}({}_R R) = \begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix}$ . However,  $\text{Soc}(R_R) = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}$ . Set  $m_1 = \text{Soc}({}_R R)$  and  $m_2 = \text{Soc}(R_R)$ . Then both  $m_1$  and  $m_2$  are maximal left and right ideals of  $R$ . A quick calculation shows that  $\text{ann}_l(m_1) = m_2$ ,  $\text{ann}_l(m_2) = 0$ ,  $\text{ann}_r(m_1) = 0$  and  $\text{ann}_r(m_2) = m_1$ . Now by Theorem 2.23,  $R_R$  is  $m_1$ -coretractable while  $R_R$  is not  $m_2$ -coretractable. Also left version of Theorem 2.23, implies that  ${}_R R$  is  $m_2$ -coretractable but it is not  $m_1$ -coretractable. Since the simple right  $R$ -module  $R/m_2$  can not be embedded in  $R_R$  and the simple left  $R$ -module  $R/m_1$  can not be embedded in  ${}_R R$ , the ring  $R$  is neither right Kasch nor left Kasch (note that since  $R$  is right GV which is not a V-ring, it can not be Kasch from Proposition 2.26).

(2) Let  $K$  be a division ring and  $R = \{A = \begin{bmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \mid a, b, c, d, e \in K\}$ . Then  $J(R) = \{A \in R \mid a = 0 = e\}$ ,  $\text{Soc}({}_R R) = \text{ann}_l(J(R)) = \{A \in R \mid a = 0\}$ ,  $\text{Soc}(R_R) = \text{ann}_r(J(R)) = J(R)$ . Since  $R/J(R) \cong K \times K$ ,  $R$  is a semilocal ring. Now  $\text{Soc}({}_R R) = \{A \in R \mid a = 0\}$  and  $\text{Soc}(R_R) = J(R)$ . From [7, Example 8.29],  $\text{Soc}({}_R R)$  is a left and right maximal ideal of  $R$ . Since  $\text{ann}_r(\text{Soc}({}_R R)) = \{A \in R \mid a = e = 0\} = J(R) \neq 0$ , it follows from [7, Corollary 8.28],  $R/\text{Soc}({}_R R)$  can be embedded in  ${}_R R$  (see also Theorem 2.23). Therefore,  ${}_R R$  is  $\text{Soc}({}_R R)$ -coretractable while  $R_R$  is not  $\text{Soc}(R_R)$ -coretractable (see also Corollary 2.9). Now an easy computation shows that  $\text{ann}_l(\text{Soc}({}_R R)) = \{A \in R \mid a = c = d = e = 0\} \neq 0$ . So  $R/\text{Soc}({}_R R)$  can be embedded in  $R_R$  by [7, Corollary 8.28]. As  $\text{Soc}(R_R)$  is a maximal right ideal of  $R$ , then  $R_R$  is  $\text{Soc}(R_R)$ -coretractable. Also from [7, Example 8.29],  $R$  is a right Kasch ring while it is not a left Kasch ring.

(3) Let  $K$  be a field and  $R = \prod_{i=1}^{\infty} K$ . It is well-known that  $R$  is a Von Neumann regular V-ring. Consider the ideal  $T_i = K \times K \times \dots \times K \times 0 \times K \times K \times \dots$ . It is clear that  $T_i$  for each  $i \in \mathbb{N}$  is a maximal

ideal of  $R$ . It is easy to see that  $\text{ann}(T_i) = 0 \times 0 \times \dots \times 0 \times K \times 0 \times \dots$  which is nonzero. Therefore, from Theorem 2.23,  $R$  is  $I$ -coretractable for each  $I \subseteq T_i$ . Now consider the ideal  $L = \bigoplus_{i=1}^{\infty} K$  of  $R$ . Then  $\text{ann}(L) = 0$  and of course  $\text{ann}(m) = 0$  for every maximal ideal  $m$  of  $R$  containing  $L$ . Hence the simple  $R$ -module  $R/m$  can not be embedded in  $R$  (see [7, Corollary 8.28]). Therefore,  $R$  is not coretractable relative to  $L$ . This means that  $R$  is not a Kasch ring.

**Proposition 2.29.** *Let  $R$  be a ring and  $I$  a right ideal of  $R$  such that every free right  $R$ -module  $R^{(A)}$  is  $(I^{(A)})$ -coretractable. Then for every right  $R$ -module  $M$  with  $I \subseteq \text{ann}_r(M)$ ,  $\text{Hom}_R(M, R) \neq 0$ .*

**Proof.** Let  $M$  be a right  $R$ -module such that  $I \subseteq \text{ann}_r(M)$ . Then there is a free right  $R$ -module  $F$  and a submodule  $K$  of  $F$  such that  $M \cong F/K$ . Since  $MI = 0$ , we have  $I^{(A)} \subseteq K$  where  $A$  is an indexed set. By assumption, there is a nonzero homomorphism  $f : F/K \rightarrow F$ . Then the homomorphism  $\pi \circ f : M \rightarrow R$  is the required one where  $\pi : F \rightarrow R$  is the natural epimorphism.  $\square$

**Proposition 2.30.** *Let  $R$  be a ring having a radical right  $R$ -module  $M$  with  $MI \neq M$  where  $I \leq R_R$ . If for every right ideal  $T$  of  $R$ ,  $\text{Rad}(T) \neq T$ , then there is a free right  $R$ -module  $R^{(A)}$  which is not  $I^{(A)}$ -coretractable.*

**Proof.** Let  $\text{Rad}(M) = M$  such that  $MI$  is a proper submodule of  $M$ . There exists a free right  $R$ -module  $F = R^{(A)}$  and a submodule  $K$  of  $F$  such that  $M/MI \cong F/K$ . Being  $M$  radical implies that  $M/MI$  is radical. So,  $\text{Hom}_R(M/MI, R) = 0$ . Since  $(F/K)I = 0$ ,  $I^{(A)} \subseteq K$ . It follows that  $\text{Hom}_R(F/K, F) = 0$  which implies  $F$  is not  $I^{(A)}$ -coretractable.  $\square$

**Proposition 2.31.** *Let  $R$  be a right max ring and  $I \leq R_R$  such that every cyclic  $R$ -module  $N$  is  $NI$ -coretractable. Then every right  $R$ -module  $M$  is  $MI$ -coretractable. In particular, if  $R$  is a (semiperfect) right perfect ring with all cyclic right  $R$ -modules coretractable, then every (finitely generated) right  $R$ -module is coretractable.*

**Proof.** Let  $M$  be a right  $R$ -module. Suppose that  $K$  is a proper submodule of  $M$  containing  $MI$ . Since  $R$  is a right max ring,  $K$  is contained in a maximal submodule  $L$  of  $M$ . For every  $x \in M \setminus L$ , we know  $M/L \cong xR/(xR \cap L)$  as  $xR + L = M$ . Note that  $MI \subseteq L$ . So that  $(xR/(xR \cap L))I = 0$ . It follows that  $(xR)I \subseteq xR \cap L$ . Being  $xR$  a  $(xR)I$ -coretractable module implies that  $\text{Hom}_R(xR/(xR \cap L), xR) \neq 0$ . Hence there is a nonzero homomorphism  $f : M/L \rightarrow M$ . Therefore,  $\text{Hom}_R(M/K, M) \neq 0$  as  $K \subseteq L$ .  $\square$

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